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On a Class of Nonlinear Fractional Schrödinger-Poisson Systems

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Abstract

In this paper, we are concerned with the following fractional Schrödinger-Poisson system:

$$\begin{cases} (-\Delta^s)u + V(x)u + \phi u = m(x)|u|^{q-2}|u| + f(x,u), & x \in \Omega, \\ (-\Delta^t)\phi = u^2, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases}$$

where $s, t \in (0, 1]$, $2t + 4s > 3$, $1 < q < 2$ and Ω is a bounded smooth domain of $\mathbb{R}^3$, and $f(x, u)$ is linearly bounded in u at infinity. Under some assumptions on m, V and f we obtain the existence of non-trivial solutions with the help of the variational methods.

Keywords: Fractional Schrödinger-Poisson systems, Non-trivial solutions, Variational methods.

1. Introduction

The fractional Schrödinger equation was introduced by Laskin [1] in the context of fractional quantum mechanics for the study of particles on stochastic fields modeled by Lévy processes. The operator $(-\Delta)^s$ can be seen as the infinitesimal generator of Lévy stable diffusion processes (see Applebaum [2]).

The aim of this paper is to investigate the existence of non-trivial solutions for the following fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta^s)u + V(x)u + \phi u = m(x)|u|^{q-2}|u| + f(x,u), & x \in \Omega, \\ (-\Delta^t)\phi = u^2, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $s, t \in (0,1]$, $2t+4s>3$, $1<q<2$, Ω is a bounded smooth domain of $\mathbb{R}^3$, $(-\Delta^s)$ is the fractional Laplacian and $f(x,u)$ is linearly bounded in u at infinity that satisfying some conditions we will precise later.

When $s=t=1$ and $V=m\equiv 1$ the equation (1.1) reduces to Schrödinger-Poisson equation, which describes quantum particles and is related to the study of nonlinear stationary Schrödinger equations interacting with the electromagnetic field generated by the motion [3, 4].
In a recent paper [5] the following Schrödinger-Poisson system was studied

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = K(x)|u|^{q-2}u + f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, \quad \lim\limits_{|x|\to\infty}\phi(x) = 0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $1<q<2$, $\lambda>0$ is a parameter and $f(x,u)$ is linearly bounded in u at infinity.

Fractional Schrödinger-Poisson equations have attracted some attention in recent years. If we only consider the first equation in (1.1) and assume that $\phi=0$, then it reduces to a fractional Schrödinger equation, which is a fundamental equation in fractional quantum mechanics [5, 6].

Recently, some authors proposed a new approach called perturbation method to study the quasi-linear elliptic equations, see [7, 8]. Kexue Li in [9] studied the nonlinear fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta^s)u + u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ (-\Delta^t)\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

and by using the perturbation method and mountain pass theorem, obtained the existence of non-trivial solutions. Motivated by the above works, we study the existence and multiplicity of solutions for the problem (1.1).

Before stating our main results, we give the following assumptions on m, V and f.

(H1) $m(x) \in L^{\frac{2}{2-q}}(\Omega)$; and $m(x)>0$ for $x\in\Omega$

(H2) $V \in C(\Omega,\mathbb{R})$ and $\inf_\Omega V(x) \geq V_0 > 0$

(H3) For every $x\in\Omega$ and $u\in\mathbb{R}$, there exist constants $C_1>0$ and $p\in[2,2_s^*)$ such that

$$|f(x,u)| \leq C_1(|u| + |u|^{p-1}),$$

where $2_s^* = \frac{6}{3-2s}$ is the fractional critical Sobolev exponent;

(H4) There exists $C>0$ such that

$$\left|\frac{f(x,u)}{u}\right| \leq C, \text{ for all } x\in\Omega, u\in\mathbb{R} \text{ and } u\neq 0.$$

(H5) $f(x,u) = o(|u|)$, $|u|\to 0$, uniformly on Ω;

(H6) There exists $\mu > 4$ such that

$$0 < \mu F(x,u) \leq uf(x,u)$$

holds for every $x \in \Omega$ and $u \in \mathbb{R}\backslash\{0\}$, where $F(x,u) = \int_0^u f(x,s)ds$;

Throughout this paper, $C > 0$ will be used indiscriminately to denote a suitable positive constant whose value may change from line to line and we will use $o(1)$ for a quantity which goes to zero. Moreover, we use $\|.\|_p$ to denote the usual norm on $L^p(\Omega)$ for $1 < p < +\infty$. Our main results reads as follows.

Theorem 1.1. *Suppose that* $(H1)-(H5)$ *hold. Then there exists* $M > 0$ *such that for every* m *with* $\|m\|_{\frac{2}{2-q}} < M$ *, problem* (1.1) *has a nontrivial solution at negative energy.*

Theorem 1.2. *Suppose that* $(H1)-(H6)$ *hold., problem* (1.1) *has a nontrivial solution at negative energy.*

Corollary 1.3. *Suppose that* $(H1)-(H6)$ *hold. Then there exists* $M > 0$ *such that for every* m *with* $\|m\|_{\frac{2}{2-q}} < M$ *, problem* (1.1) *has at least two nontrivial solutions.*

The reminder of this paper is organized as follows. In section 2, we present a suitable variational framework for our problem. In section 3, we prove Theorems 1.1-1.2.

2. Variational setting and preliminaries

For $p \in [1,\infty)$, we denote by $L^p(\Omega)$ the usual Lebesgue space with the norm $\|u\|_p = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}$. For any $p \in [1,\infty)$ and $s \in (0,1)$, we recall some definitions of fractional Sobolev spaces and the fractional Laplacian $(-\Delta)^s$, for more details, we refer to [10]. $H^s(\Omega)$ is defined as follows

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) : \int_\Omega (1 + |\xi|^{2s})|\mathcal{F}u(\xi)|^2 d\xi < \infty \right\}$$

with the norm

$$\|u\|_{H^s} = \left(|\mathcal{F}u(\xi)|^2 + |\xi|^{2s}|\mathcal{F}u(\xi)|^2\right)^{\frac{1}{2}}, \tag{2.1}$$

where $\mathcal{F}u$ denotes the Fourier transform of u. By $\mathcal{S}(\Omega)$, we denote the Schwartz space of rapidly decaying C^∞ functions in Ω. For $u \in \mathcal{S}(\Omega)$ and $s \in (0,1)$, $(-\Delta)^s$ is defined by

$$(-\Delta)^s f = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}f)), \ \forall \xi \in \Omega.$$

By Plancherel's theorem, we have $\|\mathcal{F}u\|_2 = \|u\|_2$, $\||\xi|^s \mathcal{F}u\|_2 = \|(-\Delta)^{\frac{s}{2}}u\|$. Then by (2.1), we get the equivalent norm

$$\|u\|_{H^s} = \left(\int_\Omega (|(-\Delta)^{\frac{s}{2}}u(x)|^2 + |u(x)|^2)dx\right)^{\frac{1}{2}}.$$

For $s \in (0,1)$, the fractional Sobolev space $D^{s,2}(\Omega)$ is defined as follows

$$D^{s,2}(\Omega) = \left\{ u \in L^{2_s^*}(\Omega) : |\xi|^s \mathcal{F}u(\xi) \in L^2(\Omega) \right\},$$

which is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{D^{s,2}} = \left(\int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^{\frac{1}{2}} = \left(\int_\Omega |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Lemma 2.1. (Theorem 2.1 in [11]). *For any $s \in (0, \frac{3}{2})$, $D^{s,2}(\Omega)$ is continuously embedded in $L^{2_s^*}(\Omega)$, i.e., there exists $c_s > 0$ such that*

$$\left(\int_\Omega |u|^{2_s^*} dx \right)^{2/2_s^*} \leq c_s \int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 dx, \;\; u \in D^{s,2}(\Omega).$$

We consider the variational setting of (1.1). From Theorem 6.7 and Corollary 7.2 in [10], it is known that the space $H^s(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1, 2_s^*]$ and the embedding $H^s(\Omega) \hookrightarrow L^q(\Omega)$ is locally compact for $q \in [1, 2_s^*)$.
If $2t + 4s > 3$, then $H^s(\Omega) \hookrightarrow L^{\frac{12}{3+2t}}(\Omega)$. For $u \in H^s(\Omega)$, the linear operator $T_u : D^{t,2}(\Omega) \to \mathbb{R}$ defined as

$$T_u(v) = \int_\Omega u^2 v dx.$$

By Hölder inequality and Lemma 2.1,

$$|T_u(v)| \leq \|u\|^2_{12/(3+2t)} \|v\|_{2_t^*} \leq C\|u\|^2_{H^s} \|v\|_{D^{t,2}}. \tag{2.2}$$

Set

$$\eta(u,v) = \int_\Omega (-\Delta)^{\frac{t}{2}} u \cdot (-\Delta)^{\frac{t}{2}} v dx, \;\; u,v \in D^{t,2}(\Omega).$$

It is clear that $\eta(u,v)$ is bilinear, bounded and coercive. The Lax-Milgram theorem implies that for every $u \in H^s(\Omega)$, there exists a unique $\phi_u^t \in D^{t,2}(\Omega)$ such that $T_u(v) = \eta(\phi_u, v)$ for any $v \in D^{t,2}(\Omega)$, that is

$$\int_\Omega (-\Delta)^{\frac{t}{2}} \phi_u^t (-\Delta)^{\frac{t}{2}} v dx = \int_\Omega u^2 v dx. \tag{2.3}$$

Therefore, $(-\Delta)^t \phi_u^t = u^2$ in a weak sense. Moreover,

$$\|\phi_u^t\|_{D^{t,2}} = \|T_u\| \leq C\|u\|^2_{H^s}. \tag{2.4}$$

Since $t \in (0,1]$ and $2t + 4s > 3$, then $\frac{12}{3+2t} \in (2, 2_s^*)$. From Lemma 2.1, (2.2) and (2.3), it follows that

$$\|\phi_u^t\|^2_{D^{t,2}} = \int_\Omega |(-\Delta)^{\frac{t}{2}} \phi_u^t|^2 dx = \int_\Omega u^2 \phi_u^t dx \leq \|u\|^2_{\frac{12}{3+2t}} \|\phi_u^t\|_{2_t^*} \leq C\|u\|^2_{\frac{12}{3+2t}} \|\phi_u^t\|_{D^{t,2}}. \tag{2.5}$$

Then

$$\|\phi_u^t\|_{D^{t,2}} \leq C\|u\|^2_{\frac{12}{3+2t}}. \tag{2.6}$$

For $x \in \Omega$, we have

$$\phi_u^t(x) = c_t \int_\Omega \frac{u^2(y)}{|x-y|^{3-2t}} dy, \tag{2.7}$$

which is the Riesz potential [12], where

$$c_t = \frac{\Gamma(\frac{3-2t}{2})}{\pi^{3/2} 2^{2t} \Gamma(t)}.$$

Assume that the potential $V(x)$ satisfies the condition (V_1). Let

$$E = \left\{ u \in H^s(\Omega) : \int_\Omega (|(-\Delta)^{\frac{s}{2}} u|^2 + V(x)u^2) dx < \infty \right\}.$$

Then E is a Hilbert space with the inner product

$$\langle u, v \rangle_E = \int_\Omega \big((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x)uv \big) dx$$

and the norm $\|u\|_E = \langle u, u \rangle_E^{\frac{1}{2}}$. By Lemma 2.3 in [13], it is known that E is compactly embedded in $L^p(\Omega)$ for $2 \le p < 2_s^*$ and continuously embedded in $L^p(\Omega)$ for $p \in [1, 2_s^*]$. Substituting ϕ_u^t in (1.1), we have the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u + \phi_u^t u = m(x)|u|^{q-2}|u| + f(x,u), \ x \in \Omega, \tag{2.8}$$

The energy functional $I : E \to \mathbb{R}$ corresponding to problem (2.8) is defined by

$$I(u) = \frac{1}{2} \int_\Omega (|(-\Delta)^{\frac{s}{2}} u|^2 + V(x)u^2) dx + \frac{1}{4} \int_\Omega \phi_u^t u^2 dx - \frac{1}{q} \int_\Omega m(x)|u|^q dx - \int_\Omega F(x,u) dx.$$

It is easy to see that I is well defined in E and $I \in C^1(E), \mathbb{R})$, and

$$\langle I'(u), v \rangle = \int_\Omega \big((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x)uv + \phi_u^t uv - m(x)|u|^{q-2}uv - f(x,u)v \big) \, dx, \quad v \in H^s(\Omega). \tag{2.9}$$

Definition 2.2.

(1) We call $(u, \phi) \in E \times D^{t,2}(\Omega)$ is a weak solution of (1.1) *if u is a weak solution of (2.8).*

(2) We call u is a weak solution of (2.8) if

$$\int_\Omega \big((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x)uv + \phi_u^t uv - m(x)|u|^{q-2}uv - f(x,u)v \big) \, dx = 0,$$

for any $v \in H^s(\Omega)$.

Definition 2.3. *We say a C^1 functional I satisfies Palais-Smale condition $\big($(PS) condition for short$\big)$ if any sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that*

$$I(u_n) \text{ being bounded}, \ I'(u_n) \to 0, \quad \text{as } n \to 0 \tag{2.10}$$

admits a convergent subsequence, and such a sequence is called a palais-Smale sequence $\big($(PS) sequence$\big)$.

Lemma 2.4. *Assume that* $(H1), (H2)$ *and* $(H4)$ *hold. Then any (PS) sequence of* I *is bounded in* E.

Proof . We modify the proof of [14, 15]. Let $\{u_n\}$ be a (PS) sequence of I. By contradiction, let $\|u_n\| \to \infty$. Write $v_n = \frac{u_n}{\|u_n\|}$, then we may assume that there exists $v \in E$ such that, up to subsequence,

$$v_n \rightharpoonup v \ \text{ in } \ E, \quad v_n \to v \ \text{ a.e } \ x \in \Omega, \quad \text{and} \quad v_n \to v \text{ in } L^2_{loc}(\Omega). \tag{2.11}$$

Firstly, we claim that $v = 0$. In fact , since $\|u_n\| \to \infty$, by (2.10), we have

$$\frac{\langle I'(u_n), u_n\rangle}{\|u_n\|^4} = o(1),$$

that is

$$o(1) = \frac{1}{\|u\|^2} + \int_\Omega \phi^t_{v_n} {v_n}^2 dx - \int_\Omega \frac{m(x)|u_n|^q}{\|u_n\|^4} dx - \int_\Omega \frac{f(x, u_n)u_n}{\|u_n\|^4} dx. \tag{2.12}$$

By Sobolev and Hölder inequalities, we have

$$\int_\Omega m(x)|u_n|^q dx \le \|m\|_{\frac{2}{2-q}} \|u_n\|_2^q \le C\|m\|_{\frac{2}{2-q}} \|u_n\|^q. \tag{2.13}$$

Hence

$$\int_\Omega \frac{m(x)|u_n|^q}{\|u_n\|^4} dx \to 0. \tag{2.14}$$

By $(H4)$, we get that

$$\int_\Omega \frac{|f(x, u_n)u_n|}{\|u_n\|^4} dx = \int_\Omega \left|\frac{f(x, u_n)}{u_n}\right| \frac{u_n^2}{\|u_n\|^4} dx \le \frac{C}{\|u_n\|^2} \to 0. \tag{2.15}$$

Combining with (2.12)-(2.15), we obtain that

$$\int_\Omega \phi^t_{v_n} v_n^2 dx \ge 0.$$

By Fatou's Lemma, we have

$$\int_\Omega |(-\Delta)^{\frac{t}{2}} \phi^t_v|^2 dx = \int_\Omega \phi^t_v v^2 dx \le \lim_{n\to\infty} \int_\Omega \phi^t_{v_n} v_n^2 dx = 0$$

then 2.5-2.6 implies that $v = 0$.

Next, from the fact that $v = 0$, we deduce a contradiction which implies the boundedness of $\{u_n\}$ in E. Since the embedding $E \hookrightarrow L^2(\Omega)$ is compact, we have $v_n \to 0$ in $L^2(\Omega)$. Hence (H4) yields that

$$\int_\Omega \frac{f(x, u_n)}{u_n} |v_n|^2 dx \le C \int_\Omega |v_n|^2 dx \to 0. \tag{2.16}$$

It follows from (2.14), (2.16) and $\frac{\langle I'(u_n), u_n\rangle}{\|u_n\|^2} = O(1)$, we have

$$\begin{aligned} O(1) &= 1 + \int_\Omega \phi^t_{u_n} {v_n}^2 dx - \int_\Omega \frac{m(x)|u_n|^q}{\|u_n\|^2} dx - \int_\Omega \frac{f(x, u_n)}{u_n} |v_n^2 dx \\ &= 1 + O(1) \end{aligned}$$

which is contradiction. The proof is completed. □

Lemma 2.5. *Under the assumptions of Lemma 2.4 and* $(H3)$ *any (PS) sequence of* I *has a convergent subsequence in* E.

Proof . Let $\{u_n\}$ be a (PS) sequence of I. We show that $\{u_n\}$ possesses a strong convergent subsequence. Since $\{u_n\}$ is bounded in E $\big($Lemma 2.4$\big)$, we may assume that for some $u \in E$, up to a subsequence, $u_n \rightharpoonup u$ in E. By the fact that the embedding $E \hookrightarrow L^p(\Omega)$ is compact for $p \in [2, 2_s^*)$, it is easy to see that

$$u_n \to u \quad \text{in} \quad L^p(\Omega), \quad p \in [2, 2_s^*]. \tag{2.17}$$

By (2.9), we get

$$\begin{aligned} \|u_n - u\|_E^2 &= \langle I'(u_n) - I'(u), u_n - u\rangle - \int_\Omega (\phi_{u_n}^t u_n - \phi_u^t u)(u_n - u)dx \\ &\quad + \int_\Omega (m(x)|u_n|^{q-1} - m(x)|u|^{q-1})(u_n - u) + \int_\Omega (f(x, u_n) - f(x, u))(u_n - u)dx. \end{aligned} \tag{2.18}$$

Clearly, we have

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty. \tag{2.19}$$

By the generalization of Hölder inequality, Lemma 2.1 and (2.6), it follows that

$$\begin{aligned} \left|\int_\Omega \phi_{u_n}^t u_n(u_n - u)dx\right| &\le \|\phi_{u_n}^t\|_{2_t^*}\|u_n\|_{\frac{12}{3+2t}}\|u_n - u\|_{\frac{12}{3+2t}} \\ &\le C\|\phi_{u_n}^t\|_{D^{t,2}}\|u_n\|_{\frac{12}{3+2t}}\|u_n - u\|_{\frac{12}{3+2t}} \\ &\le C\|u_n\|_{\frac{12}{3+2t}}^3\|u_n - u\|_{\frac{12}{3+2t}} \\ &\le C\|u_n\|_E^3\|u_n - u\|_{\frac{12}{3+2t}}. \end{aligned}$$

Similarly,

$$\left|\int_\Omega \phi_u^t u(u_n - u)dx\right| \le C\|u\|_E^3\|u_n - u\|_{\frac{12}{3+2t}}.$$

We have

$$\left|\int_\Omega (\phi_{u_n}^t u_n - \phi_u^t u)(u_n - u)dx\right| \le \left|\int_\Omega \phi_{u_n}^t u_n(u_n - u)dx\right| + \left|\int_\Omega \phi_u^t u(u_n - u)dx\right| \to 0 \text{ as } n \to \infty. \tag{2.20}$$

By $(H3)$, Hölder inequality and Minkowski inequality,

$$\begin{aligned} &\left|\int_\Omega (f(x, u_n) - f(x, u))(u_n - u)dx\right| \\ &\le C_1\int_\Omega (|u_n| + |u|)|u_n - u|dx + C_1\int_\Omega (|u_n|^{p-1} + |u|^{p-1})|u_n - u|dx \\ &\le C_1\||u_n| + |u|\|_2\|u_n - u\|_2 + C_1\||u_n|^{p-1} + |u|^{p-1}\|_{\frac{p}{p-1}}\|u_n - u\|_p \\ &\le C_1(\|u_n\|_2 + \|u\|_2)\|u_n - u\|_2 + C_1(\|u_n\|_p^{p-1} + \|u\|_p^{p-1})\|u_n - u\|_p \\ &\le C(\|u_n\|_E + \|u\|_E)\|u_n - u\|_2 + C(\|u_n\|_E^{p-1} + \|u\|_E^{p-1})\|u_n - u\|_p \to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.21}$$

By $(H1)$ and Hölder inequality

$$\left|\int_\Omega (m(x)|u_n|^{q-1} - m(x)|u|^{q-1})(u_n-u)dx\right| \le \int_\Omega |m(x)|\left||u_n|^{q-1} - |u|^{q-1}\right||u_n-u|dx \tag{2.22}$$

$$\le \|m\|_{\frac{2}{2-q}} \left\||u_n|^{q-1} - |u|^{q-1}\right\|_2 \|u_n-u\|_2 \to 0. \tag{2.23}$$

By (2.18), (2.19), (2.20), (2.21) and (2.22), we see that $\{u_n\}$ converges strongly in E and the proof is completed.
□

3. Existence and multiplicity results

In this section, under the assumptions on m,V and f, we give the proof of Theorems 1.1-1.2. By $(H3)$ and $(H5)$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1}, x\in\Omega,\ u\in\mathbb{R}.$$

Then

$$|F(x,u)| \le \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{p}|u|^p \tag{3.1}$$

for some $p \in (2, 2_s^*)$

Lemma 3.1. *Suppose that $(H1)-(H5)$ hold. Then There exists $M > 0$ and $\rho > 0$ such that for all m with $\|m\|_{\frac{2}{2-q}} < M$,*

$I(u) > 0$, for $u \in E$ with $\|u\| = \rho$.

Proof . It is known that E is continuously embedded into $L^q(\Omega)$ for $q \in [2, 2_s^*]$ ($2_s^* = \frac{6}{3-2s}$), then $\|u\|_q \le C_0\|u\|_E$. Since $p \in (2, 2_s^*)$, by (2.13) and (3.1) we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_\Omega \phi_u^t u^2 dx - \frac{1}{q}\int_\Omega m(x)|u|^q dx - \int_\Omega F(x,u)dx \\ &\ge \frac{1}{2}\|u\|^2 + -\frac{1}{q}\|m\|_{\frac{2}{2-q}}\|u\|^q - \varepsilon\int_\Omega u^2 dx - C_\varepsilon\int_\Omega |u|^p dx \\ &\ge C_1\|u\|^2 - C_2\|m\|_{\frac{2}{2-q}}\|u\|^q - C_\varepsilon\|u\|^p \\ &\ge \left(C_1 - C_2\|m\|_{\frac{2}{2-q}}\|u\|^{q-2} - C_\varepsilon\|u\|^{p-2}\right)\|u\|^2. \end{aligned} \tag{3.2}$$

Let

$$J(t) = C_1 - C_2\|m\|_{\frac{2}{2-q}} t^{q-2} - C_\varepsilon t^{p-2},\ \text{for } t > 0.$$

Since $1 < q < 2 < p$, the function $J(t)$ achieves its maximum on $(0,\infty)$ at $t_0 > 0$. Moreover, there exists $M > 0$ such that for $|m|_{\frac{2}{2-q}} < M$, we have

$$\max_{t\in(0,\infty)} J(t) = J(t_0) > 0.$$

By $\rho = t_0$, the proof will be completed.

□ **Proof of Theorem 1.1.** By Lemma 3.1 , we define

$$\overline{B}_\rho = \{u \in E \ : \|u\| \leq \rho\}, \qquad \partial B_\rho = \{u \in \Omega \ : \|u\| = \rho\}.$$

Then we have

$$I\big|_{\partial B_\rho} > 0. \tag{3.3}$$

Clearly $I \in C^1(\overline{B}_\rho, \mathbb{R})$, hence I is lower semicontinuous and bounded from below on $\overline{B}_\rho$. Let

$$c_1 = \inf\{I(u) : u \in \overline{B}_\rho\} > -\infty.$$

By $(H1)$, we can choose $v \in C_0^\infty(\Omega)$. Since $m(x) > 0$ on Ω and $1 < q < 2$, it is easy to obtain

$$I(tv) < 0, \quad \text{for } t \text{ small}.$$

Thus $c_1 < 0$. Now by (3.3), Lemma 2.5 and Ekeland's variational principle, c_1 can be achieved at some inner point $u_1 \in \overline{B}_\rho$ and u_1 is a critical point of I. □

Lemma 3.2. *Under the assumotions of Theorem 1.1 there exists $e \in E$ with $\|e\|_E > \rho$ such that $I_\lambda(e) < 0$ for fixed $\lambda \in (0, 1]$, where ρ is the same as in Lemma 3.1.*
Proof . *By $(H6)$, there exists a constant $C > 0$ such that*

$$F(x, u) \geq C|u|^\mu, \ u \in \mathbb{R}. \tag{3.4}$$

By (2.4), (2.5),

$$\int_\Omega \phi_u^t u^2 dx = \|\phi_u^t\|^2_{D^{t,2}} \leq C\|u\|^4_{H^s}. \tag{3.5}$$

For $\xi > 0$ and $v \in C_0^\infty(\Omega)$, by (3.4) and (3.5), we have

$$\begin{aligned} I(\xi v) &\leq \frac{\xi^2}{2}\|v\|^2 + \frac{\xi^4}{4}\int_\Omega \phi_v^t v^2 dx - \int_\Omega F(x, \xi v)dx \\ &\leq \frac{\xi^2}{2}\|v\|^2 + \frac{C\xi^4}{4}\|v\|^4_{H^s} - C\xi^\mu\|v\|^\mu_\mu \to -\infty \end{aligned}$$

as $\xi \to +\infty$. Define a path $h : [0, 1] \to E$ by $h(\eta) = \eta\xi v$. For ξ large enough, we get

$$\|h(1)\|_E = \left(\int_{R^3} \left(|(-\Delta)^{\frac{s}{2}} h(1)|^2 + V(x)h^2(1)\right)dx\right)^{\frac{1}{2}} > \rho \text{ and } I(h(1)) < 0.$$

Choose $e = h(1)$, we obtain the conclusion. □

Proof of Theorem 1.2. From Lemma 2.5, I satisfies the (PS) condition. By Lemma 3.1, 3.2 and Mountain Pass Theorem (Theorem 2.2 of [16]) we show that problem (1.1) has a nontrivial solution at positive energy. □

Proof of Corollary 1.3. It is a direct consequence of Theorem 1.1 and Theorem 1.2. □

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