On Ideal Elements in Poe-AG-groupoid

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Abstract

In this paper we introduce the concept of ideal elements in poe-AG-groupoid and give some characterizations and properties of their ideal elements. So we consider some results concerning ideals in poe-semigroups and investigate them in poe-AG-groupoids. Also, the class of ideal elements of poe-AG-groupoids are studied, certain intrinsic and basic properties of poe-AG-groupoids including: ideal, bi-ideal, interior ideal, prime, semiprime, intra-regular elements and etc. are studied as well. The corresponding results on poe-semigroups can be also obtained as application of the results of this paper.

Keywords: AG-groupoid, poe-semigroup, poe-AG-groupoid, ideal element, interior ideal element, prime, semiprime, intra-regular, g-regular, filter element, quasi-commutative.

1. Introduction

The idea of generalization of communicative semigroups was introduced in 1972 by Kazim and Naseerudin [1]. They named this structure as the left almost semigroup (LA-semigroup for short), while it was called Abel-Grassmann’s groupoid (AG-groupoid for short) in [13], if its elements satisfy the left invertive law, that is: \((ab)c = (cb)a\) for all \(a, b, c \in S\). A partial ordered groupoid (poe-groupoid for short) \(S\) is a groupoid under a multiplication "\(~\)" at the same time an ordered set under a partial order "\(\leq\)" such that \(a \leq b\) implies \(ac \leq bc\) and \(ca \leq cb\) for all \(c \in S\). If the multiplication is associative, then \(S\) is called a po-semigroup. A po-semigroup possessing the greatest element "\(e\)" (that is, \(a \leq e\) for all \(a \in S\)) is called poe-semigroup [3, 4]. In the present paper, we mean by poe-AG-groupoid the ordered AG-groupoid which has a greatest element. In this type of AG-groupoids the ideal elements (instead of ideals) play an essential role. It seems interesting to obtain the similar results of achieved results in [4, 5, 6, 7, 8, 9] and [10] with use of ideal elements in poe-AG-groupoid instead of ideals in po-AG-groupoid. The present paper shows how similar is the theory of po-AG-groupoid based on ideals with the theory of poe-AG-groupoid based on ideal elements. Infact, these are parallel.

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Recall that a po-AG-groupoid is a partial ordered groupoid (po-groupoid) that is AG-groupoid, i.e., poe-AG-groupoid is a po-AG-groupoid which has the greatest element "e" (that is, \( a \leq e \) for all \( a \in S \)). As with N. Kehayopulu and P. V. Ramad, which looked at some interesting properties of poe-semigroups in [4, 10], we look at the interesting properties of poe-AG-groupoids. In section 2, we give the definition of regular, intra-regular and semisimple poe-AG-groupoids. Then we investigate some properties of poe-AG*-groupoids and poe-AG**-groupoids. Moreover, we study the role of ideal elements to characterize some properties of poe-AG-groupoids.

2. Preliminaries

At first, recall some of necessary definitions before discussing in details of the results summarized in the abstract. Let \( S \) be a po-groupoid (or po-AG-groupoid). An element \( a \) of \( S \) is called idempotent if \( a^2 = a \) so \( S \) is called band if every their element be idempotent. An element \( a \) of \( S \) is called a left ideal element if \( xa \leq a \) for all \( x \in S \). It is called a right ideal element if \( ax \leq a \) for all \( x \in S \). It is called an ideal element if it is both a left and right ideal element. Let \( S \) be a poe-semigroup, \( b \in S \) is called bi-ideal element, if \( beb \leq b \). If \( S \) is a poe-groupoid (or poe-AG-groupoid), then \( a \) is a left (resp. right) ideal element of \( S \) if and only if \( ea \leq a \) (resp. \( ae \leq a \)) and \( a \) is called left (resp. right) regular if \( a \leq e.a \) (resp. \( a \leq a^2.e \)) for every \( a \in S \). Let \( S \) be a po-groupoid. An element \( p \) of \( S \) is called prime if for any \( a, b \) in \( S \) such that \( ab \leq p \) implies \( a \leq p \) or \( b \leq p \). It is called weakly prime if for any ideal elements \( a, b \) of \( S \) such that \( ab \leq p \) implies \( a \leq p \) or \( b \leq p \). Finally, \( p \) is called semiprime if \( a^2 \leq p \) implies \( a \leq p \).

If an ideal element of a po-groupoid is prime, then it is weakly prime and semiprime. In poe-semigroup \( S, a \in S \) is called quasi ideal element if \( ae \wedge ea \leq a \).

A poe-semigroup \( S \) is called quasi-completely regular if for all \( a \in S \), \( a \leq eae \) or \( a \leq aeae \).

A poe-semigroup is called g-regular if for all \( a \in S \), \( a \leq eae \).

A poe-semigroup \( S \) is called intra-regular (resp. regular) if \( a \leq e.a^2 \) (resp. \( a \leq a^2.e \)) for every \( a \in S \).

Clearly, if a poe-semigroup is intra regular, then it is g-regular.

If a poe-semigroup is quasi-completely regular, then it is g-regular.

If \( S \) is an AG-groupoid with left identity, then \( a(bc) = b(ac) \) for all \( a, b, c \in S \).

It is also known that in an AG-groupoid \( S \), the medial law, that is, \( (ab)(cd) = (ac)(bd) \) for all \( a, b, c, d \in S \) holds.

An AG-groupoid \( S \) is called an AG*-groupoid if it satisfies one of the following equivalent: \( (ab)c = b(ac) \), \( (ab)c = b(ca) \) implies \( b(ac) = b(ca) \) for all \( a, b, c \in S \).

An AG-groupoid \( S \) satisfying the identity \( a(bc) = b(ac) \), for all \( a, b, c \in S \) is called an AG**-groupoid. Notice that each AG-groupoid with a left identity is an AG**-groupoid.

Any AG**-groupoid is paramedial, i.e., it satisfies the identity, \( (ab)(cd) = (db)(ca) \) for all \( a, b, c, d \in S \).

3. Ideal elements in poe-AG-groupoid

The purpose of this section is to introduce the concept of poe-AG-groupoid and study their properties of ideal elements. A poe-AG-groupoid is a po-AG-groupoid that having a greatest element "e" (that is, \( a \leq e \) for all \( a \in S \)).

Definition 3.1. A poe-AG-groupoid \( S \) is called regular if \( a \leq (ae)a \) for every \( a \in S \).

Definition 3.2. A poe-AG-groupoid \( S \) is called strongly regular if \( a \leq (ae)a \) and \( ea = ae \) for every \( a \in S \).
Definition 3.3. A poe-AG-groupoid $S$ is called normal if $ea = ae$ for every $a \in S$.

Definition 3.4. A poe-AG-groupoid $S$ is called quasi commutative if $xy \leq (ye)x$ for all $x, y \in S$.

Proposition 3.5. If $S$ be quasi-commutative poe-AG-groupoid, then

1. $ey \leq (ye)e = (ee)y \leq ey$ implies $ey = ye$, i.e, $S$ is normal.
2. $xe \leq (ee)x \leq ex$ implies $xe \leq ex$.
3. $ey \leq ye = (ye)e$.
4. $ye \leq ey = (ye)e$.

Proof. Using assumption $x = e$ or $y = e$ is obvious. □

Proposition 3.6. If $S$ be quasi-commutative poe-AG*-groupoid, then $S$ is normal.

Proof. Using proposition 3.3 (5), then $ey \leq (ey)e = y(ee) \leq ye$ implies $ey = ye$, i.e, $S$ is normal. □

Corollary 3.7. If $S$ be regular quasi-commutative poe-AG*-groupoid, then $S$ is strongly regular.

Corollary 3.8. Every commutative wite left identity poe-AG-groupoid $S$ is a quasi-commutative.

Proof. If $S$ be commutative wite left identity $u$, then $u \leq e$ implies $y = uy \leq ey$ implies $xy \leq x(ey) = (ye)x$. □

Lemma 3.9. If $S$ is a poe-AG**-groupoid, $a \in S$ and $b$ a left ideal element of $S$, then $ab$ is a left ideal element of $S$.

Proof. Since $b$ is a left ideal element of $S$, $eb \leq b$, hence $e(ab) = a(eb) \leq ab$. □

Lemma 3.10. If $S$ is a poe-AG*-groupoid, $b \in S$ and $a$ a right ideal element of $S$, then $ab$ is a left ideal element of $S$.

Proof. Since $a$ is a right ideal element of $S$, $ae \leq a$, thus $e(ab) = (ae)b \leq ab$. Note that, if $S$ be a poe-semigroup and $a, b$ are ideal elements of $S$, then $ab$ is an ideal element of $S$ as well. □

Definition 3.11. A poe-AG-groupoid $S$ is called semisimple if $a \leq ((ea)e)a$ for every $a \in S$.

Proposition 3.12. Let $S$ be a poe-AG*-groupoid. If the right ideal elements of $S$ are semiprime, then they are idempotent.

Proof. Let $a$ be an right ideal element of $S$. Since $a^2 \leq a^2$ and using Lemma 3.10 $a^2$ is an right ideal element of $S$, by hypothesis $a^2 = aa \leq a^2$ implies $a \leq a^2$, hence $a \leq a^2 \leq ae \leq a$, implies $a^2 = a$. Hence $a$ is idempotent.

Note that above proposition is true for all ideal elements of $S$, if $S$ be a poe-AG**-groupoid. □
Definition 3.13. A poe-AG-groupoid $S$ is called totally ordered if for all ideal elements $a, b \in S$ either $a \leq b$ or $b \leq a$.

Proposition 3.14. Let $S$ be a poe-AG**-groupoid. If the left ideal elements of $S$ are weakly prime, then ideal elements are totally ordered.

Proof. Let $a, b$ be ideal elements of $S$. Then by Lemma 3.9 $ab$ is a left ideal element of $S$ and $a, b \leq ab$, by hypothesis $a \leq ab \leq eb \leq b$ or $b \leq ab \leq ae \leq a$, i.e., the ideal elements of $S$ are totally ordered.

Note that, if $a$ is left ideal element of poe-AG**-groupoid $S$, then $a^2$ is ideal element of $S$.

Definition 3.15. A poe-AG-groupoid $S$ is called intra-regular if $a \leq ((ea)^2)e$ for every $a \in S$.

Proposition 3.16. Let $S$ be a poe-AG-groupoid. If $S$ is intra-regular, then the ideal elements of $S$ are semiprime.

Proof. Let $u$ be an ideal element of $S$ and $a \in S$ such that $a^2 \leq u$. By hypothesis, $a \leq (ea^2)e \leq (eu)e \leq ue \leq u$. Hence $u$ is semiprime.

Lemma 3.17. Let $S$ be a poe-AG**-groupoid. If $a \in S$, then $(ea^2)e$ is left ideal element.

Proof. Since $S$ is a poe-AG**-groupoid, thus $e((ea^2)e) \leq (ea^2)(ee) \leq (ea^2)e$.

Lemma 3.18. Let $S$ be a poe-AG*-groupoid. If $a \in S$, then $a^4 \leq (ea^2)e$.

Proof. Since $e$ is greatest element of poe-AG*-groupoid $S$, thus
\[a^4 = ((aa)a)a \leq ((ee)a)a = (aa)(ee) = (e(aa))e = (ea^2)e\]

Definition 3.19. A poe-AG-groupoid $S$ is called semisimple if $a \leq (ea)(a)e$ for every $a \in S$.

Proposition 3.20. Let $S$ be a poe-AG**-groupoid. If the left ideal elements of $S$ are semiprime, then $S$ is intra-regular.

Proof. Let $a \in S$. Using Lemma 3.17, $(ea^2)e$ is a left ideal element of $S$, by hypothesis, it is semiprime. Therefore, since $a^2 \leq (ea^2)e$ and $(ea^2)e$ is semiprime, so $a \leq (ea^2)e$. Hence $S$ is intra-regular.

Proposition 3.21. Let $S$ be a poe-AG**-groupoid. If $a \in S$ is an ideal element, then $a^2$ is an ideal element.

Proof. Assumption $a$ is an ideal element of $S$, then $ea^2 = e(aa) = a(ea) \leq aa = a^2$, hence $ea^2 \leq a^2$, and $a^2e \leq (aa)e = (ea)a \leq aa = a^2$, hence $a^2e \leq a^2$. □
**Proposition 3.22.** Let $S$ be a poe-AG-groupoid. If the ideal element $t$ of $S$ be prime, then it is weakly prime and semiprime.

**Proof.** Using their definitions is obvious. □

**Corollary 3.23.** Let $S$ be a poe-AG**-groupoid. If the left ideal elements of $S$ are prime, then they form a totally ordered and $S$ is intra-regular.

**Proposition 3.24.** Let $S$ be a poe-AG-groupoid. If $S$ is semisimple, then the ideal elements of $S$ are idempotent.

**Proof.** Let $a$ be an ideal element of $S$. By hypothesis,

$$a \leq ((ea)e)a \leq ((ae)a)e = (ea)a \leq a = a^2 \leq ea \leq a$$

, hence $a^2 = a$. Therefore $a$ is idempotent. □

**Lemma 3.25.** Let $S$ be a poe-AG**-groupoid. If $S$ is quasi commutative, then for all $a \in S$, $a^2 e$ is ideal element of $S$.

**Proof.** Consider any $a \in S$, $(a^2)e = (ee)a^2 \leq ea^2 = e(aa) \leq e((ae)a) = ((ae)e)a = ((ee)a)e \leq (ea)e = (aa)e = a^2 e$. Similarly, $e(a^2)e = (a^2)e = (ee)a^2 \leq ea^2 \leq a^2 e$. □

**Proposition 3.26.** A quasi commutative poe-AG**-groupoid $S$ is regular if and only if every ideal element of $S$ be semiprime.

**Proof.** Let $S$ be regular, and $a$ is ideal element of $S$, if for every $x \in S$, $x^2 \leq a$, then $x \leq x(x) = e(x)x \leq ea \leq a \Rightarrow x \leq a$. Hence $a$ is semiprime. Conversely, let $a \in S$. Using Lemma 3.25, $a^2 e$ is an ideal element of $S$, and then is semiprime. By the assumtion we have, $(a^2)^2 = a^2 = ((aa)a)e \leq (ea)e = (aa)e = a^2 e$ and then by the semiprimness of $a^2 e$, $a^2 \leq a^2 e$, so $a \leq a^2 e$. Therefore $a \leq (aa)e = a^2 e \leq (a^2)e = (ee)a^2 \leq ea^2 = e(aa) = (ae)a$, implies $a \leq (ae)a$. Hence $S$ is regular. □

**Definition 3.27.** A poe-AG-groupoid is called left (resp. right) regular if $a \leq e.a^2$ (resp. $a \leq a^2.e$) for all $a \in S$. If $S$ is left or right regular, then it is called semiregular.

**Proposition 3.28.** Let $S$ be a poe-AG**-groupoid. Then $S$ is left regular if and only if every left ideal element of $S$ is semiprime.

**Proof.** Suppose $a \in S$ is left ideal element and $x \in S$ such that $x^2 \leq a$. By hypotheses, $x \leq ex^2 \leq ea \leq a$ implies $x \leq a$. Hence $S$ is semiprime. Conversely, consider $x \in S$, we show that $ex^2$ is left ideal element of $S$. Let $x \in S$, therefore $e(ex^2) = (ee)x^2 \leq ex^2$. Using hypotheses, $ea^2$ is semiprime. Since $x^4 = (x^2)^2 \leq ex^2$ implies $x^2 \leq ex^2$. Also $x \leq ex^2$ for every $x \in S$. Hence $S$ is left regular. □
**Definition 3.29.** Let $S$ be a poe-AG-groupoid, $a \in S$ is called an interior ideal element of $S$ if $(ea)e \leq a$.

**Proposition 3.30.** Every ideal element of a poe-AG-groupoid $S$ is an interior ideal element of $S$.

**Proof.** Suppose $a$ be right ideal element of $S$, then $(ea)e \leq ae \leq a$. □

**Definition 3.31.** A poe-AG-groupoid $S$ is called quasi-completely regular if for all $a \in S$, $a \leq ((ea)e)a$ or $a \leq ((ae)a)e$.

**Definition 3.32.** A poe-AG-groupoid $S$ is called g-regular if for all $a \in S$, $a \leq (ea)e$.

**Proposition 3.33.** If a poe-AG*-groupoid is intra regular, then it is g-regular.

**Proof.** Let $S$ be a poe-AG*-groupoid, hence $a \leq (ea)e = a^2(ee) \leq (aa)e = (ea)a \leq (ea)e$. □

**Proposition 3.34.** If a poe-AG-groupoid is quasi-completely regular, then it is g-regular.

**Proof.** Let $S$ be a poe-AG-groupoid, hence $a \leq ((ae)a)e = (ea)(ae) \leq (ea)(ee) \leq (ea)e$. □

**Definition 3.35.** A poe-AG-groupoid $S$ is called filter, if $a \leq ea$ and $a \leq ae$ for all $a \in S$.

**Proposition 3.36.** If poe-AG-groupoid $S$ is strongly regular, then $S$ is filter.

**Proof.** Consider $a \in S$, by hypothesis, $a \leq (ae)a = (ea)a = (aa)e \leq (ae)e = (ee)a \leq ea = ae$. □

**Proposition 3.37.** Every interior ideal element of a g-regular poe-AG**-groupoid $S$ is a left ideal element.

**Proof.** Consider $a \in S$. Using hypothesis, $(ea)e = a, ea = e((ea)e) = (ea)(ee) \leq (ea)e = a$. Hence $ea \leq a$. □

**Proposition 3.38.** Let $S$ is regular poe-AG-groupoid and every $a \in S$ be filter element then, $S$ is quasi-completely regular.

**Proof.** Consider $a \in S$. Using hypothesis, $a \leq (ae)a \leq (ae)(ea) = ((ea)e)a$. Semililarly, $a \leq (ae)a \leq (ae)(ae) = (aa)(ee) = ((ee)a)a = ((ae)e)a \leq ((ae)e)(ae) = ((ae)a)(ee) \leq ((ae)a)e$. Obviously, every filter element is g-regular. □
Proposition 3.39. Every semisimple poe-AG*-groupoid $S$ is a intra-regular.

Proof. Consider $a \in S$. Using hypothesis, 
\[
a \leq ((ea)e)a = ((ae)(ea))e = (e(ae))(ae) = (ea)((ae)e) = ((ea)(ea))e = (ee)(aa))e \leq (ea^2)e.
\]
□

Corollary 3.40. Let $S$ is a poe-AG-groupoid. If elements of $S$ are filter and ideal element, then it is normal.

Proposition 3.41. Let $S$ be a poe-AG**-groupoid and $e$ is idempotent, then every right ideal element is normal.

Proof. Suppose that $e^2 = e$ by hypothesis, $ae \leq a$ implies $e(ae) \leq ea$ and $e(ae) = a(ee) = ae$ implies $ae \leq ea$. Also $(ae)e \leq ae$ and $(ae)e = (ee)a = ea$ implies $ea \leq ae$. Hence $ea = ae$. □

Corollary 3.42. Every strongly regular is normal.

Proposition 3.43. Let $S$ is a poe-AG-groupoid. If $a \leq ae$ for every $a \in S$, then $a$ is filter element of $S$.

Proof. Consider $a \in S$. Using hypothesis, $a \leq ae \leq (ae)a = (ee)a \leq ea$. □

Proposition 3.44. Every poe-AG**-band is normal.

Proof. Consider $a \in S$. Using hypothesis, 
\[
ea = ea^2 = e(aa) = (ea)a = (aa)e = ae.
\]
□

Proposition 3.45. Let $S$ is a poe-AG**-groupoid. If $e$ is idempotent, then $(ea)b = e(ab)$, for all $a, b \in S$.

Proof. Using hypothesis and param medial law, 
\[
(ea)b = (ba)e = (ba)(ee) = (be)(ae) = (ee)(ab) = e(ab).
\]
□

Proposition 3.46. Let $S$ is a poe-AG*-groupoid. If $e$ is idempotent, then $S$ is normal.

Proof. Consider $b \in S$. Using hypothesis, 
\[
eb = (ee)b = (be)e = e(be) = (ee)(be) = (eb)(ee) = (eb)e = b(ee) = be.
\]
□

Definition 3.47. Let $S$ be a poe-AG-groupoid, $b \in S$ is called bi-ideal element, if $(be)b \leq b$.

Proposition 3.48. Every right ideal element of a poe-AG-groupoid $S$ is an bi-ideal element.

Proof. Suppose $b$ be a right ideal element of $S$, there for $bb \leq be \leq b$ so $(be)b \leq bb \leq b$. □

Proposition 3.49. in poe-AG**-band every bi-ideal element is ideal element.

Proof. Suppose $b$ is bi-ideal element of $S$, therefore 
\[
(be)b = (be)(bb) = (bb)(eb) = b(eb) = e(bb) = eb, \text{ hence } eb \leq b. \text{ Using proposition } 3.44, \text{ } eb = be \leq b.
\]
□
References