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On Ideal Elements in Poe-AG-groupoid

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Abstract

In this paper we introduce the concept of ideal elements in poe-AG-groupoid and give some characterizations and properties of their ideal elements. So we consider some results concerning ideals in poe-semigroups and investigate them in poe-AG-groupoids. Also, the class of ideal elements of poe-AG-groupoids are studied, certain intrinsic and basic properties of poe-AG-groupoids including: ideal, bi-ideal, interior ideal, prime, semiprime, intra-regular elements and etc. are studied as well. The corresponding results on poe-semigroups can be also obtained as application of the results of this paper.

Keywords: AG-groupoid, poe-semigroup, poe-AG-groupoid, ideal element, intrior ideal element, prime, semiprime, intra-regular, g-regular, filter element, quasi-commutative.

1. Introduction

The idea of generalization of communicative semigroups was introduced in 1972 by Kazim and Naseerudin[1]. They named this structure as the left almost semigroup (LA-semigroup for short), while it was called Abel-Grassmann's groupoid (AG-groupoid for short) in [13], if its elements satisfy the left invertive law, that is: $(ab)c = (cb)a$ for all $a, b, c \in S$. A partial ordered groupoid (po-groupoid for short) S is a groupoid under a multiplication "." at the same time an ordered set under a partial order "$\leq$" such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $c \in S$. If the multiplication is associative, then S is called a po-semigroup. A po-semigroup possessing the greatest element "e" (that is, $a \leq e$ for all $a \in S$) is called poe-semigroup [3, 4]. In the present paper, we mean by poe-AG-groupoid the ordered AG-groupoid which has a greatest element. In this type of AG-groupoids the ideal elements (instead of ideals) play an essential role. It seems interesting to obtaine the similar results of achived results in [4, 5, 6, 7, 8, 9] and [10] with use of ideal elements in poe-AG-groupoid instead of ideals in po-AG-groupoid. The present paper shows how similar is the theory of po-AG-groupoid based on ideals with the theory of poe-AG-groupoid based on ideal elements. Infact, these are parallel.

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Recall that a po-AG-groupoid is a partial ordered groupoid (po-groupoid) that is AG-groupoid, i.e., poe-AG-groupoid is a po-AG-groupoid which has the greatest element "e" (that is, $a \leq e$ for all $a \in S$). As with N.Kehayopulu and P.V.Ramand which looked at some interesting properties of poe-semigroups in [4, 10], we look at the interesting properties of poe-AG-groupoids. In section 2, we give the definition of regular, intra-regular and semisimple poe-AG-groupoids. Then we investigate to some properties of poe-AG*-groupoids and poe-AG**-groupoids. Moreover, we study the role of ideal elements to characterize some properties of poe-AG-groupoids.

2. Preliminaries

At first, recall some of necessary definitions before discussing in details of the results summarized in the abstract. Let S be a po-groupoid (or po-AG-groupoid). An element a of S is called idempotent if $a^2 = a$ so S is called band if every their element be idempotent. An element a of S is called a left ideal element if $xa \leq a$ for all $x \in S$. It is called a right ideal element if $ax \leq a$ for all $x \in S$ [11]. It is called an ideal element if it is both a left and right ideal element. Let S be a poe-semigroup, $b \in S$ is called bi-ideal element, if $beb \leq b$ [12]. If S is a poe-groupoid (or poe-AG-groupoid), then a is a left (resp. right) ideal element of S if and only if $ea \leq a$ (resp. $ae \leq a$) and a is called left (resp. right) regular if $a \leq e.a^2$ (resp. $a \leq a^2.e$)for every $a \in S$[1]. Let S be a po-groupoid. An element p of S is called prime if for any a, b in S such that $ab \leq p$ implies $a \leq p$ or $b \leq p$. It is called weakly prime if for any ideal elements a, b of S such that $ab \leq p$ implies $a \leq p$ or $b \leq p$.Finally, p is called semiprime [**?**] if $a^2 \leq p$ implies $a \leq p$.
If an ideal element of a po-groupoid is prime, then it is weakly prime and semiprime. In poe-semigroup S, $a \in S$ is called quasi ideal element if $ae \bigwedge ea \leq a$.
A poe-semigroup S is called quasi-completely regular if for all $a \in S$, $a \leq eaea$ or $a \leq aeae$.
A poe-semigroup is called g-regular if for all $a \in S$, $a \leq eae$.
A poe-semigroup S is called intra-regular (resp. regular) if $a \leq e.a^2e$ (resp. $a \leq aea$) for every $a \in S$.
Clearly, if a poe-semigroup is intra regular, then it is g-regular.
If a poe-semigroup is quasi-completely regular, then it is g-regular.
If S is an AG-groupoid with left identity, then a(bc) = b(ac);for all $a, b, c \in S$.
It is also known [1] that in an AG-groupoid S, the medial law, that is, (ab)(cd) = (ac)(bd) for all a, b, c, d $\in$ S holds.
An AG-groupoid S is called an AG*-groupoid if it satisfies one of the following equivalent: (ab)c = b(ac), (ab)c = b(ca) implies b(ac) = b(ca) for all a, b, c $\in S$.
An AG-groupoid S satisfying the identity a(bc) = b(ac), for all $a, b, c \in S$ is called an AG**-groupoid.
Notice that each AG-groupoid with a left identity is an AG**-groupoid.
Any AG**-groupoid is paramedial, i.e., it satisfies the identity, $(ab)(cd) = (db)(ca)$ for all a, b, c, d $\in S$.

3. Ideal elements in poe-AG-groupoid

The purpose of this section is to introduce the concept of poe-AG-groupoid and study their peroperties of ideal elements. A poe-AG-groupoid is a po-AG-groupoid that having a greatest element "e" (that is, $a \leq e$ for all $a \in S$).

Definition 3.1. *A poe-AG-groupoid S is called regular if $a \leq (ae)a$ for every $a \in S$.*

Definition 3.2. *A poe-AG-groupoid S is called strongly regular if $a \leq (ae)a$ and $ea = ae$ for every $a \in S$.*

Definition 3.3. *A poe-AG-groupoid S is called normal if $ea = ae$ for every $a \in S$.*

Definition 3.4. *A poe-AG-groupoid S is called quasi commutative if $xy \leq (ye)x$ for all $x, y \in S$.*

Proposition 3.5. *If S be quasi-commutative poe-AG-groupoid, then*
(1) $ey \leq (ye)e = (ee)y \leq ey$ implies $ey = (ye)e$.
(2) $xe \leq (ee)x \leq ex$ implies $xe \leq ex$.
(3) $xy \leq (ye)x \leq (ey)x$.
(4) $ye \leq ey = (ye)e$.
(5) $ey \leq (ey)e$.

Proof . Using assoumption $x = e$ or $y = e$ is obvious. □

Proposition 3.6. *If S be quasi-commutative poe-AG*-groupoid, then S is normal.*

Proof . Using proposition 3.5 (5), then $ey \leq (ey)e = y(ee) \leq ye$ implies $ey = ye$, i.e, S is normal. □

Corollary 3.7. *If S be regular quasi-commutative poe-AG*-groupoid, then S is strongly regular.*

Corollary 3.8. *Every commutative wite left identity poe-AG-groupoid S is a quasi-commutative.*

Proof . If S be commutative wite left identity u, then $u \leq e$ implies $y = uy \leq ey$ implies $xy \leq x(ey) = (ye)x$. □

Lemma 3.9. *If S is a poe-AG**-groupoid, $a \in S$ and b a left ideal element of S, then ab is a left ideal element of S.*

Proof . Since b is a left ideal element of S, $eb \leq b$, hence $e(ab) = a(eb) \leq ab$. □

Lemma 3.10. *If S is a poe-AG*-groupoid, $b \in S$ and a a right ideal element of S, then ab is a left ideal element of S.*

Proof . Since a is a right ideal element of S, $ae \leq a$, thus $e(ab) = (ae)b \leq ab$. Note that,if S be a poe-semigroup and a, b are ideal elements of S, then ab is an ideal element of S as well. □

Definition 3.11. *A poe-AG-groupoid S is called semisimple if $a \leq (((ea)e)a)e$ for every $a \in S$.*

Proposition 3.12. *Let S be a poe-AG*-groupoid. If the right ideal elements of S are semiprime, then they are idempotent.*

Proof . Let a be an right ideal element of S. Since $a^2 \leq a^2$ and using Lemma 3.10 a^2 is an right ideal element of S, by hypothesis $a^2 = aa \leq a^2$ implies $a \leq a^2$, hence $a \leq a^2 \leq ae \leq a$, implies $a^2 = a$. Hence a is idempotent.
Note that above proposition is true for all ideal elements of S, if S be a poe-AG**-groupoid. □

Definition 3.13. *A poe-AG-groupoid S is called totally ordered if for all ideal elements $a, b \in S$ either $a \leq b$ or $b \leq a$.*

Proposition 3.14. *Let S be a poe-AG**-groupoid. If the left ideal elements of S are weakly prime, then ideal elements are totally ordered.*

Proof . Let a, b be ideal elements of S. Then by Lemma 3.9 ab is a left ideal element of S and $a.b \leq ab$, by hypothesis $a \leq ab \leq eb \leq b$ or $b \leq ab \leq ae \leq a$, i.e., the ideal elements of S are totally ordered.
Note that, if a is left ideal element of poe-AG**-groupoid S, then a^2 is ideal element of S. □

Definition 3.15. *A poe-AG-groupoid S is called intra-regular if $a \leq (e.a^2)e$ for every $a \in S$.*

Proposition 3.16. *Let S be a poe-AG-groupoid. If S is intra-regular, then the ideal elements of S are semiprime.*

Proof . Let u be an ideal element of S and $a \in S$ such that $a^2 \leq u$. By hypothesis, $a \leq (ea^2)e \leq (eu)e \leq ue \leq u$. Hence u is semiprime. □

Lemma 3.17. *Let S be a poe-AG**-groupoid. If $a \in S$, then $(ea^2)e$ is left ideal element.*

Proof . Since S is a poe-AG**-groupoid, thus $e((ea^2)e) \leq (ea^2)(ee) \leq (ea^2)e$. □

Lemma 3.18. *Let S be a poe-AG*-groupoid. If $a \in S$, then $a^4 \leq (ea^2)e$.*

Proof . Since e is greatest element of poe-AG*-groupoid S, thus

$$a^4 = ((aa)a)a \leq ((ee)a)a = (aa)(ee) = (e(aa))e = (ea^2)e$$

□

Definition 3.19. *A poe-AG-groupoid S is called semisimple if $a \leq (((ea)e)a)e$ for every $a \in S$.*

Proposition 3.20. *Let S be a poe-AG**-groupoid. If the left ideal elements of S are semiprime, then S is intra-regular.*

Proof . Let $a \in S$. Using Lemma 3.17, $(ea^2)e$ is a left ideal element of S, by hypothesis, it is semiprime. Therefore, since $a^2 \leq (ea^2)e$ and $(ea^2)e$ is semiprime, so $a \leq (ea^2)e$. Hence S is intra-regular. □

Proposition 3.21. *Let S be a poe-AG**-groupoid. If $a \in S$ is an ideal element, then a^2 is an ideal element.*

Proof . Assomption a is an ideal element of S, then $ea^2 = e(aa) = a(ea) \leq aa = a^2$, hence $ea^2 \leq a^2$, and $a^2e \leq (aa)e = (ea)a \leq aa = a^2$, hence $a^2e \leq a^2$. □

Proposition 3.22. *Let S be a poe-AG-groupoid. If the ideal element t of S be prime, then it is weakly prime and semiprime.*

Proof . Using their definitions is obvious. $\square$

Corollary 3.23. *Let S be a poe-AG**-groupoid. If the left ideal elements of S are prime, then they form a totally ordered and S is intra-regular.*

Proposition 3.24. *Let S be a poe-AG-groupoid. If S is semisimple, then the ideal elements of S are idempotent.*

Proof . Let a be an ideal element of S. By hypothesis,

$$a \leq (((ea)e)a)e \leq ((ae)a)e \leq (aa)e = (ea)a \leq aa = a^2 \leq ea \leq a$$

, hence $a^2 = a$. Therefore a is idempotent. $\square$

Lemma 3.25. *Let S be a poe-AG*-groupoid. If S is quasi commutative, then for all $a \in S$, a^2e is ideal element of S.*

Proof . Consider any $a \in S$, $(a^2e)e = (ee)a^2 \leq ea^2 = e(aa) \leq e((ae)a) = ((ae)e)a = ((ee)a)a \leq (ea)a = (aa)e = a^2e$. Similarly, $e(a^2e) = (a^2e)e = (ee)a^2 \leq ea^2 \leq a^2e$. $\square$

Proposition 3.26. *A quasi commutative poe-AG*-groupoid S is regular if and only if every ideal element of S be semiprime.*

Proof . Let S be regular, and a is ideal element of S, if for every $x \in S$, $x^2 \leq a$, then $x \leq (xe)x = e(xx) \leq ea \leq a \Rightarrow x \leq a$. Hence a is semiprime. Conversely, let $a \in S$. Using Lemma 3.25, a^2e is an ideal element of S, and then is semiprime. By the assumtion we have, $(a^2)^2 = a^4 = ((aa)a)a \leq (ea)a = (aa)e = a^2e$ and then by the semiprimness of a^2e, $a^2 \leq a^2e$, so $a \leq a^2e$. Therefore $a \leq (aa)e = a^2e \leq (a^2e)e = (ee)a^2 \leq ea^2 = e(aa) = (ae)a$, implies $a \leq (ae)a$. Hence S is regular. $\square$

Definition 3.27. *A poe-AG-groupoid is called left (resp. right) regular if $a \leq e.a^2$ (resp. $a \leq a^2.e$) for all $a \in S$. If S is left or right regular, then it is called semiregular.*

Proposition 3.28. *Let S be a poe-AG*-groupoid. Then S is left regular if and only if every left ideal element of S is semiprime.*

Proof . Suppose $a \in S$ is left ideal element and $x \in S$ such that $x^2 \leq a$. By hypotheses, $x \leq ex^2 \leq ea \leq a$ implies $x \leq a$. Hence S is semiprime. Conversely, consider $x \in S$, we show that ex^2 is left ideal element of S. Let $x \in S$, therefore $e(ex^2) = (ee)x^2 \leq ex^2$. Using hypotheses, ea^2 is semiprime. Since $x^4 = (x^2)^2 \leq ex^2$ implies $x^2 \leq ex^2$. Also $x \leq ex^2$ for every $x \in S$. Hence S is left regular. $\square$

Definition 3.29. *Let S be a poe-AG-groupoid, $a \in S$ is called an intrior ideal element of S if $(ea)e \leq a$.*

Proposition 3.30. *Every ideal element of a poe-AG-groupoid S is an intrior ideal element of S.*

Proof . Suppose a be right ideal element of S, then $(ea)e \leq ae \leq a$. □

Definition 3.31. *A poe-AG-groupoid S is called quasi-completely regular if for all $a \in S$, $a \leq ((ea)e)a$ or $a \leq ((ae)a)e$.*

Definition 3.32. *A poe-AG-groupoid S is called g-regular if for all $a \in S$, $a \leq (ea)e$.*

Proposition 3.33. *If a poe-AG*-groupoid is intra regular, then it is g-regular.*

Proof . Let S be a poe-AG*-groupoid, hence
$a \leq (ea^2)e = a^2(ee) \leq (aa)e = (ea)a \leq (ea)e$. □

Proposition 3.34. *If a poe-AG-groupoid is quasi-completely regular, then it is g-regular.*

Proof . Let S be a poe-AG-groupoid, hence
$a \leq ((ae)a)e = (ea)(ae) \leq (ea)(ee) \leq (ea)e$. □

Definition 3.35. *A poe-AG-groupoid S is called filter, if $a \leq ea$ and $a \leq ae$ for all $a \in S$.*

Proposition 3.36. *If poe-AG-groupoid S is strongly regular, then S is filter.*

Proof . Consider $a \in S$, by hypothesis,
$a \leq (ae)a = (ea)a = (aa)e \leq (ae)e = (ee)a \leq ea = ae$. □

Proposition 3.37. *Every intrior ideal element of a g-regular poe-AG**-groupoid S is a left ideal element.*

Proof . Consider $a \in S$. Using hypothesis,
$(ea)e = a, ea = e((ea)e) = (ea)(ee) \leq (ea)e = a$. Hence $ea \leq a$. □

Proposition 3.38. *Let S is regular poe-AG-groupoid and every $a \in S$ be filter element then, S is quasi-completely regular.*

Proof . Consider $a \in S$. Using hypothesis, $a \leq (ae)a \leq (ae)(ea) = ((ea)e)a$. Semilarly,
$a \leq (ae)a \leq (ae)(ae) = (aa)(ee) = ((ee)a)a = ((ae)e)a \leq ((ae)e)(ae) = ((ae)a)(ee) \leq ((ae)a)e$.
Obveously, every filter element is g-regular. □

Proposition 3.39. *Every semisimple poe-AG*-groupoid S is a intra-regular.*

Proof . Consider $a \in S$. Using hypothesis,
$a \leq (((ea)e)a)e = ((ae)(ea))e = (e(ea))(ae) = (ea)((ea)e) = ((ea)(ea))e = ((ee)(aa))e \leq (ea^2)e$.
□

Corollary 3.40. *Let S is a poe-AG-groupoid. If elements of S are filter and ideal element, then it is normal.*

Proposition 3.41. *Let S be a poe-AG**-groupoid and e is idempotent, then every right ideal element is normal.*

Proof . suppose that $e^2 = e$ by hypothesis, $ae \leq a$ implies $e(ae) \leq ea$ and $e(ae) = a(ee) = ae$ implies $ae \leq ea$. Also $(ae)e \leq ae$ and $(ae)e = (ee)a = ea$ implies $ea \leq ae$. Hence $ea = ae$. □

Corollary 3.42. *Every strongly regular is normal.*

Proposition 3.43. *Let S is poe-AG-groupoid. If $a \leq ae$ for every $a \in S$, then a is filter element of S.*

Proof . Consider $a \in S$. Using hypothesis, $a \leq ae \leq (ae)e = (ee)a \leq ea$. □

Proposition 3.44. *Every poe-AG**-band is normal.*

Proof . Consider $a \in S$. Using hypothesis,
$ea = ea^2 = e(aa) = (ea)a = (aa)e = ae$. □

Proposition 3.45. *Let S is a poe-AG**-groupoid. If e is idempotent, then $(ea)b = e(ab)$, for all $a, b \in S$.*

Proof . Using hypothesis and paramedial law,
$(ea)b = (ba)e = (ba)(ee) = (be)(ae) = (ee)(ab) = e(ab)$. □

Proposition 3.46. *Let S is a poe-AG*-groupoid. If e is idempotent, then S is normal.*

Proof . Consider $b \in S$. Using hypothesis,:
$eb = (ee)b = (be)e = e(be) = (ee)(be) = (eb)(ee) = (eb)e = b(ee) = be$. □

Definition 3.47. *Let S be a poe-AG-groupoid, $b \in S$ is called bi-ideal element, if $(be)b \leq b$.*

Proposition 3.48. *Every right ideal element of a poe-AG-groupoid S is an bi-ideal element.*

Proof . Suppose b be a right ideal element of S, there for $bb \leq be \leq b$ so $(be)b \leq bb \leq b$. □

Proposition 3.49. *in poe-AG**-band every bi-ideal element is ideal element.*

Proof . Suppos b is bi-ideal element of S, therefore
$(be)b = (be)(bb) = (bb)(eb) = b(eb) = e(bb) = eb$, hence $eb \leq b$. Using proposition 3.44 , $eb = be \leq b$.
□

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