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Scalable frames in tensor product of Hilbert spaces

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Abstract

Tight frames are very similar to orthonormal bases and can be used as a good alternative to them. Scaling frames are introduced as a method to transform a general frame to a tight one. This paper investigates in under what conditions the tensor product of two frames is a scalable frame. We expand some results concerning frame operations of eigenvalues to tensor product of Hilbert spaces. Finally, we will show that if one of the frames is scalable, better conditions are obtained for the approximation of tensor product of frames that is not scalable.

Keywords: scalable frame, eigenvalue, tensor product, condition number.
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1. Introduction

If a signal is represented as a vector and transmitted by sending the sequence of coefficients of its representation then using an orthonormal basis to analyze and later reconstruct the signal can be problematic. One of the problems is that the coefficients are lost during the transmission and the reconstruction does not occur correctly. As a solution to this problem redundancy is introduced in frames so that it might be possible to reconstruct a signal if some coefficients are lost. Frame theory is a standard methodology in applied mathematics and engineering and works as an alternative to orthonormal bases in Hilbert spaces which has many advantages. Frames have had a tremendous impact on applications due to their unique ability to deliver redundant, yet stable expansions. The idea of redundancy is the crucial property in various applications [1]. The study of frames began in 1952 with their introduction by Duffin and Schaeffer [6] and then has been expanded by Daubechies and et al. [5].

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Tight frames are very similar to orthonormal bases and can be used as a good alternative to them [12]. Also, tight frames can be characterized by those frames which possess optimal numerical stability properties which can give fast convergence and recovery. All such applications require the associated algorithms to be numerically stable, which the subclass of tight frames satisfies optimally. This observation raises the question on how to carefully modify a given frame (which might be suitable for a particular application) in order to generate a tight frame. Since a frame is typically designed to accommodate certain requirements of an application, this modification process should be as careful as possible in order not to change the properties of the frame too drastically.

Trying to be as careful as possible, the most noninvasive approach seems to merely scale each frame vector, i.e., multiply it by a scalar. Indeed, almost all frame properties one can think of such as erasure resilience or sparse expansions are left untouched by this modificationite [10]. In 2013, Kutyniok [9] considered a general frame to generate a tight frame by rescaling its frame vectors and showed whether a given frame is scalable.

Due to the high utilization of tensor product in approximation theory, sampling theory, wavelets and ...[8]. This paper investigates conditions under which the tensor product of two frames is a scalable frame and is organized as follows: In Section 2 we collect results and notations that we need. In Section 3 we consider the frames and scalable frames in Hilbert spaces and extend some of the known results about tensor product of frames to scalable frames. Through an example, we show that the tensor product of a frame and a scalable frame is not generally scalable.

Scalablity of frames in measured by

$$\mathbf{r}(\Phi) = \frac{upper\ frame\ bound\ of\ \Phi}{lower\ frame\ bound\ of\ \Phi} \tag{1.1}$$

that is the same condition number of the matrix and is defined as the ratio of the largest singular value and the smallest singular value of Φ. Section 4.1 is devoted to some results concerning eigenvalues of frame operations to tensor product of Hilbert spaces and some properties of the optimal upper and lower frame bounds.

Finally, we prove that if one of the frames is scalable, better conditions are obtained for the approximation of tensor product of frames that is not scalable.

2. Notaion and Preliminaries

We begin with a brief of important and usefull definitions related to the frames, tensor product of frames. We refer to [4] and [7] for better understanding.

Throughout the paper, $\mathbf{H}$ and $\mathbf{K}$ are Hilbert spaces with infinite dimension, also $\mathbf{H}^M$ and $\mathbf{K}^M$ are M-dimensional Hilbert spaces. As usual we denote the algebra of all bounded linear operators on $\mathbf{H}$ and $\mathbf{K}$ by $\mathrm{B}(\mathbf{H})$ and $\mathrm{B}(\mathbf{K})$ respectively. We always use $E_1 = \{e_l\}_{l=1}^{\infty}$, $E_2 = \{u_k\}_{k=1}^{\infty}$ to denote orthonormal bases for $\mathbf{H}$ and $\mathbf{K}$, respectively.

Definition 2.1. *A family of vectors $\Phi = \{\varphi_i\}$ is a frame in Hilbert space **H** if there are constants $0 < A \leq B < \infty$ so that for each $f \in \boldsymbol{H}$*

$$A\|f\|^2 \leq \sum_i |\langle f, \varphi_i\rangle|^2 \leq B\|f\|^2. \tag{2.1}$$

The numbers A and B are called the frame bounds. A frame is called A-tight frame if $A = B$, when $A = B = 1$ it is called a Parseval frame. One often also write Φ for the $N \times M$ matrix whose i-th column is the vector φ_i. It is well known that Φ is A-tight if and only if

$$S := \Phi\Phi^* = \sum_{i=1}^{M} s_i \varphi_i \varphi_i^* = AI_N, \tag{2.2}$$

where I_N is the identity matrix in Hilbert space $\boldsymbol{H}^N$, and S is the frame operator of the frame $\{\varphi_i\}$[4].

2.1. Tensor product of Hilbert spaces

Tensor product in recent decades has been highly regarded. For example tensor product suggests a natural language for expressing algorithms of digital signal processing based on matrix factorization [7]. Some tensor product properties that they needed to study this section is given below [4, 8].

Let **H** and **K** be Hilbert spaces with scalar products $\langle , \rangle_{\mathbf{H}}$, and $\langle , \rangle_{\mathbf{K}}$ respectively. A mapping $f : \mathbf{H} \to \mathbf{K}$ is said to antilinear if

$$f(ax + by) = \overline{a}f(x) + \overline{b}f(y).$$

The operator norm of an antilinear map T is defined as in the linear case:

$$\|T\| = sup_{\|x\|=1}\|Tx\|.$$

The adjoint of a bounded map T is defined by

$$\langle T^*x, y\rangle_{\mathbf{K}} = \langle Ty, x\rangle_{\mathbf{H}} \ \ for\ all\ x \in \mathbf{H}, y \in \mathbf{K}.$$

Note that the map $T \to T^*$ is linear rather than antilinear. Suppose T is an antilinear map from **K** into **H** and E_1 , E_2 are orthonormal bases for **H** and **K**, respectively. Then by the Parseval identity

$$\sum_j \|Tu_j\|^2 = \sum_i \|T^*e_i\|^2.$$

This shows that $\sum_j \|Tu_j\|^2$ is independent of the choice of basis E_2.

Now, the tensor product of **H** and **K** is the set $\mathbf{H} \otimes \mathbf{K}$ of all antilinear maps $T : \mathbf{K} \to \mathbf{H}$ such that $\sum_j \|Tu_j\|^2 < \infty$ for every orthonormal basis E_2 of Hilbert space **K**. The space $\mathbf{H} \otimes \mathbf{K}$ is a Hilbert space [7] with the norm $|||T|||^2 = \sum_j \|Tu_j\|^2$ and associated inner product

$$\langle Q, T\rangle = \sum_j \langle Qu_j, Tu_j\rangle. \tag{2.3}$$

Let $x, x' \in \mathbf{H}$ and $y, y' \in \mathbf{K}$ and λ is scalar,then it is defined the map $x \otimes y$ as follows

$$(x \otimes y)(y') = \langle y, y' \rangle x, \forall y' \in \mathbf{K}. \tag{2.4}$$

Let $T \in \mathbf{H} \otimes \mathbf{K}$, then by (2.3)

$$|||T||| = |||T^*||| \tag{2.5}$$

$$|||x \otimes y||| = \|x\|\|y\| \tag{2.6}$$

$$\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle. \tag{2.7}$$

Also, for all $U, U' \in \mathbf{B(H)}$ and $V, V' \in \mathbf{B(K)}$ we have

$$U \otimes V \in B(\mathbf{H} \otimes \mathbf{K}) \ \textit{and} \ \|U \otimes V\| = \|U\|\|V\| \tag{2.8}$$

$$(U \otimes V)(x \otimes y) = Ux \otimes Vy \ \textit{for all} \ x \in \mathbf{H} \ , \ y \in \mathbf{K}, [7]. \tag{2.9}$$

Theorem 2.2. *[8] Let $\{\varphi_i\}$, $\{\psi_j\}$ be frames for $\boldsymbol{H}$ and $\boldsymbol{K}$ with frame bounds A_1, B_1 and A_2, B_2, respectively. Then $\varphi_i \otimes \psi_j$ is a frame for $\boldsymbol{H} \otimes \boldsymbol{K}$ with frame bounds A_1A_2, B_1B_2.*

3. Scalable frame for tensor product of Hilbert spaces

It is desirable to construct tight frames by just scaling each frame vector as it is noninvasive, and frame properties such as erasure resilience or sparse expansions are left untouched by this modification. This procedure is called frame scaling [9].

In 2013, Kutyniok [9] considered a general frame to generate a tight frame by rescaling its frame vectors and showed whether a given frame is scalable. Most of the centralization was on identifying frames whose vectors can be rescaled resulting in a tight frame.

A frame $\Phi = \{\varphi_i\}$ for $\mathbf{H}$ is said to be scalable if there exists a collection of scalars $c_i \geq 0$ such that $\{c_i\varphi_i\}$ is a Parseval frame. If $c_i > 0$ for all $i = 1, ...,$ then Φ is called positively scalable. If there exists $\delta > 0$, such that $c_i \geq \delta$ for all $j \in J$, then Φ is called strictly scalable.

By (2.2), a frame is scalable if and only if there exists $c_i \geq 0$ for $i = 1, ..., M$ such that

$$I_N = \sum_{i=1}^{M} c_i^2 \varphi_i \varphi_i^*. \tag{3.1}$$

It is shown through an example tensor product of a scalable frame with a frame which is not generally scalable.

Example 3.1. *Okoudjou [11] by considering convex polytopes associated to scalable frames suggests that this matrix produces:*

$$\begin{bmatrix} 1 & cos2\theta_2 & ... & cos2\theta_M \\ 0 & sin2\theta_2 & ... & sin2\theta_M \end{bmatrix}. \tag{3.2}$$

He showed that in the case $M = 3$ by putting conditions for θ_k, the frame can be scalable.
a. If we put $\theta_2 = \frac{\pi}{2}, \theta_3 = \frac{2\pi}{3}$ in (4.4), we get the following scalable frame:

$$\Phi_1 = \begin{bmatrix} 1 & -1 & -\frac{1}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix} \tag{3.3}$$

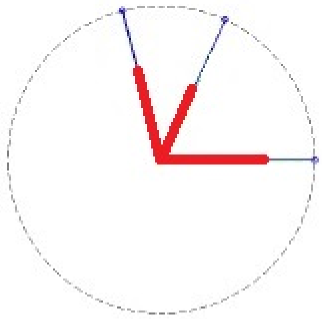

Figure 1: A scalable frame with 3 vectors in $\mathbf{R}^2$. The original frames are in blue, the frames obtained by scaling are in red.

b. If we put $\theta_2 = \frac{\pi}{6}, \theta_3 = \frac{\pi}{3}$ in (4.4), we get the following frame which is not scalable:

$$\Phi_2 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \tag{3.4}$$

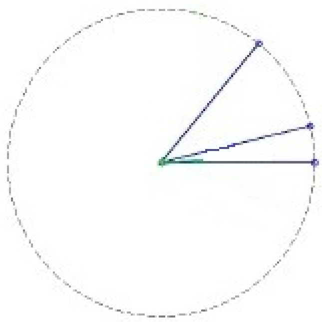

Figure 2: A non scalable frame with 3 vectors in $\mathbf{R}^2$. The original frames are in blue, the frames obtained by the scalable ones do not exist.

It is simply demonstrated that $\Phi_1 \otimes \Phi_2$ not scalable.

In the next theorems, we put a condition on a frame until the tensor product with a scalable frame is scalable frame for $\mathbf{H} \otimes \mathbf{K}$. We have a scalable frame for $\mathbf{H}$, we need conditions on another frame for $\mathbf{K}$ so that tensor product of frames is scalable frame for $\mathbf{H} \otimes \mathbf{K}$.

Theorem 3.2. *Suppose $\{\varphi_i\}$ is a Scalable frame for $\boldsymbol{H}$ and let $\{\psi_j\}$ is a tight frame for $\boldsymbol{K}$. Then $\{\varphi_i \otimes \psi_j\}$ is scalable frame for $\boldsymbol{H} \otimes \boldsymbol{K}$.*

Proof . Let $\{\psi_j\}$ be a tight frame and there exists scalars $c_i \geq 0 \ \ , i = 1, ...$ such that $\{c_i\varphi_i\}$ is a Parseval frame.

For all $T \in \mathbf{H} \otimes \mathbf{K}$ of (2.3) we have

$$\langle T, c_i\varphi_i \otimes \psi_j\rangle = \sum_k \langle Tu_k, (c_i\varphi_i \otimes \psi_j)(u_k)\rangle$$

$$= \sum_k \langle Tu_k, \langle \psi_j, u_k\rangle c_i\varphi_i\rangle = \sum_k \overline{\langle \psi_j, u_k\rangle} \langle Tu_k, c_i\varphi_i\rangle,$$

since T is antilinear map

$$= \langle \sum_k \overline{< \psi_j, u_k >} Tu_k, c_i\varphi_i\rangle = \langle T(\sum_k \langle \psi_j, u_k\rangle u_k), c_i\varphi_i\rangle = \langle T\psi_j, c_i\varphi_i\rangle.$$

Finally, since $\{c_i\varphi_i\}$ is a Parseval frame so

$$\sum_i \sum_j |\langle T, c_i\varphi_i \otimes \psi_j\rangle|^2 = \sum_i \sum_j |\langle T\psi_j, c_i\varphi_i\rangle|^2 = \sum_j \|T\psi_j\|^2. \tag{3.5}$$

By using the Parseval identity, we have

$$\|T\psi_j\|^2 = \sum_l |\langle T\psi_j, e_l\rangle|^2 = \sum_l |\langle T^*e_l, \psi_j\rangle|^2.$$

Since $\{\psi_j\}$ is tight frame for $\mathbf{K}$, then we have

$$\sum_j \|T\psi_j\|^2 = \sum_j \sum_l |\langle T\psi_j, e_l\rangle|^2 = \sum_l \sum_j |\langle T^*e_l, \psi_j\rangle|^2$$

$$A \sum_l \|T^*e_l\|^2 = A \sum_k \|Tu_k\|^2 = A|\|T\||^2,$$

by (3.5)

$$\sum_i \sum_j |\langle T, c_i\varphi_i \otimes \psi_j\rangle|^2 = A|\|T\||^2,$$

now,

$$\frac{1}{A} \sum_i \sum_j |\langle T, c_i\varphi_i \otimes \psi_j\rangle|^2 = |\|T\||^2,$$

therefore

$$\sum_i \sum_j |\langle T, \frac{c_i}{A}\varphi_i \otimes \psi_j\rangle|^2 = |\|T\||^2,$$

set $\dfrac{c_i}{A} = d_i$ thus $\{\varphi_i \otimes \psi_j\}$ is scalable frame. $\square$

Corollary 3.3. *Let $\{\varphi_i\}$, $\{\psi_j\}$ be two frames in $\boldsymbol{H}$, $\boldsymbol{K}$. The frame $\{\varphi_i \otimes \psi_j\}$ is scalable frame for $\boldsymbol{H} \otimes \boldsymbol{K}$ if any one of the following conditions holds:*

(a) If $\{\varphi_i\}$ is a Parseval frame for $\boldsymbol{H}$ and $\{\psi_j\}$ is a scalable frame for $\boldsymbol{K}$.

(b) If $\{\varphi_i\}$ is a scalable frame for $\boldsymbol{H}$ and $\{\psi_j\}$ is a scalable frame for $\boldsymbol{K}$.

The following theorem states that from the effect of scalable frame in the tensor space on a unit sphere, from each of the spaces **H** and **K** can be found scalable frames for spaces **H** and **K**.

Theorem 3.4. *Let $\{T_i\}$ is scalable frame for $\boldsymbol{H} \otimes \boldsymbol{K}$. Then for each $0 \neq x_0 \in \boldsymbol{H}$ and $0 \neq y_0 \in \boldsymbol{K}$, $\{\frac{T_i(y_0)}{\|y_0\|}\}$ and $\{\frac{T_i^*(x_0)}{\|x_0\|}\}$ are scalable frame for $\boldsymbol{H}$ and $\boldsymbol{K}$, respectively.*

Proof . We assume that $\{T_i\}$ is scalable frame for $\mathbf{H} \otimes \mathbf{K}$ then there exists scalars $c_i \geq 0 \ \ , i = 1, ...$ such that $\{c_i T_i\}$ is the Parseval frame. Let $x \in \mathbf{H}$, by (2.3) we have

$$\langle x \otimes y_0, T_i \rangle = \sum_j \langle x \otimes y_0(u_j), T_i u_j \rangle$$

by (2.4)

$$\sum_j \langle \langle y_0, u_j \rangle x, T_i u_j \rangle = \sum_j \langle x, \overline{\langle y_0, u_j \rangle} T_i u_j \rangle$$

$$= \langle x, \sum_j \overline{\langle y_0, u_j \rangle} T_i u_j \rangle \quad (T_i \, is \ an \ antilinear \ map)$$

$$= \langle x, T_i (\sum_j \langle y_0, u_j \rangle u_j) \rangle = \langle x, T_i y_o \rangle.$$

Since $\{c_i T_i\}$ is Parseval frame, then

$$\sum_i |\langle x, c_i T_i y_0 \rangle|^2 = \sum_i |\langle x \otimes y_0, c_i T_i \rangle|^2 = |||x \otimes y_0|||^2 = \|x\|^2 \|y_0\|^2.$$

So,

$$\sum_i |\langle x, \frac{c_i T_i y_0}{\|y_0\|} \rangle|^2 = \|x\|^2.$$

Thus $\{\frac{c_i T_i(y_0)}{\|y_0\|}\}$ is Parseval frame for **H**. Similarly, since for all $y \in \mathbf{K}$

$$\langle y, T_i^* x_0 \rangle = \langle x_0, T_i y \rangle = \langle x_0 \otimes y, T_i \rangle.$$

Therefore $\{\frac{T_i^*(x_0)}{\|x_0\|}\}$ is also a scalable frame for **K**. □

The following theorem proves that adjoint of scalable frames in tensor product of Hilbert spaces is also a scalable frame.

Theorem 3.5. *Suppose $\{T_i\}$ is a scalable frame for $\boldsymbol{H} \otimes \boldsymbol{K}$ then $\{T_i^*\}$ is scalable frame for $\boldsymbol{K} \otimes \boldsymbol{H}$.*

Proof . Since the sequence $\{T_i\}$ is scalable frame for $\mathbf{H} \otimes \mathbf{K}$ then there exists scalars $c_i \geq 0 \ \ , i = 1, ...$ such that $\{c_i T_i\}$ is Parseval frame for $\mathbf{H} \otimes \mathbf{K}$. By using (2.5) and Parseval identity it is clear that

$$\langle T^*, T_n \rangle = \langle T, T_n^* \rangle$$

then

$$\sum_i |\langle T^*, c_i T_i \rangle|^2 = |||T^*|||^2 = |||T|||^2 = \sum_i |\langle T, \overline{c_i} T_i^* \rangle|^2.$$

Therefore $\{T_i^*\}$ is scalable frame for $\mathbf{K} \otimes \mathbf{H}$. □

4. Non scalable of tensor product of frames

Example 3.1, part (b) shows that all frames are not scalable frames. By (3.1) if a frame is not scalable frame, then

$$I_N \neq \sum_{i=1}^{M} s_i^2 \varphi_i \varphi_i^*.$$

When a frame is not scalable, scalars $\{s_i\}_{i=1}^M$ should be found such that $\{s_i\varphi_i\}_{i=1}^M$ is as tight as possible. This should naturally mean that $\{s_i\varphi_i\}_{i=1}^M$ is the best condition in the sense that the ratio of upper and lower frame bounds is the closest to 1. Chen [3] studies the case when a frame is not scalable by measuring

$$min\|I_n - \sum_{i=1}^{M} s_i^2 \varphi_i \varphi_i^* \|_F \tag{4.1}$$

and the minimal ellipsoid of the convex hull of the frame vectors, where $\|.\|_F$ is the Frobenious norm. However, it is not clear whether solving (4.1) gives the best conditioned frame. Recently, in [2] has studied the case in which a frame is not scalable by measuring

$$min\ \mathbf{r}(\sum_{i=1}^{M} s_i^2 \varphi_i \varphi_i^*) \tag{4.2}$$

where for a given frame $\Phi = \{\varphi_i\}_{i=1}^M$, $\mathbf{r}(\Phi) = \dfrac{upper\ frame\ bound\ of\ \Phi}{lower\ frame\ bound\ of\ \Phi}$, which is the same as the ratio of the largest singular value and the smallest singular value of Φ.

4.1. Tensor product of eigenvalues

A completely different application of eigenvectors and eigenvalues is that they can be used in a theory of systems in diffrential equations [4]. Due to in (4.2) properties of eigenvalues are required, some results are developed concerning eigenvalues of frame operators to Hilbert spaces tensor product and some properties of the optimal upper and lower frame bounds [4].

Let Q be a bounded operator, then the trace of Q is defined by

$$TrQ = \sum_{i=1}^{N} \langle Qe_i, e_i \rangle. \tag{4.3}$$

Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathbf{H}^L$, denote $(\lambda_i)_{i=1}^L$ the eigenvalues for S. Then

$$\sum_{i=1}^{L} \lambda_i = \sum_{i=1}^{M} \|\varphi_i\|^2. \tag{4.4}$$

Theorem 4.1. *Let $(\varphi_i)_{i=1}^M$ and $(\psi_j)_{j=1}^N$ be frames for Hilbert spaces $\boldsymbol{H}^L$ and $\boldsymbol{H}^K$ with frame operators S_1, S_2 having normalized eigenvectors $(e_i)_{i=1}^L$, $(u_j)_{j=1}^k$ and respective eigenvalues $(\lambda_i)_{i=1}^L$, $(\gamma_j)_{j=1}^K$. Then for all $j = 1, 2, ..., N$, $i = 1, 2, ..., M$ we have*

$$Tr(S_1 \otimes S_2) = \sum_{i=1}^{M} \sum_{j=1}^{N} \|\varphi_i \otimes \psi_j\|^2.$$

Proof . Assume that $(\varphi_i)_{i=1}^M$ and $(\psi_j)_{j=1}^N$ be frames for $\mathbf{H}^L$ and $\mathbf{H}^K$, respectively. The spaces $\mathbf{H}^L$, $\mathbf{H}^K$ have orthonormal bases consisting of eigenvectors $(e_i)_{i=1}^L$, $(u_j)_{j=1}^K$ for S_1, S_2. We denote the corresponding eigenvalues by $(\lambda_i)_{i=1}^L$, $(\gamma_j)_{j=1}^K$, then we have

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\|\varphi_i\otimes\psi_j\|^2=\sum_{i=1}^{M}\sum_{j=1}^{N}\langle\varphi_i\otimes\psi_j,\varphi_i\otimes\psi_j\rangle$$
$$=\sum_{i=1}^{M}\sum_{j=1}^{N}\langle\varphi_i,\varphi_i\rangle\langle\psi_j,\psi_j\rangle=\sum_{i=1}^{M}\|\varphi_i\|^2\sum_{j=1}^{N}\|\psi_j\|^2.$$

By using (4.3) and (4.4) we have

$$=\sum_{i=1}^{L}\lambda_i\sum_{j=1}^{K}\gamma_j=TrS_1TrS_2 \tag{4.5}$$

also,

$$\begin{aligned}\sum_{i=1}^{L}\lambda_i\sum_{j=1}^{K}\gamma_j&=\sum_{i=1}^{L}\langle\lambda_ie_i,e_i\rangle\sum_{j=1}^{K}\langle\gamma_ju_j,u_j\rangle\\&=\sum_{i=1}^{L}\langle S_1e_i,e_i\rangle\sum_{j=1}^{K}\langle S_2u_j,u_j\rangle\\&=\sum_{i=1}^{L}\sum_{j=1}^{K}\langle S_1e_i\otimes S_2u_j,e_i\otimes u_j\rangle\\&=\sum_{i=1}^{L}\sum_{j=1}^{K}\langle(S_1\otimes S_2)(e_i\otimes u_j),e_i\otimes u_j\rangle\end{aligned}$$
$$=Tr(S_1\otimes S_2). \tag{4.6}$$

Then the equations (4.5) and (4.6) imply that

$$Tr(S_1\otimes S_2)=\sum_{i=1}^{M}\sum_{j=1}^{N}\|\varphi_i\otimes\psi_j\|^2=TrS_1TrS_2.$$

This completes the proof. □

Convergence rate in numerical algorithms involving a strictly positive definite matrix depends heavily on the condition number of the matrix, which is defined as the ratio between the largest eigenvalue λ_{max} and the smallest eigenvalue λ_{min}, Christensen [4] considers the case of the frame operator, these eigenvalues correspond to the optimal frame bounds. We will expand its theorem to tensor product space.

Theorem 4.2. *Let $(\varphi_i)_{i=1}^M$ and $(\psi_j)_{j=1}^N$ be frames for $\boldsymbol{H}^L$ and $\boldsymbol{H}^K$, respectively with frame operators S_1, S_2 having eigenvalues $\lambda_1 \geq ... \geq \lambda_L$ and $\lambda'_1 \geq ... \geq \lambda'_K$. Then $\lambda_1\lambda'_1$ conicides with the optimal upper frame bound and $\lambda_L\lambda'_K$ is the optimal lower frame bound for $\{\varphi_i \otimes \psi_j\}_{i,j}$ that $i = 1, ..., M$ and $j = 1, ..., N$.*

Proof . Assume that $\{e_j\}_{j=1}^{L}$ and $\{e_j\}_{j=1}^{K}$ are normalized eigenvectors corresponding to eigenvalues $\{\lambda_i\}_{i=1}^{L}$ and $\{\lambda'_j\}_{j=1}^{K}$, respectively. Also $f_1 \in \mathbf{H}^L$, $f_2 \in \mathbf{H}^K$,

$$f_1 = \sum_{j=1}^{L} \langle f_1, e_j \rangle e_j, f_2 = \sum_{j=1}^{K} \langle f_2, e_j \rangle e_j$$

and,

$$Sf_1 = \sum_{j=1}^{L} \langle f_1, e_j \rangle Se_j, Sf_2 = \sum_{j=1}^{K} \langle f_2, e_j > Se_j$$

so

$$\begin{aligned} \sum_{i=1}^{M}\sum_{j=1}^{N} |\langle f_1 \otimes f_2, \varphi_i \otimes \psi_j \rangle|^2 &=< (S_1 \otimes S_2)(f_1 \otimes f_2), (f_1 \otimes f_2)\rangle \\ &= \langle (S_1 f_1 \otimes S_2 f_2), (f_1 \otimes f_2)\rangle = \langle S_1 f_1, f_1 \rangle \langle S_2 f_2, f_2 \rangle \\ &= \sum_{i=1}^{L} \lambda_i |\langle f_1, e_i \rangle|^2 \sum_{j=1}^{K} \lambda'_j |\langle f_2, e_j \rangle|^2 \leq \lambda_1 \lambda'_1 \|f_1 \otimes f_2\|^2. \end{aligned}$$

Therefore $\lambda_1 \lambda'_1$ is an upper frame bound. Similarly, the lower bound is proved. □

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4.2. Optimaly conditioned on tensor product of scaled frames

In particular, Casazza [2] and Chen [3] have shown that the problem of minimizing the condition number of a scaled frame (4.2) is equivalent to solving the minimization problem

$$min \|I_n - \sum_{i=1}^{M} s_i^2 \varphi_i \varphi_i^* \|_2, \tag{4.7}$$

where $\|.\|_2$ is the operator norm of a matrix. They have shown that the optimal solution to (4.1) does not even have to be a frame, and so it would yield an undefined condition number for the corresponding system.

In Example 3.1, we simply see that the tensor product of a frame and a scalable frame $(\Phi_1 \otimes \Phi_2)$ is not generally scalable. We consider optimally conditioned of tensor product of two frames. If tensor product of frame $(\varphi_i)_{i=1}^{M}$ with scalable frame $(\psi_j)_{j=1}^{N}$ are not scalabe, then how "not scalable" is a frame can be measured by measuring

$$min \|I_n \otimes I_m - \sum_{i=1}^{M}\sum_{j=1}^{N} s_j^2 \varphi_i \varphi_i^* \otimes \psi_j \psi_j^* \|_2, \tag{4.8}$$

where $\|.\|_2$ is the operator norm of a matrix that is equivalent the problem of minimizing the condition number

$$min \ \mathbf{r}(\sum_{i=1}^{M} s_j^2 \varphi_i \varphi_i^* \otimes \psi_j \psi_j^*). \tag{4.9}$$

The next theorem states that better conditions for the approximation of non-scalable tensor product of two frames Φ , Ψ are obtained if one of these frames is scalable.

Proposition 4.3. *Let $\Phi = (\varphi_i)_{i=1}^M$ be a frame in $\boldsymbol{H}^L$ and $\Psi = (\psi_j)_{j=1}^N$ be scalable frame in $\boldsymbol{H}^K$. Then*

$$min\|I_L \otimes I_K - \sum_{i=1}^{M}\sum_{j=1}^{N} s_j^2 \varphi_i\varphi_i^* \otimes \psi_j\psi_j^*\|_2 \leq min\|I_L - \sum_{i=1}^{M}\varphi_i\varphi_i^*\|_2.$$

Proof .

$$min\|I_L \otimes I_K - \sum_{i=1}^{M}\sum_{j=1}^{N} s_j^2 \varphi_i\varphi_i^* \otimes \psi_j\psi_j^*\|_2$$

$$\leq \|I_L \otimes (I_K - \sum_{j=1}^{N} s_j^2\psi_j\psi_j^*)\|_2 + \|(I_L - \sum_{i=1}^{M}\varphi_i\varphi_i^*) \otimes I_K\|_2$$

$$= \|I_L\|_2\|I_K - \sum_{j=1}^{N} s_j^2\psi_j\psi_j^*\|_2 + \|I_L - \sum_{i=1}^{M}\varphi_i\varphi_i^*\|_2\|I_K\|_2$$

since $\Psi = (\psi_j)_{j=1}^N$ is scalable frame $\|I_K - \sum_{j=1}^{N} s_j^2\psi_j\psi_j^*\|_2 = 0$ then,

$$min\|I_L \otimes I_K - \sum_{i=1}^{M}\sum_{j=1}^{N} s_j^2 \varphi_i\varphi_i^* \otimes \psi_j\psi_j^*\|_2 \leq min\|I_L - \sum_{i=1}^{M}\varphi_i\varphi_i^*\|_2.$$

This completes the proof. □

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