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Application of frames of subspaces in conjugate gradient method for solving operator equations

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Abstract

This paper is concerned with the conjugate gradient method for solving an operator equation on Hilbert spaces by using frames of subspaces. We design an algorithm, based on the bounds of a frame of subspaces and conjugate gradient method, and investigate its convergence and optimality.

Keywords: Hilbert spaces, dual space, frame, frame of subspaces, conjugate gradient method.
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1. Introduction and preliminaries

The area of conjugate direction algorithms has been one of great creativity in the non linear programming field. Hereof the conjugate gradient method is the conjugate direction method that is obtained by selecting the successive direction vectors as a conjugate version of the successive gradients obtained as the method progresses. This method is extremely effective in dealing with general objective functions is consider among the best general purpose methods. On the other hand the potential of frames in numerical analysis is an almost unexplored field. On the one hand the redundancy of a frame can give the freedom to implemented further properties, which would be mutually exclusive in the Riesz bases case, e.g. both high smoothness and small support. On the other hand, since one is working with a weaker concept, the concrete construction of a frame is usually much simpler where compared to stable multiscale bases. Consequently, there is some hope that the frame approach might simplify the geometrical construction on bounded domains. To handle this emerging applications of frames, new methods have to be developed. One starting points is to

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first build frames "locally" and then piece them together to obtain frames for the whole space. One advantage of this idea is that it would facilitate the construction of frames for special applications, since we can first construct frames or choose already known frames for smaller spaces. And on a second step one could construct a frame for the whole space from them. This arise the concept of the frame of subspaces.

In this paper we will use frames of subspaces to get some approximated solutions for the operator equation

$$Lu = f, \tag{1.1}$$

where $L : H \to H$ is a bounded, invertible and self adjoint linear operator on a separable Hilbert space H. A natural approach to constructing an approximate solution is to solve a finite dimensional analog of the problem (1.1). In [8, 1] you can see the development of numerical methods for solving the problem by using frames.

First we will briefly recall the definitions and basic properties of frames and frames of subspaces. For more information we refer to the survey articles by Cassaza and Gitta Kutyniok [5] and the book by Christensen [7]. Throughout this paper H shall always denote an arbitrary separable Hilbert space. Furthermore all subspaces are assumed to be closed. Moreover, Λ denotes a countable indexing set and I denotes the identity operator. Also if W is a subspace of a Hilbert space H, we let π_W denote the orthogonal projection of H onto W.

Assume that H is a separable Hilbert space, Λ is a countable set of indices and $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ is a frame for H. This means that there exist constants $0 < A_\Psi \leq B_\Psi < \infty$ such that

$$A_\Psi \|f\|_H^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \psi_\lambda\rangle|^2 \leq B_\Psi \|f\|_H^2, \quad \forall f \in H. \tag{1.2}$$

For a frame Ψ, the operator $S : H \to H$ defined by

$$S(f) = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda\rangle \psi_\lambda,$$

is called the frame operator. It was shown in [7], for the frame $(\psi_\lambda)_{\lambda\in\Lambda}$, S is a positive invertible operator satisfying $A_\Psi I_H \leq S \leq B_\Psi I_H$ and $B_\Psi^{-1} I_H \leq S^{-1} \leq A_\Psi^{-1} I_H$. Also, the sequence

$$\widetilde{\Psi} = (\widetilde{\psi}_\lambda)_{\lambda\in\Lambda} = (S^{-1}\psi_\lambda)_{\lambda\in\Lambda},$$

is a frame (called the canonical dual frame) for H with bounds $B_\Psi^{-1}, \ \ A_\Psi^{-1}$. Every $f \in H$ has the expansion

$$f = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda\rangle \widetilde{\psi}_\lambda = \sum_{\lambda\in\Lambda} \langle f, \widetilde{\psi}_\lambda\rangle \psi_\lambda.$$

For an index set $\widetilde{\Lambda} \subset \Lambda$, $(\psi_\lambda)_{\lambda\in\widetilde{\Lambda}}$ is called a frame sequence if it is a frame for its closed span.

Now let H be a separable Hilbert space and Λ be a countable indexing set. For a family of weights $\{v_\lambda\}_{\lambda\in\Lambda}$, i.e, $v_\lambda > 0$ for all $\lambda \in \Lambda$, a family of subspaces $\{H_\lambda\}_{\lambda\in\Lambda}$ of a Hilbert space H is called a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for H, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{\lambda\in\Lambda} v_\lambda^2 \|\pi_{H_\lambda}(f)\|^2 \leq B\|f\|^2 \quad \forall f \in H, \tag{1.3}$$

where π_{H_λ} denotes the orthogonal projection onto the subspace H_λ.

The constants A and B is called the frame bounds of the frame of subspaces. If $A = B$ then the frame $\{H_\lambda\}_{\lambda\in\Lambda}$ with respect to$\{v_\lambda\}_{\lambda\in\Lambda}$, is called a A -tight frame of subspaces. It is proved that [5], the family $\{H_\lambda\}_{\lambda\in\Lambda}$ of the frame of subspaces is complete, in the sense that $\overline{span}_{\lambda\in\Lambda}\{H_\lambda\} = H$. The following theorem [5], shows that how we able to string together frames for each of the subspaces H_λ to get a frame for H.

Theorem 1.1. *Let Λ be an index set, $v_\lambda > 0$ for each $\lambda \in \Lambda$, and $\{\psi_{\lambda_i}\}_{i\in I_\Lambda}$ be a frame sequence in H with frame bounds A_λ and B_λ. Define $H_\lambda = \overline{span}_{i\in I_\Lambda}\{\psi_{\lambda i}\}$ for all $\lambda \in \Lambda$, and suppose that $0 < A = \inf_{\lambda\in\Lambda} A_\lambda \le B = \sup_{\lambda\in\Lambda} B_\lambda < \infty$. Then $\{v_\lambda\psi_{i_\lambda}\}_{\lambda\in\Lambda, i\in I_\Lambda}$ is a frame for H if and only if $\{H_\lambda\}_{\lambda\in\Lambda}$ is a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for H.*

As in the well know frame situation, the frame operator $S_{H,v}$ for $\{H_\lambda\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$ is defined by

$$S_{H,v}(f) = \sum_{\lambda\in\Lambda} v^2 \pi_{H_\lambda}(f).$$

The frame operator $S_{H,v}$ for $\{H_\lambda\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$ is a operator, self-adjoint, invertible on H with $AI \le S_{H,v} \le BI$, where A and B are the bounded of the frame of subspaces. Further, the following reconstruction formula satisfies:

$$f = \sum_{\lambda\in\Lambda} v_\lambda^2 S_{H,v}^{-1} \pi_{H_\lambda}(f) \quad \forall f \in H.$$

It is proved that $\{S_{H,v}^{-1} H_\lambda\}_{\lambda\in\Lambda}$ is a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$.

2. Some basic facts

The most straight forward approach to an iterative solution of a linear system is to rewrite the equation (1.1) as a linear fixed-point iteration. One way to do this is the Richardson iteration. The abstract method reads as follows:
write $Lu = f$ as

$$u = (I - L)u + f.$$

For given $u_0 \in H$, define for $k \ge 0$,

$$u_{k+1} = (I - L)u_k + f. \tag{2.1}$$

Since $Lu - f = 0$,

$$\begin{aligned} u_{k+1} - u &= (I - L)u_k + f - u - (f - Lu) \\ &= (I - L)u_k - u + Lu \\ &= (I - L)(u_k - u). \end{aligned}$$

Hence

$$\|u_{k+1} - u\|_H \le \|I - L\|_{H\to H} \|u_k - u\|_H,$$

so that (2.1) converges if

$$\|I - L\|_{H\to H} < 1.$$

It is sometimes possible to precondition (1.1) by multiplying both sides by a matrix B,

$$BLu = Bf,$$

such that convergence of iterative methods is improved. This is very effective technique for solving differential equations, integral equations, and related problems [2, 3]. The following proposition is useful to use frames of subspaces in Richardson iterative method.

Proposition 2.1. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$, and let $L : H \to H$ be a bounded invertible operator on H. Then $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ is a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$.*

Proof . See [5]. □

In this case if u is the solution of equation (1.1) and S' is the frame operator of the frame of subspaces $\{LH_\lambda\}$ then

$$u = \sum_{\lambda\in\Lambda} v_\lambda^2 S'^{-1}_{H,v} \pi_{LH_\lambda} u.$$

Now since L is invertible then $\pi_{LH_\lambda} = L\pi_{H_\lambda} L^{-1}$. Therefore

$$u = \sum_{\lambda\in\Lambda} v_\lambda^2 S'^{-1}_{H,v} L\pi_{H_\lambda} L^{-1} u = \sum_{\lambda\in\Lambda} v_\lambda^2 S'^{-1}_{H,v} L\pi_{H_\lambda} (L^{-1})^2 f.$$

Since H is a separable Hilbert space (infinite dimension), it is difficult to obtain L^{-1} and S'^{-1}. Our goal is to find a sequence u_i of approximated solutions, related to a frame of subspaces, such that converges to the solution u of the equation (1.1). Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for a separable Hilbert space H with the frame operator $S_{H,v}$. By Proposition 2.1, $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ also is a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$. We denote the frame operator for $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$, by $S'_{H,v}$. Also since L is bounded invertible then there exist two positive constants c_1 and c_2 such that

$$c_1\|u\|_H \le \|Lu\|_H \le c_2\|u\|_H, \quad \forall u \in H. \tag{2.2}$$

Theorem 2.2. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for H and let L be bounded, invertible and self adjoint operator in H. If $S'_{H,v}$ is the frame operator for the frame of subspaces $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ with bounds A, B, and c_1, c_2 as in (2.2). Then*

$$\|I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L\| \le \frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}. \tag{2.3}$$

Proof . for every $v \in H$ we have

$$\begin{aligned}
\langle (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L)v, v\rangle &= \|v\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B}\langle S'_{H,v} Lv, Lv\rangle \\
&= \|v\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B}\langle \sum_{\lambda\in\Lambda} v_\lambda^2 \pi_{LH_\lambda}(Lv), Lv\rangle \\
&= \|v\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B}\sum_{\lambda\in\Lambda} v_\lambda^2 \|\pi_{LH_\lambda}(Lv)\|_H^2 \\
&\le \|v\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B}\|Lv\|_H^2 \\
&\le \|v\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B} c_1^2 \|v\|_H^2 \\
&= (\frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B})\|v\|_H^2,
\end{aligned}$$

where in the first inequality we used the property of the lower bound of the frame of subspaces and in the second inequality we used the property of c_1 in (2.2). Similarly we have

$$-(\frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B})\|v\|_H^2 \leq \langle (I - \frac{2}{c_1^2 A + c_2^2 B} LS'_{H,v} L)v, v\rangle.$$

Therefore

$$\|I - \frac{2}{c_1^2 A + c_2^2 B} LS'_{H,v} L\| \leq \frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}.$$

□

3. Conjugate gradient method based on the upper and lower bounds of a frame of subspaces

We define $LS'L$-norm as

$$\|f\|_{LS'L} = \langle f, LS'Lf\rangle^{\frac{1}{2}} = \|(LS'L)^{\frac{1}{2}} f\|, \qquad \forall f \in H.$$

In fact the corresponding inner product is $\langle f, g\rangle_{LS'L} = \langle f, LS'Lg\rangle, \quad \forall f, g \in H$. Since $LS'L$ is positive and invertible, this is indeed a new norm in H. First of all we note that if $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces and A, B are the frame bounds of the frame of subspaces $\{LH_\lambda\}_{\lambda\in\Lambda}$ with frame operator S', that is $AI \leq S' \leq BI$, and u be the solution of the equation (1.1), then

$$\|u\|_{LS'L}^2 = \langle u, LS'Lu\rangle = \langle Lu, S'Lu\rangle = \langle f, S'f\rangle,$$

therefore

$$\sqrt{A}\|f\| \leq \|u\|_{LS'L} \leq \sqrt{B}\|f\|. \tag{3.1}$$

Now let V_n be the subspace generated by the vectors $(LS'L)^j u, \quad j = 1, \ldots, n$,

$$V_n = span\{(LS'L)^j u, \quad j = 1, \ldots, n\},$$

and let $P_{-1} = 0, P_0 = \beta LS'f$, where $\beta = \frac{2}{c_1^2 A + c_2^2 B}$, and for $k \geq 0$,

$$P_{k+1} = LS'LP_k - \frac{\langle LS'LP_k, LS'LP_k\rangle}{\langle P_k, LS'LP_k\rangle} P_k - \frac{\langle LS'LP_k, LS'LP_{k-1}\rangle}{\langle P_{k-1}, LS'LP_{k-1}\rangle} P_{k-1}.$$

In this case the following lemma holds.

Lemma 3.1. $\{P_0, P_1, \ldots, P_{n-1}\}$ *is an orthogonal basis for* V_n *with respect to the inner product* $\langle f, g\rangle_{Ls'L} = \langle f, Ls'Lg\rangle$.

Proof . First we note that $\{P_0, P_1, \ldots, P_{n-1}\} \subseteq V_n$. We verify this by induction. Clearly it is true for $n = 1$. Assuming that it is true for all $k \leq n$, then for $k = n+1$, by the definition of P_n, we have

$$P_n = LS'LP_{n-1} - \frac{\langle LS'LP_{n-1}, Ls'LP_{n-1}\rangle}{\langle P_{n-1}, LS'LP_{n-1}\rangle} P_{n-1} - \frac{\langle LS'LP_{n-1}, LS'LP_{n-2}\rangle}{\langle P_{n-2}, LS'LP_{n-2}\rangle} P_{n-2}$$

$$\in LS'L(V_n) + V_n \subseteq V_{n+1},$$

as we desired.
Then we only have to show $\{P_0, P_1, \ldots, P_{n-1}\}$ is an orthogonal set. It is clear for $n = 1$. For $n = 2$ we have,

$$\langle P_0, P_1\rangle_{Ls'L} = \langle Ls'LP_0, P_1\rangle$$
$$= \langle LS'LP_0, LS'LP_0 - \frac{\langle LS'LP_0, LS'LP_0\rangle}{\langle P_0, LS'LP_0\rangle}P_0\rangle$$
$$= \langle LS'LP_0, LS'LP_0\rangle - \frac{\langle LS'LP_0, LS'LP_0\rangle}{\langle P_0, Ls'LP_0\rangle}\langle LS'LP_0, P_0\rangle = 0.$$

Now, arguing by induction, assume that we know already that $\langle P_n, LS'LP_j\rangle = 0$ for $j = 0, 1, \ldots, n-1$ and then $\{P_0, P_1, \ldots, P_n\}$ is an $LS'L$-orthogonal basis for V_{n+1}. We have to show that $\langle P_{n+1}, LS'LP_j\rangle = 0$ for $j = 0, 1, \ldots, n$.
For $j = n$,

$$\begin{aligned}\langle P_{n+1}, LS'LP_n\rangle &= \langle LS'LP_n - \frac{\langle LS'LP_n, LS'LP_n\rangle}{\langle P_n, LS'LP_n\rangle}P_n \\ &- \frac{\langle LS'LP_n, LS'LP_{n-1}\rangle}{\langle P_{n-1}, LS'LP_{n-1}\rangle}P_{n-1}, LS'LP_n\rangle \\ &= \langle LS'LP_n, LS'LP_n\rangle - \langle LS'LP_n, LS'LP_n\rangle \\ &- \frac{\langle LS'LP_n, LS'LP_{n-1}\rangle}{\langle P_{n-1}, LS'LP_{n-1}\rangle}\langle P_{n-1}, LS'LP_n\rangle = 0.\end{aligned}$$

Similarity this argument also holds for $j = n - 1$.
For $j < n - 1$ we observe that $LS'LP_j \in LS'LV_{n-1} \subseteq V_n$ and by induction hypothesis $LS'LP_j = \sum_{i=1}^{n-1} c_iP_i$.
Now because of orthogonality of P_j for $j \le n$ (induction hypothesis) and $\langle P_{n+1}, P_n\rangle = \langle P_{n+1}, P_{n-1}\rangle = 0$, then

$$\begin{aligned}\langle P_{n+1}, LS'LP_j\rangle &= \langle LS'LP_n - \frac{\langle LS'LP_n, LS'LP_n\rangle}{\langle P_n, LS'LP_n\rangle}P_n \\ &- \frac{\langle LS'LP_n, LS'LP_{n-1}\rangle}{\langle P_{n-1}, LS'LP_{n-1}\rangle}P_{n-1}, LS'LP_j\rangle \\ &= \langle LS'LP_n, LS'LP_j\rangle = \langle LS'LP_n, \sum_{i=1}^{n-1} c_iP_i\rangle = 0.\end{aligned}$$

□

Now, we can design the following algorithm based on the conjugate gradient method and using frame of subspaces in order to give an approximated solution to the equation(1.1). Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces and S' be the frame operator of the frame of subspaces $\{LH_\lambda\}_{\lambda\in\Lambda}$ with lower and upper bounds A, B respectively. Also let c_1, c_2 be as in the equation (2.2) and let $\sigma = \frac{c_2\sqrt{B}-c_1\sqrt{A}}{c_2\sqrt{B}+c_1\sqrt{A}}$.

Algorithm$[A, B, c_1, c_2, \epsilon] \to u_\epsilon$
(i)$Put\ h_0 = 0,\ P_{-1} = 0,\ n = 0,\ r_0 = LS'f,\ P_0 = \beta LS'f$
(ii) While $\frac{2\sigma^n}{1+\sigma^{2n}}\sqrt{B}\|f\| > \epsilon$

(1) $n := n + 1$

(2) $\lambda_{n-1} = \frac{\langle r_{n-1}, P_{n-1}\rangle}{\langle P_{n-1}, LS'LP_{n-1}\rangle}$

(3) $h_n = h_{n-1} + \lambda_{n-1}P_{n-1}$

(4) $P_n = LS'LP_{n-1} - \frac{\langle LS'LP_{n-1}, LS'LP_{n-1}\rangle}{\langle P_{n-1}, LS'LP_{n-1}\rangle}P_{n-1} - \frac{\langle LS'LP_{n-1}, LS'LP_{n-2}\rangle}{\langle P_{n-2}, LS'LP_{n-2}\rangle}P_{n-2}$

(5) $r_n = r_{n-1} - \lambda_{n-1}LS'LP_{n-1}$

(iii) $u_\epsilon := h_n$.

Theorem 3.2. *The approximated solution h_n in the **Algorithm**$[A, B, c_1, c_2, \epsilon]$ is the orthogonal projection of the solution u of the problem (1.1) onto V_n, with respect to the $LS'L-$inner product. That is*

$$\|u - h_n\|_{LS'L} \leq \|u - g\|_{LS'L}, \qquad \forall g \in V_n.$$

Proof . Since $h_n = \sum_{k=0}^{n-1} \lambda_k P_k \in V_n$, then it is enough to show that $\langle u - h_n, h_n\rangle_{LS'L} = 0$. By Lemma 3.1 we have

$$\langle h_j, P_j\rangle_{LS'L} = \langle \sum_{k=0}^{j-1} \lambda_k P_k, P_j\rangle_{LS'L} = 0. \tag{3.2}$$

Rewriting r_j as

$$\begin{aligned} r_j &= r_{j-1} - \lambda_{j-1}LS'LP_{j-1} = \cdots = r_0 - \sum_{k=0}^{j-1} \lambda_k LS'LP_k \\ &= r_0 - LS'L(\sum_{k=0}^{j-1} \lambda_k P_k) = r_0 - LS'Lh_j = LS'f - LS'Lh_j \\ &= LS'Lu - LS'Lh_j = LS'L(u - h_j), \end{aligned} \tag{3.3}$$

we obtain

$$\lambda_j = \frac{\langle r_j, P_j\rangle}{\langle P_j, LS'LP_j\rangle} = \frac{\langle LS'L(u - h_j), P_j\rangle}{\langle P_j, P_j\rangle_{LS'L}},$$

and by using (3.2) and (3.3) we conclude

$$\begin{aligned} \langle u - h_n, h_n\rangle_{LS'L} &= \langle u - \sum_{j=0}^{n-1} \lambda_j P_j, \sum_{j=0}^{n-1} \lambda_j P_j\rangle_{LS'L} \\ &= \sum_{j=0}^{n-1} \overline{\lambda}_j \langle u, P_j\rangle_{LS'L} - \sum_{j=0}^{n-1} \lambda_j \overline{\lambda}_j \langle P_j, P_j\rangle_{LS'L} \\ &= \sum_{j=0}^{n-1} \overline{\lambda}_j (\langle u, P_j\rangle_{LS'L} - \lambda_j \langle P_j, P_j\rangle_{LS'L}) \\ &= \sum_{j=0}^{n-1} \overline{\lambda}_j (\langle u, P_j\rangle_{LS'L} - \frac{\langle LS'L(u - h_j), P_j\rangle}{\langle P_j, P_j\rangle_{LS'L}} \langle P_j, P_j\rangle_{LS'L}) \\ &= \sum_{j=0}^{n-1} \overline{\lambda}_j \langle LS'Lu - LS'L(u - h_j), P_j\rangle = \sum_{j=0}^{n-1} \overline{\lambda}_j \langle LS'Lh_j, P_j\rangle = 0. \end{aligned}$$

□

By definition of V_n we observe that

$$h_n = q_{n-1}(LS'L)\beta LS'Lu,$$

where $q_{n-1}(x)$ is a polynomial of degree $n-1$. Therefore

$$u - h_n = (I - q_{n-1}(LS'L)\beta LS'L)u = \phi_n(I - \beta LS'L)u,$$

where $\phi_n(x) = 1 - (1-x)q_{n-1}(\frac{1-x}{\beta})$ is a polynomial of degree n and $\phi_n(1) = 1$.
For the error estimate we have

$$\begin{aligned}\|u - h_n\|_{LS'L} &= \|\phi_n(I - \beta LS'L)u\|_{LS'L}\\ &= \|(LS'L)^{\frac{1}{2}}\phi_n(I - \beta LS'L)(LS'L)^{-\frac{1}{2}}(LS'L)^{\frac{1}{2}}u\|\\ &\le \|(LS'L)^{\frac{1}{2}}\phi_n(I - \beta LS'L)(LS'L)^{-\frac{1}{2}}\|\|(LS'L)^{\frac{1}{2}}u\|\\ &= \|\phi_n(I - \beta LS'L)\|\|u\|_{LS'L}\\ &\le \max_{|x|\le\alpha_0}|\phi_n(x)|\|u\|_{LS'L}.\end{aligned}$$

So

$$\|u - h_n\|_{LS'L} \le \max_{|x|\le\alpha_0}|\phi_n(x)|\|u\|_{LS'L}, \tag{3.4}$$

where $\alpha_0 = \frac{c_2^2B - c_1^2A}{c_2^2B + c_1^2A}$.
The aim is to minimize this error. Therefore we try to find

$$\min_{\phi_n(x)}\max_{|x|\le\alpha_0}|\phi_n(x)|, \tag{3.5}$$

where min is considered on all polynomials of degrees less than or equal n such that $\phi_n(1) = 1$. This is done by Chebyshev polynomials, satisfying the recurrence relation

$$C_0(x) = 1,\ C_1(x) = x,\ C_n(x) = 2xC_{n-1}(x) - C_{n-2}(x), \quad \forall n \ge 2. \tag{3.6}$$

In fact

$$C_n(x) = \begin{cases} \cos(n\cos^{-1}(x)), & |x| \le 1\\ \cosh(\cosh^{-1}(x)) = \frac{1}{2}((x+\sqrt{x^2-1})^n + (x+\sqrt{x^2-1})^{-n}), & |x| \ge 1.\end{cases}$$

For more details see [6].
In this case, the following lemma holds [6].

Lemma 3.3. *Given $a < b < 1$ set*

$$P_n(x) = \frac{C_n(\frac{2x-a-b}{b-a})}{C_n(\frac{2-a-b}{b-a})}.$$

Then

$$\max_{a\le x\le b}|P_n(x)| \le \max_{a\le x\le b}|\phi_n(x)|,$$

for all $\phi_n(x)$ of degree less than or equal n, satisfying $\phi_n(1) = 1$.
Furthermore

$$max_{a\le x\le b}|P_n(x)| = \frac{1}{C_n(\frac{2-a-b}{b-a})}.$$

Theorem 3.4. *The approximated solution h_n in the* ***Algorithm***$[A, B, c_1, c_2, \epsilon]$ *satisfies*

$$\|u - h_n\|_{LS'L} \leq \frac{2\sigma^n}{1+\sigma^{2n}}\sqrt{B}\|f\|.$$

Proof . Based on the above argument, by setting $a = -\alpha_0$ and $b = \alpha_0$ in Lemma 3.3, the polynomial

$$P_n(x) = \frac{C_n(\frac{2x+\alpha_0-\alpha_0}{\alpha_0+\alpha_0})}{C_n(\frac{2+\alpha_0-\alpha_0}{\alpha_0+\alpha_0})} = \frac{C_n(\frac{x}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}. \tag{3.7}$$

Solves (3.5) and minimize the error $\|u - h_n\|_{LS'L}$ in (3.4). Also the maximum is

$$\frac{1}{C_n(\frac{2+\alpha_0-\alpha_0}{\alpha_0+\alpha_0})} = C_n(\frac{1}{\alpha_0}) = C_n(\frac{c_2^2B + c_1^2A}{c_2^2B - c_1^2A})$$

$$= \frac{1}{2}((\frac{c_2^2B + c_1^2A}{c_2^2B - c_1^2A} + \sqrt{\frac{(c_2^2B + c_1^2A)^2}{(c_2^2B - c_1^2A)^2} - 1})^n + \frac{1}{(\frac{c_2^2B+c_1^2A}{c_2^2B-c_1^2A} + \sqrt{\frac{(c_2^2B+c_1^2A)^2}{(c_2^2B-c_1^2A)^2} - 1})^n})$$

$$= \frac{1}{2}((\frac{c_2^2B + c_1^2A}{c_2^2B - c_1^2A} + \sqrt{\frac{4(c_2^2B)(c_1^2A)}{(c_2^2B - c_1^2A)^2}})^n + \frac{1}{(\frac{c_2^2B+c_1^2A}{c_2^2B-c_1^2A} + \sqrt{\frac{4(c_2^2B)(c_1^2A)}{(c_2^2B-c_1^2A)^2}})^n})$$

$$= \frac{1}{2}((\frac{(\sqrt{c_2^2B} + \sqrt{c_1^2A})^2}{c_2^2B - c_1^2A})^n + \frac{1}{(\frac{(\sqrt{c_2^2B}+\sqrt{c_1^2A})^2}{c_2^2B-c_1^2A})^n})$$

$$= \frac{1}{2}((\frac{c_2\sqrt{B} + c_1\sqrt{A}}{c_2\sqrt{B} - c_1\sqrt{A}})^n + \frac{1}{(\frac{c_2\sqrt{B}+c_1\sqrt{A}}{c_2\sqrt{B}-c_1\sqrt{A}})^n})$$

$$= \frac{1}{2}(\frac{1}{\sigma^n} + \sigma^n) = \frac{1+\sigma^{2n}}{2\sigma^n}.$$

Finally, by the inequalities (3.4) and (3.1) we obtain

$$\|u - h_n\|_{LS'L} \leq \frac{1}{C_n(\frac{1}{\alpha_0})}\|u\|_{LS'L} = (C_n(\frac{1}{\alpha_0}))^{-1}\|u\|_{LS'L}$$

$$= \frac{2\sigma^n}{1+\sigma^{2n}}\|u\|_{LS'L} \leq \frac{2\sigma^n}{1+\sigma^{2n}}\sqrt{B}\|f\|,$$

as we desired. □

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