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q-Analogue of Liu-Srivastava operator on meromorphic functions based on subordination

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Abstract

In this paper, the authors investigate a new subclass of meromorphic functions associated with q-Analogue of Liu-Srivastava operator and differential subordination. Some properties in the form of coefficient inequality, Integral representation, Radii of starlikeness and convexity, and partial sum concept are introduced.

Keywords: Meromorphic function, q-Analogue of Liu-Srivastava operator, Coefficient bounds, Radii properties, Partial sum, Neighborhoods, Hadamard product.
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1. Introduction

Studying the theory of analytical functions has been an area of concern for many researchers. A more specific field is the study of inequalities in complex analysis. Literature review indicates lots of studies based on the classes of analytical functions. The q-Analogue of Liu-Srivastava operator and differential subordination a very important aspect in complex function theory study.
The q-analogue of derivative and integral operators were introduced by Jackson [6, 7] along with some applications of q-calculus.Purohit and Raina [15], Juma, Abdulhussain and Al-khafaji [8] used fractional q-calculus operator investigating certain classes of functions which are analytic in the open disk. Kanas and Raducanu [9] gave the q-analogue of Ruscheweyh differential operator using the concepts of convolution and then studied some of its properties. More applications of this operator can be seen in the paper [2].

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The theory of q-analogues or q-extensions of classical formulas and functions based on the observation that

$$\lim_{q\to 1}\frac{1-q^{\alpha}}{1-q}=\alpha,\quad |q|<1,$$

therefore the number $\frac{1-q^{\alpha}}{1-q}$ is sometimes called the basic number $[\alpha]_q$. In this work we derive q-analogue of Liu-Srivastava operator and employ this new differential operator to define an integral operator for meromorphic functions.

Let Σ denote the class of meromorphic functions of the type

$$f(z)=\frac{1}{z}+\sum_{k=1}^{+\infty}a_k z^{k-1}, \tag{1.1}$$

which are analytic in the punctured open disk

$$\mathbb{U}^{*}=\{z\in\mathbb{C}:\ 0<|z|<1\}.$$

If $f\in\Sigma$ is given by (1.1) and $g\in\Sigma$ given by

$$g(z)=\frac{1}{z}+\sum_{k=1}^{+\infty}b_k z^{k-1},$$

then the Hadamard product (or convolution) $f*g$ of f and g is defined by

$$(f*g)(z)=\frac{1}{z}+\sum_{k=1}^{+\infty}a_k b_k z^{k-1}=(g*f)(z).$$

The q-shifted factorial is defined for $w,q\in\mathbb{C}$ as a product of n factors by:

$$(w,q)_n=\begin{cases}1 & ,\quad n=0\\ (1-w)(1-wq)\dots(1-wq^{n-1}) & ,\quad n\in\mathbb{N}=\{1,2,\dots\}.\end{cases} \tag{1.2}$$

In view of the relation (1.2), we get

$$\lim_{q\to 1^{-}}\frac{(q^{w},q)_n}{(1-q)^n}=(w)_n. \tag{1.3}$$

where $(w)_n=w(w+1)\cdots(w+n-1)$ is the familiar Pochhammer symbol. For complex parameters

$$\alpha_i,\ \beta_j,\quad (i=1,\cdots,t,\ \ j=1,\cdots,m,\ \ \alpha_i\in\mathbb{C},\ \beta_j\in\mathbb{C}\backslash\{0,-1,-2,\cdots\}),$$

the q-hypergeometric function is the q-Analogue of the hypergeometric function and it is introduced as follow:

$$\Psi(\alpha_1,\dots,\alpha_t,\beta_1,\dots,\beta_m,q,z)=\sum_{k=0}^{\infty}\frac{(\alpha_1,q)_k\dots(\alpha_t,q)_k}{(q,q)_k(\beta_1,q)_k\dots(\beta_m,q)_k}\times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+m-t}z^k, \tag{1.4}$$

where $\binom{k}{2} = \dfrac{k(k-1)}{2}$, $q \neq 0$, $t > m+1$ $(t, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ and $(w,q)_k$ is the q-analogue of the Pochhammer symbol $(w)_k$ defined in (1.2) see [4].
For $z \in \mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$, $|q| < 1$ and $t = m+1$, the q-Analogue of the hypergeometric function defined in (1.4) takes the from

$$ {}_t\Phi_m(\alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_m, q, z) = \sum_{k=0}^{\infty} \frac{(\alpha_1, q)_k \dots (\alpha_t, q)_k}{(q,q)_k (\beta_1, q)_k \dots (\beta_m, q)_k} \ z^k, \tag{1.5} $$

which converges absolutely in the open unit disk $\mathbb{U}$.
Also corresponding to the function defined in (1.5), consider

$$ \begin{aligned} \frac{1}{z} \ {}_t\Phi_m(\alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_m, q, z) &= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\alpha_1, q)_{k+1} \dots (\alpha_t, q)_{k+1}}{(q,q)_{k+1} (\beta_1, q)_{k+1} \dots (\beta_m, q)_{k+1}} z^k \\ &= {}_t\mathcal{G}_m(\alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_m, q, z). \end{aligned} \tag{1.6} $$

Now we consider the linear operator

$$ \mathcal{L}_m^t(\alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_m, q) : \Sigma \longrightarrow \Sigma $$

by

$$ \begin{aligned} \mathcal{L}_m^t(\alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_m, q) f(z) &= {}_t\Phi_m(\alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_m, q, z) * f(z) \\ &= \frac{1}{z} + \sum_{k=1}^{+\infty} X_m^t(k) a_k z^k, \end{aligned} \tag{1.7} $$

where

$$ X_m^t(k) = \frac{(\alpha_1, q)_{k+1} \dots (\alpha_t, q)_{k+1}}{(q,q)_{k+1} (\beta_1, q)_{k+1} \dots (\beta_m, q)_{k+1}}, \tag{1.8} $$

see [3].
For the sake of simplicity we write

$$ \mathcal{L}_m^t(\alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_m, q) f(z) = \mathcal{L}_m^t[\alpha_i, \beta_j, q] f(z). \tag{1.9} $$

In special case, when

$$ \alpha_i = q^{\alpha_i}, \ \beta_j = q^{\beta_j}, \ \alpha_i > 0, \ \beta_j > 0 \quad (i = 1, \dots, t, \ j = 1, \dots, m, \ t = m+1) $$

and $q \to 1$, the operator

$$ \mathcal{L}_m^t[\alpha_i, \beta_j, q] f(z) = \mathcal{H}_m^t[\alpha_i] f(z), $$

was introduced by Liu and Srivastara [11].
Also for $t = 2, \ m = 1, \ \alpha_2 = q$ and $q \to 1$ the operator investigated in [10].
Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}^*$, then we say that $f(z)$ is subordinate to $g(z)$, if there exists an analytic function $w(z)$ with $w(0) = 0$ and $\mid w(z) \mid < 1$, such that $f(z) = g(w(z))$.
We denote this subordination by $f(z) \prec g(z)$.
We denote the subclass $\Sigma_q^{\alpha_t, \beta_m}(A, B, C, \theta)$ of Σ consisting of function $f \in \Sigma$ for which

$$ -\frac{z[\mathcal{L}_m^t[\alpha_i, \beta_j, q] f(z)]''}{[\mathcal{L}_m^t[\alpha_i, \beta_j, q] f(z)]'} \prec 2\frac{1 + Az}{1 + Bz}, \tag{1.10} $$

where

$$ A = B + (C - B)(1 - \theta), 0 \le \theta < 1, -1 \le B < C \le 1 \ \ and \ \ -1 \le B < A \le 1. $$

several other classes studied by various authors, for example see [1], [12] and [13].

2. Main Results

In this section, we obtain coefficient bounds and some properties for the class $\Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$.

Theorem 2.1. *Let $f(z)\in\sum$, then $f(z)\in\Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$ if and only if*

$$\sum_{k=1}^{+\infty}[k^2(1+B)+k(B+2(C-B)(1-\theta))]X_m^t(k)a_k<2(C-B)(1-\theta), \tag{2.1}$$

where $X_m^t(k)$ is given in (1.8).
The result is sharp for the function $F(z)$ given by

$$F(z)=\frac{1}{z}+\frac{2(C-B)(1-\theta)}{[k^2(1+B)+k(B+2(C-B)(1-\theta))]X_m^t(k)}z^k, \quad k=1,2,\dots, \tag{2.2}$$

and $X_m^t(k)$ is given in (1.8).

Proof . Let $f(z)\in\Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$, then the subordination relation (1.9) or equivalently

$$\left|\frac{z[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]''+2[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'}{zB[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]''+2(B+(C-B)(1-\theta)[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'}\right|<1, \tag{2.3}$$

holds true , therefore by making use of (1.8) and (1.9) we have

$$\left|\frac{\sum_{k=1}^{+\infty}k^2X_m^t(k)a_kz^{k-1}}{-2(C-B)(1-\theta)z^{-2}+\sum_{k=1}^{+\infty}k(B(k-1)+2A)X_m^t(t)a_zz^{k-1}}\right|<1.$$

Since $\Re(z)\le|z|$ for all z, therefore

$$\Re\left\{\frac{\sum_{k=1}^{+\infty}k^2X_m^t(k)a_kz^{k-1}}{2(C-B)(1-\theta)z^{-2}-\sum_{k=1}^{+\infty}k(B(k-1)+2A)X_m^t(t)a_kz^{k-1}}\right\}<1.$$

By letting $z\to\overline{1}$ through real values, we conclude

$$\textstyle\sum_{k=1}^{+\infty}[k^2(1+B)+k(B+2(C-B)(1-\theta))]X_m^t(k)a_k<2(C-B)(1-\theta),$$

where $X_m^t(k)$ is defined in (1.8).
Conversely, let (2.1) holds true, it we let $z\in\partial\mathbb{U}^*$, where $\partial\mathbb{U}^*$ denotes the boundary of $\mathbb{U}^*$, then we have

$$\begin{aligned}&\left|\frac{z[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]''+2[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'}{zB[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]''+2(B+(C-B)(1-\theta)[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'}\right|\\ &\le\frac{\sum_{k=1}^{+\infty}k^2X_m^t(k)\mid a_k\mid}{2(C-B)(1-\theta)-\sum_{k=1}^{+\infty}k(B(k-1)+2A)X_m^t(k)\mid a_k\mid}<1,\end{aligned}$$

(by (2.1)).
Thus by the maximum modulus theorem we conclude $f(z)\in\Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$. □

Remark 2.2. *Theorem 2.1 shows that if* $f(z) \in \Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$*, then*

$$| a_k | \leq \frac{2(C-B)(1-\theta)}{[k^2(1+B)+k(B+2(C-B)(1-\theta))]X_m^t(k)}, k = 1, 2, \dots, \tag{2.4}$$

where $X_m^t(k)$ *is given in* (1.8).

Now we obtain integral representation for $\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)$.

Theorem 2.3. *if* $f(z) \in \Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$ *then*

$$\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z) = \int_0^z \exp\left\{\int_0^z \frac{2[(B+(C-B)(1-\theta))\mathcal{M}(\nu)-1]}{\nu(1-B\mathcal{M}(\nu))}d\nu\right\} d\omega \tag{2.5}$$

where $| \mathcal{M}(z) | < 1$.

Proof . since $f(z) \in \Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$, so (2.1) holds true or equivalently we have

$$|\mathcal{M}(z)| = \left| \frac{z[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'' + 2[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'}{zB[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'' + 2(B+(C-B)(1-\theta))[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'} \right| < 1.$$

Hence

$$\frac{[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]''}{[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'} = \frac{2[(B+(C-B)(1-\theta))\mathcal{M}(\nu)-1]}{z(1-B\mathcal{M}(\nu))},$$

where $| \mathcal{M}(z) | < 1, \quad z \in \mathbb{U}^*$.
After integration we get the required result. □

3. Radii and partial sum properties

In the last section we introduce Radii of starlikeness and convexity. Also partial sum property is considered.

Theorem 3.1. *if* $f(z) \in \Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$ *then,*

(i) f *is meromorphically univalent starlike of order* $\lambda(0 \leq \lambda < 1)$ *in disk* $|z| < R_1$*, where*

$$R_1 = \inf_k \left\{ \frac{(1-\lambda)[k^2(1+B)+k(B+2(C-B)(1-\theta))]X_m^t(k)}{2(C-B)(1-\theta)(k+2-\lambda)} \right\}^{\frac{1}{k+1}}, \tag{3.1}$$

and $X_m^t(k)$ *is given in* (1.8).

(ii) f *is meromerphically univalent convex of order* $\lambda(0 \leq \lambda < 1)$ *in disk* $|z| < R_2$ *where*

$$R_2 = \inf_k \left\{ \frac{(1-\lambda)[k^2(1+B)+k(B+2(C-B)(1-\theta))]X_m^t(k)}{2k(C-B)(1-\theta)(k+2-\lambda)} \right\}^{\frac{1}{k+1}}. \tag{3.2}$$

$X_m^t(k)$ *is given in* (1.8).

Proof . (i) For starlikeness it is enough to show that

$$\left|\frac{zf(z)'}{f(z)} + 1\right| < 1 - \lambda.$$

but

$$\left|\frac{zf(z)'}{f(z)}+1\right| = \left|\frac{\sum_{k=1}^{+\infty}(k+1)a_k z^{k+1}}{1+\sum_{k=1}^{+\infty} a_k z^{k+1}}\right| \leq \frac{\sum_{k=1}^{+\infty}(k+1)a_k|z|^{k+1}}{1-\sum_{k=1}^{+\infty} a_k|z|^{k+1}} \leq 1-\lambda,$$

or

$$\sum_{k=1}^{+\infty}\frac{k+2-\lambda}{1-\lambda}a_k|z|^{k+1} \leq 1.$$

By using (2.4), we obtain

$$\begin{aligned}&\sum_{k=1}^{+\infty}\frac{k+2-\lambda}{1-\lambda}a_k|z|^{k+1}\\ &\leq \sum_{k=1}^{+\infty}\frac{2(C-B)(1-\theta)(k+2-\lambda)}{(1-\lambda)[k^2(1+B)+k(B+2(C-B)(1-\theta))]X_m^t(k)}|z|^{k+1}\\ &\leq 1.\end{aligned}$$

So, it is enough to suppose

$$|z|^{k+1} \leq \frac{(1-\lambda)[k^2(1+B)+k(B+2(C-B)(1-\theta))]X_m^t(k)}{2(C-B)(1-\theta)(k+2-\lambda)}.$$

(ii) For convexity by using the fact that "f is convex if and only if zf' is starlike" and by an easy calculation we conclude the required result. □

Theorem 3.2. *Let $f(z) \in \sum$, and define*

$$S_1(z) = \frac{1}{z}, \quad S_m(z) = \frac{1}{z} + \sum_{k=1}^{m-1} a_k z^k, \quad (m = 2, 3, \dots). \tag{3.3}$$

Also suppose $\sum_{k=1}^{+\infty} d_k a_k \leq 1$, where

$$d_k = \frac{[k^2(1+B)+k(B+2(C-B)(1-\theta))]X_m^t(k)}{2(C-B)(1-\theta)},$$

then

$$\Re\left\{\frac{f(z)}{S_m(z)}\right\} > 1 - \frac{1}{d_m}, \tag{3.4}$$

and

$$\Re\left\{\frac{S_m(z)}{f(z)}\right\} > \frac{d_m}{1+d_m}. \tag{3.5}$$

Proof . Since $\sum_{k=1}^{+\infty} d_k a_k \leq 1$, they by Theorem 2.1, $f(z) \in \Sigma_q^{\alpha_t,\beta_m}(A, B, C, \theta)$.
Also by $k \geq 1$, we conclude and $\{d_k\}$ is an increasing sequence, therefore we obtain

$$\sum_{k=1}^{m-1} a_k + d_m \sum_{k=m}^{+\infty} a_k \leq 1. \tag{3.6}$$

Now by putting

$$V(z) = d_m\left[\frac{f(z)}{S_m(z)} - (1 - \frac{1}{x_m})\right],$$

and making use of (3.6) we obtain

$$\Re\left\{\frac{V(z)-1}{V(z)+1}\right\} \leq \left|\frac{V(z)-1}{V(z)+1}\right| = \left|\frac{d_m f(z) - d_m S_m(z)}{d_m f(z) - d_m S_m(z) + 2S_m(z)}\right|$$
$$= \left|\frac{d_m \sum_{k=m}^{+\infty} a_k z^k}{d_m \sum_{k=m}^{+\infty} a_k z^k + 2(\frac{1}{z} + \sum_{k=1}^{m-1} a_k z^k)}\right|$$
$$\leq \frac{d_m \sum_{k=m}^{+\infty} |a_k|}{2 - \sum_{k=1}^{m-1} |a_k| - d_m \sum_{k=m}^{+\infty} |a_k|} \leq 1.$$

By a simple calculation we conclude $\Re\{V(z)\} > 0$, therefore $\Re\left\{\frac{V(z)}{d_m}\right\} > 0$, or equivalently

$$\Re\left\{\frac{f(z)}{S_m(z)} - (1 - \frac{1}{d_m})\right\} > 0,$$

and this gives the first inequality in(3.4).
For the second inequality (3.5), we consider

$$W(z) = (1 + d_m)\left[\frac{S_m(z)}{f(z)} - \frac{d_m}{1 + d_m}\right],$$

and by using (3.6) we have $\left|\frac{W(z)-1}{W(z)+1}\right| \leq 1$, and hence $\Re\{W(z)\} > 0$, therefore $\Re\left\{\frac{W(z)}{1+d_m}\right\} > 0$, or equivalently

$$\Re\left\{\frac{S_m(z)}{f(z)} - \frac{d_m}{1 + d_m}\right\} > 0,$$

and this shows the second inequality in (3.5). So the proof is complete. □

4. Neighborhoods and Hadamard product

In this section, we start by introducing the $\delta-$neighborhood of a function $f \in \Sigma$, for more detils see [5, 14, 16, 17]. To do this, we assume that $-1 \leq B < A \leq 1$, $-1 \leq B < C \leq 1$, $A = B + (C - B)(1 - \theta)$, $0 \leq \theta < 1$ and $\delta \geq 0$. Define $\delta-$neighborhood of a function $f \in \Sigma$ of the from of (1.1) as:

$$N_\delta(f) = \left\{g(z) : g(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} b_k z^{k-1} \in \Sigma \textit{ and } M \leq \delta\right\},$$

where, for $i = 1, \cdots, t,\ \ j = 1, \cdots, m,\ \ \alpha_i \in \mathbb{C},\ \beta_j \in \mathbb{C}\backslash\{0, -1, -2, \cdots\}$, we have

$$M = \sum_{k=1}^{+\infty} \frac{[k^2(1 + B) + k(|B| + 2(C - B)(1 - \theta))]X_m^t(k)}{2(C - B)(1 - \theta)}|a_k - b_k|,$$

and

$$X_m^t(k) = \frac{(\alpha_1, q)_{k+1} \cdots (\alpha_t, q)_{k+1}}{(q, q)_{k+1}(\beta_1, q)_{k+1} \cdots (\beta_m, q)_{k+1}}.$$

Theorem 4.1. *Let the function $f(z)$ defined by (1.1) be in the class $\Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$. If $f(z)$ satisfies the following condition:*

$$\frac{f(z)+\nu z^{-1}}{1+\nu} \in \Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta) \quad , \quad (\nu\in\mathbb{C}, |\nu|<\delta, \delta>0),$$

then $N_\delta(f) \subset \Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$.

Proof . By using (2.3), we obtain $f \in \Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$ if and only if,

$$\frac{z[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'' + 2[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'}{zB[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'' + 2(B+(C-B)(1-\theta)[\mathcal{L}_m^t[\alpha_i,\beta_j,q]f(z)]'} \neq 1,$$

which is equivalent to

$$\frac{(f*Q)(z)}{z^{-1}} \neq 0 \quad , \quad (z\in U^*), \tag{4.1}$$

where

$$Q(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} e_k z^{k-1} \quad , \quad (z\in U^*),$$

such that

$$e_k = \frac{[k^2(1+B)+k(B+2(C-B)(1-\theta))]X_m^t(k)}{2(C-B)(1-\theta)}. \tag{4.2}$$

It follows from (4.2) that

$$\begin{aligned} |e_k| &= \Big|\frac{[k^2(1+B)+k(B+2(C-B)(1-\theta))]X_m^t(k)}{2(C-B)(1-\theta)}\Big| \\ &\leq \frac{[k^2(1+B)+k(|B|+2(C-B)(1-\theta))]X_m^t(k)}{2(C-B)(1-\theta)}. \end{aligned}$$

Since $\frac{f(z)+\nu z^{-1}}{1+\nu} \in \Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$ by (4.1), we get

$$\frac{\Big(\frac{f(z)+\nu z^{-1}}{1+\nu} * Q\Big)(z)}{z^{-1}} \neq 0. \tag{4.3}$$

Now assume that $\Big|\frac{(f*Q)(z)}{z^{-1}}\Big| < \delta$. Then, by (4.3), we get

$$\Big|\frac{1}{1+\nu}\frac{(f*Q)(z)}{z^{-1}} + \frac{\nu}{1+\nu}\Big| \geq \frac{1}{|1+\nu|}(|\nu|-1)\Big|\frac{(f*Q)(z)}{z^{-1}}\Big| > \frac{|\nu|-\delta}{|1+\nu|} \geq 0.$$

This is a contradiction with $|\nu|<\delta$. Therefore $\Big|\frac{(f*Q)(z)}{z^{-1}}\Big| \geq \delta$. Now, if we suppose that $g(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} b_k z^{k-1} \in N_\delta(f)$ then

$$\begin{aligned} \Big|\frac{(f-g)(z)*Q)(z)}{z^{-1}}\Big| &= \Big|\sum_{k=1}^{+\infty}(a_k-b_k)e_k z^{k-1}\Big| \leq \sum_{k=1}^{+\infty}|a_k-b_k||e_k||z^{k-1}| \\ &\leq |z^{k-1}| \times \sum_{k=1}^{+\infty}\frac{[k^2(1+B)+k(|B|+2(C-B)(1-\theta))]X_m^t(k)}{2(C-B)(1-\theta)}|a_k-b_k| \leq \delta. \end{aligned}$$

Thus, we have

$$\frac{(g*Q)(z)}{z^{-1}} \neq 0 \quad , \quad (z \in U^*),$$

which implies that $g \in \Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$. So $N_\delta(f) \subset \Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$. □

Theorem 4.2. *If $f(z), g(z) \in \Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$ then Hadamard product of f and g defined by*

$$f*g(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} a_k b_k z^{k-1}$$

is in the class $\Sigma_q^{\alpha_t,\beta_m}(A,B,C,\phi)$ such that

$$\phi \leq 1 - \frac{\Big[k^2(1+B) + k(B+2(C-B)(1-\theta))\Big]^2 X_m^t(k)}{4k(C-B)^2(1-\theta)} + \frac{k(1+B)+kB}{2(C-B)}.$$

Proof . Since $f(z), g(z) \in \Sigma_q^{\alpha_t,\beta_m}(A,B,C,\theta)$, so by Theorem (2.1), we have

$$\sum_{k=1}^{+\infty} [k^2(1+B) + k(B+2(C-B)(1-\theta))] X_m^t(k) a_k < 2(C-B)(1-\theta),$$

and

$$\sum_{k=1}^{+\infty} [k^2(1+B) + k(B+2(C-B)(1-\theta))] X_m^t(k) b_k < 2(C-B)(1-\theta).$$

Therefore, we must find the smallest ϕ such that

$$\sum_{k=1}^{+\infty} [k^2(1+B) + k(B+2(C-B)(1-\phi))] X_m^t(k) a_k b_k < 2(C-B)(1-\theta).$$

By using the Cauchy-Schwarts inequality, we have

$$\sum_{k=1}^{+\infty} [k^2(1+B) + k(B+2(C-B)(1-\theta))] X_m^t(k) \sqrt{a_k b_k} < 2(C-B)(1-\theta). \tag{4.4}$$

Now, it is enough to show that

$$\begin{aligned} &[k^2(1+B) + k(B+2(C-B)(1-\phi))] X_m^t(k) a_k b_k \\ &\leq [k^2(1+B) + k(B+2(C-B)(1-\theta))] X_m^t(k) \sqrt{a_k b_k}, \end{aligned}$$

which is equivalent to

$$\sqrt{a_k b_k} \leq \frac{[k^2(1+B) + k(B+2(C-B)(1-\theta))]}{[k^2(1+B) + k(B+2(C-B)(1-\phi))]}. \tag{4.5}$$

But from equation (4.4), we have

$$\sqrt{a_k b_k} \leq \frac{2(C-B)(1-\theta)}{[k^2(1+B) + k(B+2(C-B)(1-\theta))] X_m^t(k)}. \tag{4.6}$$

In view of equations (4.5) and (4.6), this is equivalent to showing that

$$\frac{2(C-B)(1-\theta)}{[k^2(1+B)+k(B+2(C-B)(1-\theta))]X_m^t(k)} \leq \frac{[k^2(1+B)+k(B+2(C-B)(1-\theta))]}{[k^2(1+B)+k(B+2(C-B)(1-\phi))]},$$

which yields the following inequality

$$2(C-B)(1-\theta)[k^2(1+B)+k(B+2(C-B)(1-\phi))] \leq \Big[k^2(1+B)+k(B+2(C-B)(1-\theta))\Big]^2 X_m^t(k).$$

Therefore

$$\phi \leq 1-\frac{\Big[k^2(1+B)+k(B+2(C-B)(1-\theta))\Big]^2 X_m^t(k)}{4k(C-B)^2(1-\theta)}+\frac{k(1+B)+kB}{2(C-B)}.$$

This completes the proof of the theorem. □

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