Int. J. Nonlinear Anal. Appl. 11 (2020) No. 2, 351-361
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.20368.2148

Endpoint results for weakly contractive mappings in $\mathcal{F}$-metric spaces with an application

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(Communicated by Choonkil Park)

Abstract

Many researchers have provided certain interesting results for endpoints of some contractions in metric spaces. In this paper, we introduce α-ζ-contractive multivalued mappings in $\mathcal{F}$-metric spaces and establish some endpoint results in this framework. An illustrative example is given to elaborate the usability of our main result. In the sequel, we give some endpoint theorems for Suzuki-type contractive multivalued mappings and provide an application to integral equations.

Keywords: $\mathcal{F}$-metric spaces, Endpoint, Multivalued mappings, Integral equations.
2010 MSC: 47H10, 54H25.

1. Introduction

Let $(\mathfrak{W}, d)$ be a metric space, $2^{\mathfrak{W}}$ be the set of all nonempty subsets of $\mathfrak{W}$ and $\mathcal{CB}(\mathfrak{W})$ be the set of all nonempty closed bounded subsets of $\mathfrak{W}$. As we know, the Hausdorff metric $\mathcal{H}$ on $\mathcal{CB}(\mathfrak{W})$ is defined by $\mathcal{H}(\Gamma,\Delta) = \max\{\sup_{\gamma\in\Gamma} d(\gamma,\Delta), \sup_{\delta\in\Delta} d(\delta,\Gamma)\}$. An element $\gamma \in \mathfrak{W}$ is said to be a fixed point of the multivalued mapping $\Upsilon : \mathfrak{W} \to 2^{\mathfrak{W}}$, provided that $\gamma \in \Upsilon\gamma$. Also, an element $\gamma \in \mathfrak{W}$ is called an endpoint of Υ provided that $\Upsilon\gamma = \{\gamma\}$. We say that Υ enjoys approximate fixed point property provided that $\inf_{\gamma\in\mathfrak{W}} \sup_{\delta\in\Upsilon\gamma} d(\gamma,\delta) = 0$.

In 2010, Amini-Harandi proved that some multivalued mappings have unique endpoint if and only if they have the approximate endpoint property ([3]). Afterwards, Moradi and Khojasteh

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Received: February 2019 *Revised:* June 2020

[20] obtained a consequence for generalized weak contractive multifunctions. The approach of α-ζ-contractive mappings has been introduced by Samet et al. in 2012 ([25]). Later, some authors used this concept in fixed point theory (see, [5, 8, 10, 16, 23]) or generalized it for some contractive multivalued mappings (see, [2, 9, 13, 19]).

Assume that Ψ is the collection of all nondecreasing functions $\zeta : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \zeta^n(t) < \infty$ for all $t > 0$ ([25]). Evidently, $\zeta(t) < t$ for all $t > 0$ ([25]).

A multivalued mapping $\Upsilon : \mathfrak{W} \to \mathcal{CB}(\mathfrak{W})$ enjoys property $(\mathcal{BS})$, provided that for each $\gamma \in \mathfrak{W}$ there exists $\delta \in \Upsilon\gamma$ such that $\mathcal{H}(\Upsilon\gamma, \Upsilon\delta) = \sup_{b\in\Upsilon\delta} d(\delta, b)$. In fact, there are many multifunctions which possess the property $(\mathcal{BS})$. For instance, let $\mathfrak{W} = [0,\infty)$, $d(\gamma,\delta) = |\gamma - \delta|$, $s, t > 0$, $\Upsilon_1, \Upsilon_2 : \mathfrak{W} \to \mathcal{CB}(\mathfrak{W})$ be defined by $\Upsilon_1\gamma = [0, s\gamma]$ and $\Upsilon_2\gamma = [\gamma, \gamma+t]$. It is easy to check that the multivalued mappings Υ_1 and Υ_2 have the property $(\mathcal{BS})$. Also, we say that the multivalued mapping Υ enjoys property $(\mathcal{SBS})$, provided that for each sequence $\{\rho_n\}$ with $d(\rho_n, \Upsilon\rho_n) \le d(\rho_n, \rho_{n+1}) + \zeta(d(\rho_n, \rho_{n+1}))$ for all n and $\rho_n \to \gamma$, there exists $N \in \mathbb{N}$ such that $d(\rho_n, \Upsilon\rho_n) \le d(\rho_n, \gamma) + \zeta(d(\rho_n, \gamma))$ for all $n \ge N$ (see [18]).

In 2013, Asl et al. [4] introduced α^*-admissible multivalued mappings as follows.

Definition 1.1 ([4]). *Let $\alpha : \mathfrak{W}\times\mathfrak{W} \to [0,\infty)$ be a mapping and $\Upsilon : \mathfrak{W} \to \mathcal{CB}(\mathfrak{W})$ be a multivalued mapping. Then Υ is called α^*-admissible provided that for each $\gamma, \delta \in \mathfrak{W}$, $\alpha(\gamma,\delta) \ge 1$ implies $\alpha^*(\Upsilon\gamma, \Upsilon\delta) \ge 1$, where $\alpha^*(\Upsilon\gamma, \Upsilon\delta) = \inf\{\alpha(a,b) : a \in \Upsilon\gamma, b \in \Upsilon\delta\}$.*

Definition 1.2 ([18]). *Let $\alpha : \mathfrak{W} \times \mathfrak{W} \to [0,\infty)$ be a mapping and $\Upsilon : \mathfrak{W} \to 2^{\mathfrak{W}}$ a multivalued mapping. Υ is α-admissible provided that for each $\gamma \in \mathfrak{W}$ and $\delta \in \Upsilon\gamma$ with $\alpha(\gamma,\delta) \ge 1$, then $\alpha(\delta, z) \ge 1$ for all $z \in \Upsilon\delta$.*

Obviously, any α^*-admissible multivalued mapping is α-admissible, but the converse may not be true. Also, we say that $\mathfrak{W}$ is α-regular, provided that for each sequence $\{\rho_n\}$ in $\mathfrak{W}$ with $\alpha(\rho_n, \rho_{n+1}) \ge 1$ for all n and $\rho_n \to \gamma$, then $\alpha(\rho_n, \gamma) \ge 1$ for all $n \in \mathbb{N}$ (see also, [25]).

In 2015, Mohammadi et al. [18] proved the existence of at least one endpoint for α-ζ-contractions using the property $(\mathcal{BS})$. Recently, Jleli and Samet [12] nominated an inspiring generalization of the notion of metric space in the following manner.

Suppose that $\mathcal{F}$ is the set of all functions $f : (0,+\infty) \to \mathbb{R}$ verifying the following assumptions:

$(\mathcal{F}_1)$ f is non-decreasing;

$(\mathcal{F}_2)$ for all sequences $\{t_n\} \subseteq (0,+\infty)$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} f(t_n) = -\infty$.

Definition 1.3 ([12]). *Let $\mathfrak{W}$ be a nonempty set and $\mathfrak{F} : \mathfrak{W}\times\mathfrak{W} \to [0,+\infty)$ be a mapping. Suppose that there exist $f \in \mathcal{F}$ and $\sigma \in [0,+\infty)$ such that*

(D_1) $\mathfrak{F}(\gamma,\delta) = 0 \iff \gamma = \delta$, for all $(\gamma,\delta) \in \mathfrak{W} \times \mathfrak{W}$;

(D_2) $\mathfrak{F}(\gamma,\delta) = \mathfrak{F}(\delta,\gamma)$, for all $(\gamma,\delta) \in \mathfrak{W} \times \mathfrak{W}$;

(D_3) for any $(\gamma,\delta) \in \mathfrak{W}\times\mathfrak{W}$, for any $N \in \mathbb{N}$, $N \ge 2$ and for any $(\rho_i)_{i=1}^{N} \subset \mathfrak{W}$ with $(\rho_1, \rho_N) = (\gamma,\delta)$, we have

$$\mathfrak{F}(\gamma,\delta) > 0 \Rightarrow f(\mathfrak{F}(\gamma,\delta)) \le f\left(\sum_{i=1}^{N-1} \mathfrak{F}(\rho_i, \rho_{i+1})\right) + \sigma.$$

Then $\mathfrak{F}$ is called an $\mathcal{F}$-metric on $\mathfrak{W}$, and the pair $(\mathfrak{W}, \mathfrak{F})$ is called an $\mathcal{F}$-metric space (shortly, $\mathcal{F}$-MS).

Remark 1.4. *Jleli and Samet [12] showed that any metric space is an $\mathcal{F}$-MS, but the converse is not true in general, which corroborates that this concept is more general than the standard metric concept.*

Definition 1.5 ([12]). *Let $(\mathfrak{W}, \mathfrak{F})$ be an $\mathcal{F}$-MS and $\{\rho_n\}$ be a sequence in $\mathfrak{W}$. Then,*

(i) $\{\rho_n\}$ is $\mathcal{F}$-convergent to $\gamma \in \mathfrak{W}$, if $\lim_{n\to\infty} \mathfrak{F}(\rho_n, \gamma) = 0$;

(ii) $\{\rho_n\}$ is $\mathcal{F}$-Cauchy, if $\lim_{n,m\to\infty} \mathfrak{F}(\rho_n, \rho_m) = 0$;

(iii) $(\mathfrak{W}, \mathfrak{F})$ is $\mathcal{F}$-complete, if any $\mathcal{F}$-Cauchy sequence in $\mathfrak{W}$ is $\mathcal{F}$-convergent to an element in $\mathfrak{W}$.

Theorem 1.6 ([12]). *Let $(\mathfrak{W}, \mathfrak{F})$ be an $\mathcal{F}$-MS and $g : \mathfrak{W} \to \mathfrak{W}$ be a given mapping. Suppose that the following assertions are satisfied:*

(i) $(\mathfrak{W}, \mathfrak{F})$ is $\mathcal{F}$-complete,

(ii) there exists $k \in (0, 1)$ such that

$$\mathfrak{F}(g(\gamma), g(\delta)) \leq k\mathfrak{F}(\gamma, \delta).$$

Then g possesses a unique fixed point $\gamma^* \in \mathfrak{W}$. Moreover, for any $\rho_0 \in \mathfrak{W}$, the sequence $\{\rho_n\} \subset \mathfrak{W}$ defined by

$$\rho_{n+1} = g(\rho_n), n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to γ^*.

Hussain and Kanwal [7] considered the notion of α-ζ-contraction in the setting of $\mathcal{F}$-metric spaces and proved the following fixed point theorem.

Theorem 1.7 ([7]). *Let $(\mathfrak{W}, \mathfrak{F})$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS and $\Upsilon : \mathfrak{W} \to \mathfrak{W}$ be a α-admissible mapping. Suppose that the following assertions are satisfied:*

(i) there exists $\zeta \in \Psi$ such that

$$\alpha(\gamma, \delta)\mathfrak{F}(\Upsilon\gamma, \Upsilon\delta) \leq \zeta(M(\gamma, \delta)),$$

where

$$M(\gamma, \delta) = \max\{\mathfrak{F}(\gamma, \delta), \mathfrak{F}(\gamma, \Upsilon\gamma), \mathfrak{F}(\delta, \Upsilon\delta)\};$$

(ii) there exists $\rho_0 \in \mathfrak{W}$ such that $\alpha(\rho_0, \Upsilon\rho_0) \geq 1$.

Then Υ possesses a unique fixed point $\gamma^* \in \mathfrak{W}$.

In this paper, we obtain some endpoint consequences for multivalued mappings in the framework of $\mathcal{F}$-MS, partially ordered $\mathcal{F}$-metric spaces and graphical $\mathcal{F}$-metric spaces.

2. The Results

Here, we provide our main consequences. We assume that the function f used in the definition of $\mathcal{F}$-MS enjoys further supposition $f(\inf \Gamma) = \inf(f(\Gamma))$ for any bounded subset of $\mathbb{R}^+$.

Let $(\mathfrak{W}, \mathfrak{F})$ be an $\mathcal{F}$-MS. The Hausdorff metric $\mathcal{H}_\mathcal{F}$ on $\mathcal{CB}(\mathfrak{W})$ induced by $\mathcal{F}$-metric $\mathfrak{F}$ is defined by

$$\mathcal{H}_\mathcal{F}(\Gamma, \Delta) = \max\left\{\sup_{\gamma\in\Gamma}\mathfrak{F}(\gamma, \Delta), \sup_{\delta\in\Delta}\mathfrak{F}(\delta, \Gamma)\right\},$$

for all $\Gamma, \Delta \in \mathcal{CB}(\mathfrak{W})$, where $\mathfrak{F}(\gamma, \Delta) = \inf_{\delta\in\Delta} \mathfrak{F}(\gamma, \delta)$.

We will need the following lemma in the sequel.

Lemma 2.1. *Suppose that* $(\mathfrak{W}, \mathfrak{F})$ *is an* $\mathcal{F}$*-MS. Then* $(\mathcal{CB}(\mathfrak{W}), \mathcal{H}_{\mathcal{F}})$ *is also an* $\mathcal{F}$*-MS.*

Proof . The properties (D_1) and (D_2) in Definition 1.3 are obvious. It is sufficient to prove only (D_3). Suppose that $U, V \in \mathcal{CB}(\mathfrak{W})$ and $(\Gamma_i)_{i=1}^n \subset \mathfrak{W}$ with $(\Gamma_1, \Gamma_n) = (U, V)$. Assume that $\mathcal{H}_{\mathcal{F}}(U, V) > 0$. We shall show that

$$f(\mathcal{H}_{\mathcal{F}}(\Gamma_1, \Gamma_n)) = f(\mathcal{H}_{\mathcal{F}}(U, V)) \leq f\left(\sum_{i=1}^{n-1} \mathcal{H}_{\mathcal{F}}(\Gamma_i, \Gamma_{i+1})\right) + \sigma.$$

From the definition of $\mathcal{H}_{\mathcal{F}}$, without loss of generality we can assume that $\mathcal{H}_{\mathcal{F}}(\Gamma_1, \Gamma_n) = \sup_{a_1 \in \Gamma_1} \mathfrak{F}(a_1, \Gamma_n)$. Suppose that $\varepsilon > 0$ is arbitrary. Then there exists $a_1 \in \Gamma_1$ such that

$$\mathcal{H}_{\mathcal{F}}(\Gamma_1, \Gamma_n) \leq \mathfrak{F}(a_1, \Gamma_n) + \varepsilon. \tag{2.1}$$

Suppose that $a_2, ..., a_n$ are arbitrary points in $\Gamma_2, ..., \Gamma_n$, respectively. If $\mathfrak{F}(a_1, \Gamma_n) = 0$, then $\mathcal{H}_{\mathcal{F}}(\Gamma_1, \Gamma_n) \leq \varepsilon$. Otherwise, we have

$$\begin{aligned} f(\mathfrak{F}(a_1, a_n)) &\leq f\left(\sum_{i=1}^{n-1} \mathfrak{F}(a_i, a_{i+1})\right) + \sigma \\ &= f(\mathfrak{F}(a_1, a_2) + ... + \mathfrak{F}(a_{n-2}, a_{n-1}) + \mathfrak{F}(a_{n-1}, a_n)) + \sigma. \end{aligned}$$

Taking inf in both sides of the above inequality as $a_n \in \Gamma_n$, we obtain that

$$\begin{aligned} f(\mathfrak{F}(a_1, \Gamma_n)) &\leq f(\mathfrak{F}(a_1, a_2) + ... + \mathfrak{F}(a_{n-2}, a_{n-1}) + \mathfrak{F}(a_{n-1}, \Gamma_n)) + \sigma \\ &\leq f(\mathfrak{F}(a_1, a_2) + ... + \mathfrak{F}(a_{n-2}, a_{n-1}) + \mathcal{H}_{\mathcal{F}}(\Gamma_{n-1}, \Gamma_n)) + \sigma. \end{aligned}$$

Taking inf in both sides of the above inequality as $a_{n-1} \in \Gamma_{n-1}$, we obtain

$$\begin{aligned} f(\mathfrak{F}(a_1, \Gamma_n)) &\leq f(\mathfrak{F}(a_1, a_2) + ... + \mathfrak{F}(a_{n-2}, \Gamma_{n-1}) + \mathcal{H}_{\mathcal{F}}(\Gamma_{n-1}, \Gamma_n)) + \sigma \\ &\leq f(\mathfrak{F}(a_1, a_2) + ... + \mathcal{H}_{\mathcal{F}}(\Gamma_{n-2}, \Gamma_{n-1}) + \mathcal{H}_{\mathcal{F}}(\Gamma_{n-1}, \Gamma_n)) + \sigma. \end{aligned}$$

Continuing in this manner we get

$$\begin{aligned} f(\mathfrak{F}(a_1, \Gamma_n)) &\leq f(\mathfrak{F}(a_1, \Gamma_2) + ... + \mathcal{H}_{\mathcal{F}}(\Gamma_{n-2}, \Gamma_{n-1}) + \mathcal{H}_{\mathcal{F}}(\Gamma_{n-1}, \Gamma_n)) + \sigma \\ &\leq f(\mathcal{H}_{\mathcal{F}}(\Gamma_1, \Gamma_2) + ... + \mathcal{H}_{\mathcal{F}}(\Gamma_{n-2}, \Gamma_{n-1}) + \mathcal{H}_{\mathcal{F}}(\Gamma_{n-1}, \Gamma_n)) + \sigma \\ &= f\left(\sum_{i=1}^{n-1} \mathcal{H}_{\mathcal{F}}(\Gamma_i, \Gamma_{i+1})\right) + \sigma. \end{aligned}$$

From (2.1),

$$\begin{aligned} \mathcal{H}_{\mathcal{F}}(\Gamma_1, \Gamma_n) \leq \mathfrak{F}(a_1, \Gamma_n) + \varepsilon &= f^{-1}(f(\mathfrak{F}(a_1, \Gamma_n))) + \varepsilon \\ &\leq f^{-1}\left(f\left(\sum_{i=1}^{n-1} \mathcal{H}_{\mathcal{F}}(\Gamma_i, \Gamma_{i+1})\right) + \sigma\right) + \varepsilon. \end{aligned}$$

Since ε was arbitrary, we get that

$$\mathcal{H}_{\mathcal{F}}(\Gamma_1, \Gamma_n) \leq f^{-1}\left(f\left(\sum_{i=1}^{n-1} \mathcal{H}_{\mathcal{F}}(\Gamma_i, \Gamma_{i+1})\right) + \sigma\right).$$

Therefore,

$$f(\mathcal{H}_{\mathcal{F}}(\Gamma_1,\Gamma_n)) \leq f\left(\sum_{i=1}^{n-1}\mathcal{H}_{\mathcal{F}}(\Gamma_i,\Gamma_{i+1})\right)+\sigma,$$

and the proof is completed. □

Definition 2.2. *Suppose that $(\mathfrak{W},\mathfrak{F})$ is an $\mathcal{F}$-MS. A closed-valued multifunction $\Upsilon:\mathfrak{W}\to 2^{\mathfrak{W}}$ is called a generalized α-ζ-contraction, if there exist two functions $\alpha:\mathfrak{W}\times\mathfrak{W}\to[0,+\infty)$ and $\zeta\in\Psi$ such that*

$$\gamma,\delta\in\mathfrak{W},\ \alpha(\gamma,\delta)\geq 1 \Longrightarrow\ \mathcal{H}_{\mathcal{F}}(\Upsilon\gamma,\Upsilon\delta)\leq\zeta(\mathfrak{F}(\gamma,\delta)). \tag{2.2}$$

Theorem 2.3. *Let $(\mathfrak{W},\mathfrak{F})$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS and $\Upsilon:\mathfrak{W}\to\mathcal{CB}(\mathfrak{W})$ be a generalized α-ζ-contraction such that Υ enjoys property $(\mathcal{BS})$. Also, assume that the following assertions hold:*

(i) Υ is α-admissible;

(ii) $\alpha(\rho_0,\rho_1)\geq 1$ for an $\rho_0\in\mathfrak{W}$ and $\rho_1\in\Upsilon\rho_0$;

(iii) $\mathfrak{W}$ is α-regular.

Then Υ possesses an endpoint.

Proof . Choose $\rho_0\in\mathfrak{W}$ and $\rho_1\in\Upsilon\rho_0$ such that $\alpha(\rho_0,\rho_1)\geq 1$. Since Υ enjoys property $(\mathcal{BS})$, there exists $\rho_2\in\Upsilon\rho_1$ such that $\mathcal{H}_{\mathcal{F}}(\Upsilon\rho_1,\Upsilon\rho_2)=\sup_{b\in\Upsilon\rho_2}\mathfrak{F}(\rho_2,b)$. Since Υ is α-admissible, $\alpha(\rho_1,\rho_2)\geq 1$. Pursuing this process, we obtain a sequence $\{\rho_n\}$ such that $\rho_{n+1}\in\Upsilon\rho_n$, $\alpha(\rho_n,\rho_{n+1})\geq 1$ and $\mathcal{H}_{\mathcal{F}}(\Upsilon\rho_n,\Upsilon\rho_{n+1})=\sup_{b\in\Upsilon\rho_{n+1}}\mathfrak{F}(\rho_{n+1},b)$ for all n. If $\rho_n=\rho_{n+1}$ for some $n\in\mathbb{N}$, then we obtain that $\mathcal{H}_{\mathcal{F}}(\{\rho_{n+1}\},\Upsilon\rho_{n+1})=\sup_{b\in\Upsilon\rho_{n+1}}\mathfrak{F}(\rho_{n+1},b)=\mathcal{H}_{\mathcal{F}}(\Upsilon\rho_n,\Upsilon\rho_{n+1})=0$. This implies that ρ_{n+1} is an endpoint of Υ. Thus, we may assume that $\rho_n\neq\rho_{n+1}$ for all $n\in\mathbb{N}$.
From (2.2), we have

$$\begin{aligned}\mathfrak{F}(\rho_n,\rho_{n+1}) &\leq \sup_{b\in\Upsilon\rho_n}\mathfrak{F}(\rho_n,b)\\ &= \mathcal{H}_{\mathcal{F}}(\Upsilon\rho_{n-1},\Upsilon\rho_n)\\ &\leq \zeta(\mathfrak{F}(\rho_{n-1},\rho_n))\leq\zeta^2(\mathfrak{F}(\rho_{n-2},\rho_{n-1}))\leq\cdots\leq\zeta^n(\mathfrak{F}(\rho_0,\rho_1)),\end{aligned}$$

for all $n\in\mathbb{N}$. Suppose that $(f,\alpha)\in\mathcal{F}\times[0,+\infty)$ such that (D_3) is satisfied, and fix $\epsilon>0$. By $(\mathcal{F}_2)$, there exists $\delta>0$ such that

$$0<t<\delta\Longrightarrow f(t)<f(\epsilon)-\sigma. \tag{2.3}$$

Consider $N\in\mathbb{N}$ such that $0<\sum_{n\geq N}\zeta^n(\mathfrak{F}(\rho_0,\rho_1))<\delta$. Hence, by (2.3) and $(\mathcal{F}_1)$, we have

$$f\left(\sum_{i=n}^{m-1}\mathfrak{F}(\rho_i,\rho_{i+1})\right)\leq f\left(\sum_{i=n}^{m-1}\zeta^i(\mathfrak{F}(\rho_0,\rho_1))\right)\leq f\left(\sum_{n\geq N}\zeta^n(\mathfrak{F}(\rho_0,\rho_1))\right)<f(\epsilon)-\sigma, \tag{2.4}$$

for $m>n\geq N$. Using (D_3) and (2.4), we obtain that $\mathfrak{F}(\rho_n,\rho_m)>0$ where $m>n\geq N$ which implies that

$$f(\mathfrak{F}(\rho_n,\rho_m))\leq f\left(\sum_{i=n}^{m-1}\mathfrak{F}(\rho_i,\rho_{i+1})\right)+\sigma<f(\epsilon),$$

which implies by $(\mathcal{F}_1)$ that $\mathfrak{F}(\rho_n, \rho_m) < \epsilon$, for all $m > n \geq N$. This proves that $\{\rho_n\}$ is $\mathcal{F}$-Cauchy. Because of $\mathcal{F}$-completeness of $\mathfrak{W}$, there exists $\gamma^\star \in \mathfrak{W}$ such that $\rho_n \to \gamma^\star$. We shall show that γ^* is an endpoint of Υ. Suppose to the contrary that $\Upsilon\gamma^* \neq \{\gamma^*\}$. Then $\mathcal{H}_{\mathcal{F}}(\{\gamma^*\}, \Upsilon\gamma^*) > 0$. Since $\mathfrak{W}$ is α-regular, $\alpha(\rho_n, \gamma^*) \geq 1$ for all $n \in \mathbb{N}$. Then, by (2.2) and $(\mathcal{F}_1)$

$$\begin{aligned} f(\mathcal{H}_{\mathcal{F}}(\{\rho_n\}, \Upsilon\rho_n)) &= f(\mathcal{H}_{\mathcal{F}}(\Upsilon\rho_{n-1}, \Upsilon\rho_n)) \\ &\leq f(\mathcal{H}_{\mathcal{F}}(\Upsilon\rho_{n-1}, \Upsilon\gamma^*) + \mathcal{H}_{\mathcal{F}}(\Upsilon\rho_n, \Upsilon\gamma^*)) + \sigma \\ &\leq f(\zeta(\mathfrak{F}(\rho_{n-1}, \gamma^*)) + \zeta(\mathfrak{F}(\rho_n, \gamma^*))) + \sigma \\ &< f(\mathfrak{F}(\rho_{n-1}, \gamma^*) + \mathfrak{F}(\rho_n, \gamma^*)) + \sigma \to -\infty, \end{aligned}$$

as $n \to \infty$. Thus, $\lim_{n\to\infty} \mathcal{H}_{\mathcal{F}}(\{\rho_n\}, \Upsilon\rho_n) = 0$. On the other hand,

$$\begin{aligned} f(\mathcal{H}_{\mathcal{F}}(\{\gamma^*\}, \Upsilon\gamma^*)) &\leq f(\mathcal{H}_{\mathcal{F}}(\{\gamma^*\}, \{\rho_n\}) + \mathcal{H}_{\mathcal{F}}(\{\rho_n\}, \Upsilon\rho_n) + \mathcal{H}_{\mathcal{F}}(\Upsilon\rho_n, \Upsilon\gamma^*)) + \sigma \\ &\leq f(d(\gamma^*, \rho_n) + \mathcal{H}_{\mathcal{F}}(\{\rho_n\}, \Upsilon\rho_n) + \mathcal{H}_{\mathcal{F}}(\Upsilon\rho_n, \Upsilon\gamma^*)) + \sigma \\ &\leq f(d(\gamma^*, \rho_n) + \mathcal{H}_{\mathcal{F}}(\{\rho_n\}, \Upsilon\rho_n) + \zeta(\mathfrak{F}(\rho_n, \gamma^*))) + \sigma \\ &< f(d(\gamma^*, \rho_n) + \mathcal{H}_{\mathcal{F}}(\{\rho_n\}, \Upsilon\rho_n) + \mathfrak{F}(\rho_n, \gamma^*)) + \sigma \to -\infty, \end{aligned}$$

as $n \to \infty$, which is a contradiction. Thus, $\{\gamma^*\} = \Upsilon\gamma^*$. □

Example 2.4. *Consider the set* $\mathfrak{W} = \{\lambda, \mu, \nu\}$. *Let* $\mathfrak{F} : \mathfrak{W} \times \mathfrak{W} \to [0, +\infty)$ *be given by*

$$\mathfrak{F}(\lambda, \mu) = \frac{\lambda}{\mu},$$

$$\mathfrak{F}(\mu, \nu) = \frac{\mu}{\nu},$$

$$\mathfrak{F}(\lambda, \nu) = \frac{4}{\nu},$$

$\mathfrak{F}(\gamma, \gamma) = 0$ *and* $\mathfrak{F}(\gamma, \delta) = \mathfrak{F}(\delta, \gamma)$ *for all* $\gamma, \delta \in \mathfrak{W}$.
Since $\mathfrak{F}(\lambda, \nu) = \frac{4}{\nu} > \frac{7}{6} = \frac{\lambda}{\mu} + \frac{\mu}{\nu} = \mathfrak{F}(\lambda, \mu) + \mathfrak{F}(\mu, \nu)$, $\mathfrak{F}$ *is not a metric. To prove that* $\mathfrak{F}$ *is an* $\mathcal{F}$*-metric, take* $f(t) = \ln(\sqrt{t})$ *and* $\sigma = \ln\sqrt{\frac{8}{7}}$. *Then,*

$$f(\mathfrak{F}(\lambda, \mu)) \leq f(\mathfrak{F}(\lambda, \nu) + \mathfrak{F}(\mu, \nu)) \leq f(\mathfrak{F}(\lambda, \nu) + \mathfrak{F}(\mu, \nu)) + \sigma,$$

$$f(\mathfrak{F}(\mu, \nu)) \leq f(\mathfrak{F}(\lambda, \mu) + \mathfrak{F}(\lambda, \nu)) \leq f(\mathfrak{F}(\lambda, \mu) + \mathfrak{F}(\lambda, \nu)) + \sigma,$$

$$f(\mathfrak{F}(\lambda, \nu)) = \ln\sqrt{\frac{8}{6}} = \ln\sqrt{\frac{7}{6}} + \ln\sqrt{\frac{8}{7}} \leq f(\mathfrak{F}(\lambda, \mu) + \mathfrak{F}(\mu, \nu)) + \sigma.$$

Then, $(\mathfrak{W}, \mathfrak{F})$ *is an* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space on* $\mathfrak{W}$. *Define* $\Upsilon : \mathfrak{W} \to \mathcal{CB}(\mathfrak{W})$ *by* $\Upsilon(\lambda) = \Upsilon(\mu) = \{\lambda\}$ *and* $\Upsilon(\nu) = \{\lambda, \mu\}$. *Taking* $\zeta(t) = \frac{\nu}{4}t$, *we have*

$$\mathcal{H}_{\mathcal{F}}(\Upsilon(\lambda), \Upsilon(\mu)) = 0,$$

$$\mathcal{H}_{\mathcal{F}}(\Upsilon(\lambda), \Upsilon(\nu)) = \mathfrak{F}(\lambda, \mu) = \frac{\lambda}{\mu} \leq \frac{\nu}{4}\frac{4}{\nu} = k\mathfrak{F}(\lambda, \nu),$$

$$\mathcal{H}_{\mathcal{F}}(\Upsilon(\mu), \Upsilon(\nu)) = \mathfrak{F}(\lambda, \mu) = \frac{\lambda}{\mu} \leq \frac{\nu}{4}\frac{\mu}{\nu} = k\mathfrak{F}(\mu, \nu).$$

Therefore, $\mathcal{H}_{\mathcal{F}}(\Upsilon(\gamma), \Upsilon(\delta)) \leq \zeta(\mathfrak{F}(\gamma, \delta))$, *for all* $\gamma, \delta \in \mathfrak{W}$. *Taking* $\alpha(\gamma, \delta) = \lambda$ *for all* $\gamma, \delta \in \mathfrak{W}$, Υ *satisfies all of the assertions of Theorem 2.3 and so* Υ *possesses an endpoint. Here,* $\Upsilon\lambda = \{\lambda\}$.

3. The consequences in graphical $\mathcal{F}$-metric spaces

Jachymski [11] has obtained an extension of BCP in a graphical metric space. Later, Dinevari and Frigon [6] extended his consequences to multivalued mappings.

In this section, we give the existence of endpoints on an $\mathcal{F}$-MS endowed with a graph ($\mathcal{F}$-GMS). The following notions and definitions are indispensable.

Suppose that $(\mathfrak{W},\mathfrak{F})$ is an $\mathcal{F}$-MS. A set $\{(\gamma,\gamma) : \gamma \in \mathfrak{W}\}$ is called a *diagonal* of $\mathfrak{W}\times\mathfrak{W}$, and denoted by Γ. Consider a graph G such that the set $\mathcal{V}(\mathcal{G}) = \mathfrak{W}$ which is the set of its vertices and the set $\mathcal{E}(\mathcal{G})$ of its edges contains all loops, i.e., $\Gamma \subseteq \mathcal{E}(\mathcal{G})$.

Definition 3.1 ([15]). *Let $\mathfrak{W}$ be a nonempty set endowed with a graph G and $\Upsilon : \mathfrak{W} \to 2^{\mathfrak{W}}$ be a multivalued mapping. The mapping Υ is called* preserves edges weakly *if, for each $\gamma \in \mathfrak{W}$ and $\delta \in \Upsilon\gamma$ with $(\gamma,\delta) \in \mathcal{E}(\mathcal{G})$, we have $(\delta, z) \in \mathcal{E}(\mathcal{G})$ for all $z \in \Upsilon\delta$.*

Motivated by [15], we present the following definitions.

Definition 3.2. *Let $(\mathfrak{W},\mathfrak{F})$ be an $\mathcal{F}$-GMS and $\Upsilon : \mathfrak{W} \to \mathcal{CB}(\mathfrak{W})$ be a given multivalued mapping. Then,*

(i) $\mathfrak{W}$ is called $\mathcal{E}(\mathcal{G})$-complete, if any Cauchy sequence $\{\rho_n\}$ in $\mathfrak{W}$ with $(\rho_n,\rho_{n+1}) \in \mathcal{E}(\mathcal{G})$ for all $n \in \mathbb{N}$ converges in $\mathfrak{W}$;

(ii) Υ is called a generalized $(\mathcal{G},\zeta)$-contraction, if there exists a function $\zeta \in \Psi$ such that

$$\gamma,\delta \in \mathfrak{W},\ (\gamma,\delta) \in \mathcal{E}(\mathcal{G}) \Longrightarrow\ \mathcal{H}_{\mathcal{F}}(\Upsilon\gamma,\Upsilon\delta) \le \zeta(\mathfrak{F}(\gamma,\delta)). \tag{3.1}$$

Theorem 3.3. *Let $(\mathfrak{W},\mathfrak{F})$ be an $\mathcal{E}(\mathcal{G})$-complete $\mathcal{F}$-GMS and $\Upsilon : \mathfrak{W} \to \mathcal{CB}(\mathfrak{W})$ be a generalized $(\mathcal{G},\zeta)$-contraction. Suppose that the following conditions hold:*

(S_1) Υ preserves edges weakly;

(S_2) there exist $\rho_0 \in \mathfrak{W}$ and $\rho_1 \in \Upsilon\rho_0$ such that $(\rho_0,\rho_1) \in \mathcal{E}(\mathcal{G})$;

(S_3) if $\{\rho_n\}$ is a sequence in $\mathfrak{W}$ with $\rho_n \to \gamma \in \mathfrak{W}$ as $n \to \infty$ and $(\rho_n,\rho_{n+1}) \in \mathcal{E}(\mathcal{G})$ for all $n \in \mathbb{N}$, then $(\rho_n,\gamma) \in \mathcal{E}(\mathcal{G})$ for all $n \in \mathbb{N}$.

Then Υ possesses an endpoint point in $\mathfrak{W}$.

Proof . This consequence can be obtained from Theorem 2.3, if we define a mapping $\alpha : \mathfrak{W}\times\mathfrak{W} \to [0,+\infty)$ by $\alpha(\gamma,\delta) = 1$ if $(\gamma,\delta) \in \mathcal{E}(\mathcal{G})$, and $\alpha(\gamma,\delta) = 0$ otherwise. □

4. The consequences in ordered $\mathcal{F}$-metric spaces

Fixed point theorems in ordered metric spaces have wide applications in differential and integral equations and other branches in mathematical analysis(see [1, 21, 22]). From Theorem 2.3, we derive the following new consequences in the setting of $\mathcal{F}$-metric spaces endowed with an ordered ($\mathcal{F}$-OMS), i.e., spaces of the type $(\mathfrak{W},\mathfrak{F},\preceq)$ where $(\mathfrak{W},\mathfrak{F})$ is an $\mathcal{F}$-MS and $\preceq$ is a partial order on $\mathfrak{W}$. Recall that $\Upsilon : \mathfrak{W} \to \mathfrak{W}$ is nondecreasing, if $\forall\gamma,\delta \in \mathfrak{W},\ \gamma \preceq \delta \Rightarrow \Upsilon(\gamma) \preceq \Upsilon(\delta)$.

Motivated by [14], we introduce the following concepts in an $\mathcal{F}$-OMS.

Definition 4.1. *Let $\mathfrak{W}$ be an ordered nonempty set and $\Upsilon : \mathfrak{W} \to 2^{\mathfrak{W}}$ be a given multivalued mapping. The mapping Υ is called* weakly increasing *if, for each $\gamma \in \mathfrak{W}$ and $\delta \in \Upsilon\gamma$ with $\gamma \preceq \delta$, one obtains that $\delta \preceq z$ for all $z \in \Upsilon\delta$.*

Definition 4.2. *Let $(\mathfrak{W}, \mathfrak{F}, \preceq)$ be an $\mathcal{F}$-OMS and $\Upsilon : \mathfrak{W} \to \mathcal{CB}(\mathfrak{W})$ be a given multivalued mapping.*

- *(i) $\mathfrak{W}$ is called $\preceq$-*complete*, if any Cauchy sequence $\{\rho_n\}$ in $\mathfrak{W}$ with $\rho_n \preceq \rho_{n+1}$ for all $n \in \mathbb{N}$ converges in $\mathfrak{W}$;*
- *(ii) Υ is called an ordered generalized ζ-contraction, if there exists a function $\zeta \in \Psi$ such that*

$$\gamma, \delta \in \mathfrak{W},\ \gamma \preceq \delta \Longrightarrow\ \mathcal{H}_{\mathcal{F}}(\Upsilon\gamma, \Upsilon\delta) \leq \zeta((\mathfrak{F}(\gamma, \delta)). \tag{4.1}$$

Theorem 4.3. *Let $(\mathfrak{W}, \mathfrak{F}, \preceq)$ be a $\preceq$-complete $\mathcal{F}$-OMS and $\Upsilon : \mathfrak{W} \to \mathcal{CB}(\mathfrak{W})$ be an ordered generalized ζ-contraction. Suppose that the following assertions hold:*

- *(S_1) Υ is weakly increasing;*
- *(S_2) there exist $\rho_0 \in \mathfrak{W}$ and $\rho_1 \in \Upsilon\rho_0$ such that $\rho_0 \preceq \rho_1$;*
- *(S_3) if $\{\rho_n\}$ is a sequence in $\mathfrak{W}$ with $\rho_n \to \gamma \in \mathfrak{W}$ as $n \to \infty$ and $\rho_n \preceq \rho_{n+1}$ for all $n \in \mathbb{N}$, then $\rho_n \preceq \gamma$ for all $n \in \mathbb{N}$.*

Then Υ possesses an endpoint point in $\mathfrak{W}$.

Proof . This consequence can be obtained from Theorem 2.3, if we define a mapping $\alpha : \mathfrak{W} \times \mathfrak{W} \to [0, +\infty)$ by $\alpha(\gamma, \delta) = 1$ if $\gamma \preceq \delta$, and $\alpha(\gamma, \delta) = 0$ otherwise. □

5. Suzuki type endpoint consequences in $\mathcal{F}$-MS

Theorem 5.1. *Let $(\mathfrak{W}, \mathfrak{F})$ be a complete $\mathcal{F}$-MS, $\zeta \in \Psi$ and $\Upsilon : \mathfrak{W} \to \mathcal{CB}(\mathfrak{W})$ a multivalued mapping such that $\mathfrak{F}(\gamma, \Upsilon\gamma) \leq \mathfrak{F}(\gamma, \delta) + \zeta(\mathfrak{F}(\gamma, \delta))$ implies that $\mathcal{H}(\Upsilon\gamma, \Upsilon\delta) \leq \zeta(\mathfrak{F}(\gamma, \delta))$ for all $\gamma, \delta \in \mathfrak{W}$ and Υ enjoys ($\mathcal{BS}$) property. If Υ satisfies the condition ($\mathcal{SBS}$), then Υ possesses an endpoint in $\mathfrak{W}$.*

Proof . Define $\alpha : \mathfrak{W} \times \mathfrak{W} \to [0, +\infty)$ by

$$\alpha(\gamma, \delta) = \begin{cases} 1, & \mathfrak{F}(\gamma, \Upsilon\gamma) \leq \mathfrak{F}(\gamma, \delta) + \zeta(\mathfrak{F}(\gamma, \delta)), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that Υ is α-admissible. Also, for any $\rho_0 \in \mathfrak{W}$ and $\rho_1 \in \Upsilon\rho_0$, we have $\mathfrak{F}(\rho_0, \Upsilon\rho_0) \leq \mathfrak{F}(\rho_0, \rho_1) \leq \mathfrak{F}(\rho_0, \rho_1) + \zeta(\mathfrak{F}(\rho_0, \rho_1))$. Hence, $\alpha(\rho_0, \rho_1) = 1$. Also, it is easy to check that $\mathcal{H}(\Upsilon\gamma, \Upsilon\delta) \leq \zeta(\mathfrak{F}(\gamma, \delta))$ for all $\gamma, \delta \in \mathfrak{W}$. Note that the property ($\mathcal{SBS}$) leads to α-regularity of $\mathfrak{W}$. Therefore, by Theorem 2.3, Υ possesses an endpoint. □

Corollary 5.2. *Let $(\mathfrak{W}, \mathfrak{F})$ be a complete $\mathcal{F}$-MS, $r \in [0, 1)$ and $\Upsilon : \mathfrak{W} \to \mathcal{CB}(\mathfrak{W})$ a multivalued mapping such that $\frac{1}{1+r}\mathfrak{F}(\gamma, \Upsilon\gamma) \leq \mathfrak{F}(\gamma, \delta)$ implies that $\mathcal{H}(\Upsilon\gamma, \Upsilon\delta) \leq r\mathfrak{F}(\gamma, \delta)$ for all $\gamma, \delta \in \mathfrak{W}$ and Υ enjoys property ($\mathcal{BS}$). If Υ satisfies the condition ($\mathcal{SBS}$), then Υ possesses an endpoint in $\mathfrak{W}$.*

6. Application to nonlinear integral equations

Denote $\mathcal{CB}(\mathbb{R})$ the collection of all nonempty closed and bounded subsets of $\mathbb{R}$. Let $\mathcal{X} := \mathcal{C}(\mathcal{I}, \mathbb{R})$ be the space of all real-valued continuous functions on $\mathcal{I} = [0, 1]$. Evidently, $\mathcal{X}$ endowed with the $\mathfrak{F}$-metric $\mathfrak{F} : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ defined by

$$\mathfrak{F}(\gamma, \delta) = \begin{cases} e^{||\gamma - \delta||}, & \text{if } \gamma \neq \delta, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$||\gamma - \delta|| = \sup_{t\in\mathcal{I}}[\,|\gamma(t) - \delta(t)|\,],$$

is a complete $\mathfrak{F}$-metric space(see [12]).

In this setting, motivated by what have been done in [24], we consider the problem of solving the integral equation

$$\{\gamma(t)\} = \int_0^t K(t, s, \gamma(s))ds + g(t), \quad t \in \mathcal{I}, \tag{6.1}$$

where $\gamma \in \mathcal{X}$, $K : \mathcal{I} \times \mathcal{I} \times \mathbb{R} \to \mathcal{CB}(\mathbb{R})$ is a set-valued operator and $g : \mathcal{I} \to \mathbb{R}$ is a continuous function.

By an application of Theorem 2.3, we establish the existence of a solution of (6.1) as follows.

Theorem 6.1. *By the above mentioned notations, assume that the following assertions hold:*

(i) *for each $\gamma \in \mathcal{X}$, the set-valued operator $K : \mathcal{I} \times \mathcal{I} \times \mathbb{R} \to \mathcal{CB}(\mathbb{R})$ is such that $K(t, s, \gamma(s))$ is continuous in $\mathcal{I} \times \mathcal{I}$,*

(ii) *there exists a continuous function $L : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ with the property $\inf_{t\in\mathcal{I}} \int_0^1 L(t, s)ds = \tau > 0$ such that for any $\gamma, \delta \in X$ and each $k_\gamma(t, s) \in K_\gamma(t, s) = K(t, s, \gamma(s))$, there exists $k_\delta(t, s) \in K_\delta(t, s)$ such that*

$$|k_\gamma(t, s) - k_\delta(t, s)| \le |\gamma(s) - \delta(s)| - L(t, s), \quad \textit{for all } t, s \in \mathcal{I}. \tag{6.2}$$

Then, the integral equation (6.1) possesses at least one solution in $\mathcal{X}$.

Proof . Suppose that $\Upsilon : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ is the set-valued operator specified by

$$\Upsilon\gamma = \{v \in \mathcal{X} : v(t) \in \int_0^1 K(t, s, \gamma(s))ds + g(t), t \in \mathcal{I}\},$$

for each $\gamma \in \mathcal{X}$. Evidently, each endpoint of Υ is a solution of (6.1).
Next, consider the set-valued operator $K_\gamma : \mathcal{I} \times \mathcal{I} \to \mathcal{CB}(\mathbb{R})$, defined by $K_\gamma(t, s) = K(t, s, \gamma(s))$. By Michael's selection theorem, we get that there exists a continuous operator $k_\gamma : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ such that $k_\gamma(t, s) \in K_\gamma(t, s) = K(t, s, \gamma(s))$, for all $t, s \in \mathcal{I}$. This implies that $\int_0^t k_\gamma(t, s)ds + g(t) \in \Upsilon\gamma$ and so $\Upsilon\gamma$ is a nonempty set.
Next, we show that the set-valued operator Υ satisfies all the assertions of Theorem 2.3.
Suppose that $\gamma, \delta \in \mathcal{X}$ and $v(t) \in \Upsilon\gamma$. Then there exists $k_\gamma(t, s) \in K_\gamma(t, s)$ with $t, s \in \mathcal{I}$ such that $v(t) = \int_0^1 k_\gamma(t, s)ds + g(t)$, $t \in \mathcal{I}$. On the other hand, by hypothesis (ii), there exists $k_\delta(t, s) \in K_\delta(t, s)$ such that (6.2) holds. Now taking $z(t) = \int_0^1 k_\delta(t, s)ds + g(t)$, we get

$$z(t) \in \int_0^1 K(t, s, \delta(s))ds + g(t) = \Upsilon\delta, \quad t \in \mathcal{I}.$$

Then,

$$
\begin{aligned}
\mathfrak{F}(v,z) &= e^{||v-z||} \\
&\le e^{\sup_{t\in\mathcal{I}} |\int_0^t k_\gamma(t,s)ds - \int_0^1 k_\delta(t,s)ds|} \\
&\le e^{\sup_{t\in\mathcal{I}} \int_0^t |k_\gamma(t,s)-k_\delta(t,s)|ds} \\
&\le e^{\sup_{t\in\mathcal{I}} \int_0^t (|\gamma(s)-\delta(s)|-L(t,s))ds} \\
&= e^{\sup_{t\in\mathcal{I}} \int_0^t |\gamma(s)-\delta(s)|ds - \int_0^t L(t,s)ds} \\
&\le e^{||\gamma(s)-\delta(s)|| - \inf_{t\in\mathcal{I}} \int_0^t L(t,s)ds} \\
&\le e^{||\gamma(s)-\delta(s)||-\tau} \\
&= e^{||\gamma(s)-\delta(s)||} \cdot e^{-\tau} \\
&= \zeta(\mathfrak{F}(\gamma,\delta)),
\end{aligned}
$$

where $\zeta(t) = e^{-\tau}t$. Interchanging the roles of γ and δ, we obtain that $\mathcal{H}_{\mathcal{F}}(\Upsilon\gamma,\Upsilon\delta) \le \zeta(\mathfrak{F}(\gamma,\delta))$ for all $\gamma,\delta \in \mathcal{X}$. Taking $\alpha(\gamma,\delta) = 1$ for all $\gamma,\delta \in \mathcal{X}$, all of the assertions of Theorem 2.3 are satisfied and accordingly Υ possesses an endpoint, which is a solution of integral equation (6.1). □

Acknowledgement. This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-23-130-38). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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