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On some fixed point results for (α, β)-Berinde-φ-Contraction mapppings with applications

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Abstract

The aim of this paper is to introduce a new class of mappings called (α, β)-Berinde-φ-contraction mappings and to establish some fixed point results for this class of mappings in the frame work of metric spaces. Furthermore, we applied our results to the existence of solution of second order differential equations and the existence of a solution for the following nonlinear integral equation:

$$x(t) = g(t) + \int_a^b M(t,s)K(t,x(s))ds,$$

where $M : [a,b] \times [a,b] \to \mathbb{R}^+$, $K : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous functions. Our results improve, extend and generalize some known results in the literature. In particular, our main result is a generalization of the fixed point result of R. Pant, Fixed point theorem for nonlinear contractions with application to iterated function system, Appl. Gen. Topol. 19 (1) (2018), 163-172.

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1. Introduction and Premilinaries

For the past three to four decades, the theory of fixed point has played an important role in nonlinear functional analysis and known to be very useful in establishing the existence and uniqueness theorems for nonlinear differential and integral equations. In fact, the theory of fixed point has been applied to solve many real life problems, such as, equilibrium problems, variational inequalities, and optimization problems. Banach [9] in 1922 proved the well celebrated Banach contraction principle in the frame work of metric spaces. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. Due to its importance and useful applications, several authors have generalized this result by considering different classes of nonlinear mappings and spaces which are more general than contraction mappings and metric spaces, (see [1, 23, 25, 26, 27, 28, 29, 32, 34] and the references therein). Also, over the years, several authors have developed several iterative schemes for solving fixed point problem for different operators and in Hilbert, Banach, p-uniformly convex metric and Hadamard spaces, see for example, [2, 3, 4, 5, 6, 7, 8, 15, 16, 17, 18, 19, 20, 21, 22, 24, 37, 38, 39]. For example, Boyd and Wong [13] introduced a class of mappings called the φ-contraction mapping in the frame work of metric spaces and obtained some fixed point results for this class of mappings.

Definition 1.1. *[13] Let (X, d) be a metric space and $\varphi : [0, \infty) \to [0, \infty)$ a function such that $\varphi(t) < t$ for $t > 0$. A selfmap $T : X \to X$ is called φ-contraction if*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \tag{1.1}$$

for all $x, y \in X$.

Theorem 1.2. *[13] Let (X, d) be a complete metric space and $T : X \to X$ a φ-contraction such that φ is upper semicontinuous from the right on $[0, \infty)$ and satisfies $\varphi(t) < t$ for all $t > 0$. Then T has a unique fixed point.*

Remark 1.3. *It is clear that if $\varphi(x) \leq \delta x$ for $\delta \in [0, 1)$, then we obtain the Banach contraction mapping.*

More so, Berinde [11, 12] introduced and studied the following class of contraction mappings:

Definition 1.4. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a generalized almost contraction if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \tag{1.2}$$

for all $x, y \in X$.

Furthermore, in 2008, Suzuki [35] introduced a class of mappings satisfying condition (C) which is also known as Suzuki-type generalized nonexpansive mapping and proved some fixed point theorems for this class of mappings.

Definition 1.5. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to satisfy condition (C) if for all $x, y \in X$,*

$$\frac{1}{2} d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

Theorem 1.6. *Let (X, d) be a compact metric space and $T : X \to X$ be a mapping satisfying condition (C). Then T has a unique fixed point.*

Samet et al. [33] introduced another class of mappings, called the α-admissible mappings and obtain some fixed point results for this class of mappings.

Definition 1.7. *[33] Let $\alpha : X \times X \to [0, \infty)$ be a function. We say that a self mapping $T : X \to X$ is α-admissible if for all $x, y \in X$,*

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 1.8. *[23] Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is a triangular α-admissible if*

1. *T is α-admissible and*
2. *$\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$ for all $x, y, z \in X$.*

Theorem 1.9. *[33] Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Suppose that the following conditions hold:*

1. *for all $x, y \in X$, we have $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$, where $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
3. *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$.*

Then T has a fixed point.

In 2016, Chandok et al. [14] introduced another class of mappings, called the TAC-contractive and established some fixed point results in the frame work of complete metric spaces.

Definition 1.10. *Let $T : X \to X$ be a mapping and let $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. Then T is called a cyclic (α, β)-admissible mapping, if*

1. *$\alpha(x) \geq 1$ for some $x \in X$ implies that $\beta(Tx) \geq 1$,*
2. *$\beta(x) \geq 1$ for some $x \in X$ implies that $\alpha(Tx) \geq 1$.*

Definition 1.11. *Let (X, d) be a metric space and let $\alpha, \beta : X \to [0, \infty)$ be two mappings. We say that T is a TAC-contractive mapping, if for all $x, y \in X$,*

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))),$$

where ψ is a continuous and nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$, ϕ is continuous with $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$ and $f : [0, \infty)^2 \to \mathbb{R}$ is continuous, $f(a, t) \leq a$ and $f(a, t) = a \Rightarrow a = 0$ or $t = 0$ for all $s, t \in [0, \infty)$.

Theorem 1.12. *Let (X, d) be a complete metric space and let $T : X \to X$ be a cyclic (α, β)-admissible mapping. Suppose that T is a TAC-contraction mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1, \beta(x_0) \geq 1$ and either of the following conditions hold:*

1. *T is continuous,*

2. *if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.*

In addition, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in F(T)$ (where $F(T)$ denotes the set of fixed points of T), then T has a unique fixed point.

Very recently, Pant [31] introduce the notion of Suzuki type generalized φ-contractive mapping and also established some fixed point results for this class of mappings.

Definition 1.13. *[31] Let (X, d) be a metric space. A selfmap $T : X \to X$ is called a Suzuki type generalized φ-contractive if for all $x, y \in X$,*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \varphi(m(x, y)), \tag{1.3}$$

where $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ and $\varphi : [0, \infty) \to [0, \infty)$ is a function such that $\varphi(t) < t$ and $\limsup_{s \to t^+} \varphi(s) < t$ for all $t > 0$.

Theorem 1.14. *[31] Let (X, d) be a complete metric space and $T : X \to X$ a Suzuki type generalized φ-contractive mapping. Then T has a unique fixed point in X.*

Lemma 1.15. *[10] Suppose that (X, d) is a metric space and $\{x_n\}$ is a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_{k-1}}) < \epsilon$ and*

1. $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$,
2. $\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = \epsilon$,
3. $\lim_{k\to\infty} d(x_{m_{k-1}}, x_{n_k}) = \epsilon$,
4. $\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = \epsilon$.

Inspired by the works of Boyd and Wong [13], Pant [31], Samet et al. [33] and Chandok et al. [14], we introduced a new class of (α, β)-cyclic admissible mappings and a new class of (α, β)-Berinde-φ-contraction mappings and establish the existence and uniqueness theorems for fixed points of the class of (α, β)-Berinde-φ-contraction mappings in the frame work of complete metric spaces. Furthermore, we present some examples and applications of our main results.

2. Main Result

In this section, we introduce a new class of mappings and prove the existence and uniqueness theorems for fixed points of this class of mappings.

Definition 2.1. *Let X be a nonempty set, $T : X \to X$ be a mapping and $\alpha, \beta : X \times X \to \mathbb{R}^+$ be two functions. We say that T is an (α, β)-cyclic admissible mapping, if for all $x, y \in X$*

1. $\alpha(x, y) \geq 1 \Rightarrow \beta(Tx, Ty) \geq 1$,
2. $\beta(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.

Remark 2.2. *Clearly, if $\beta(x, y) = \alpha(x, y)$, we obtain Definition 1.7.*

Definition 2.3. *Let (X,d) be a metric space, $\alpha,\beta : X\times X\to[0,\infty)$ be two functions and T be a self map on X. The mapping T is said to be an (α,β)-Berinde-φ-contraction mapping, if there exists $L\geq 0$ such that for all $x,y\in X$ with $Tx\neq Ty$, we have*

$$\alpha(x,Tx)\beta(y,Ty)\geq 1\Rightarrow d(Tx,Ty)\leq\varphi(d(x,y))+L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\},\tag{2.1}$$

where $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function which satisfies $\varphi(t)<t$ for all $t>0$ and $\varphi(0)=0$.

Example 2.4. *Let $X=[0,\infty)$ and $d : X\times X\to[0,\infty)$ be defined as $d(x,y)=|x-y|$ for all $x,y\in X$. It is clear that (X,d) is a metric space. We define $T:X\to X$ by*

$$Tx=\begin{cases}\frac{x}{7} & if \quad x\in[0,1]\\ 5x & if \quad x\in(1,\infty),\end{cases}$$

$\alpha,\beta : X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}1 & if \quad x,y\in[0,1]\\ 0 & if \quad x,y\in(1,\infty),\end{cases}$$

$$\beta(x,y)=\begin{cases}2 & if \quad x,y\in[0,1]\\ 0 & if \quad x,y\in(1,\infty),\end{cases}$$

and $\varphi:[0,\infty)\to[0,\infty)$ by $\varphi(t)=\frac{t}{2}$. Then T is an (α,β)-Berinde-φ-contraction mapping but not an φ-contraction as defined by Boyd et al. [13] and Suzuki type generalized φ-contraction as defined by Rant [31].

Proof . Clearly, for any $x,y\in[0,1]$, we have that $\alpha(x,Tx)=1$ and $\beta(y,Ty)=2$, as such we have that $\alpha(x,Tx)\beta(y,Ty)>1$. Since $\alpha(x,Tx)\beta(y,Ty)>1$ if $x,y\in[0,1]$, we need to show that $d(Tx,Ty)\leq\varphi(d(x,y))+L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$ for any $x,y\in[0,1]$. Observe that for $L\geq 0$, we have

$$\begin{aligned}d(Tx,Ty)&=|\frac{x}{7}-\frac{y}{7}|=\frac{1}{7}|x-y|\leq\frac{1}{2}|x-y|+L\min\{|x-\frac{x}{7}|,|y-\frac{y}{7}|,|x-\frac{y}{7}|,|y-\frac{x}{7}|\}\\ &=\varphi(d(x,y))+L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}.\end{aligned}$$

Thus, T is an (α,β)-Berinde-φ-contraction mapping. However T is not an φ-contraction mapping as defined by Boyd et al. [13] and Pant [31]. Suppose $x=0$ and $y=4$. Observe that

$$d(Tx,Ty)=20>2=\varphi(d(x,y)).$$

Also, we have

$$\frac{1}{2}d(x,Tx)=0<4=d(x,y),$$

but

$$d(Tx,Ty)=20>8=\varphi(d(y,Ty))=\varphi(m(x,y)),$$

where $m(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}$. $\square$

Remark 2.5. *In Definition 2.3, if we suppose that $\varphi(t) = \delta t$, where $\delta \in [0, 1)$. Then, we obtain a new type of contractive mapping*

$$\alpha(x, Tx)\beta(y, Ty) \geq 1 \Rightarrow d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \quad (2.2)$$

Using the above example, it is easy to see that (2.2) *is a generalization of* (1.2).

Lemma 2.6. *Let X be a nonempty set and $T : X \to X$ be an (α, β)-cyclic admissible mapping. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, then $\alpha(x_m, x_{m+1}) \geq 1$ implies that $\beta(x_n, x_{n+1}) \geq 1$ and $\beta(x_m, x_{m+1}) \geq 1$ implies that $\alpha(x_n, x_{n+1}) \geq 1$, for all $n, m \in \mathbb{N} \cup \{0\}$ with $m < n$.*

Proof . Using our hypothesis and the fact that T is an (α, β)-cyclic admissible mapping, we have that there exists $x_0 \in X$ such that

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \beta(Tx_0, Tx_1) = \beta(x_1, x_2) \geq 1$$

and

$$\beta(x_1, x_2) \geq 1 \Rightarrow \alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1.$$

Continuing this way, we obtain that

$$\alpha(x_{2n}, x_{2n+1}) \geq 1 \text{ and } \beta(x_{2n+1}, x_{2n+2}) \geq 1, \ \forall n \in \mathbb{N}.$$

Using similar approach, we obtain

$$\beta(x_{2n}, x_{2n+1}) \geq 1 \text{ and } \alpha(x_{2n+1}, x_{2n+2}) \geq 1, \forall n \in \mathbb{N}.$$

In similar sense, we obtain the same result for all $m \in \mathbb{N}$. That is

$$\alpha(x_{2m}, x_{2m+1}) \geq 1 \text{ and } \beta(x_{2m+1}, x_{2m+2}) \geq 1$$

and

$$\beta(x_{2m}, x_{2m+1}) \geq 1 \text{ and } \alpha(x_{2m+1}, x_{2m+2}) \geq 1, \forall m \in \mathbb{N}.$$

In addition, since

$$\alpha(x_m, x_{m+1}) \geq 1 \Rightarrow \beta(x_{m+1}, x_{m+2}) \geq 1 \Rightarrow \alpha(x_{m+2}, x_{m+3}) \geq 1 \cdots$$

with $m < n$, we deduce that

$$\alpha(x_m, x_{m+1}) \geq 1 \Rightarrow \beta(x_n, x_{n+1}) \geq 1.$$

Using similar approach, we have that

$$\beta(x_m, x_{m+1}) \geq 1 \Rightarrow \alpha(x_n, x_{n+1}) \geq 1.$$

□

Theorem 2.7. *Let (X,d) be a complete metric space and $T : X \to X$ be an (α,β)-Berinde-φ-contraction mapping. Suppose the following conditions hold:*

1. *T is an (α,β)-cyclic admissible mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\beta(x_0,Tx_0) \geq 1$,*
3. *T is continuous.*

Then T has a fixed point.

Proof . We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}\cup\{0\}$. If we suppose that $x_{n+1} = x_n$, we obtain the desired result. Now, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}\cup\{0\}$. Since T is an (α,β)-cyclic admissible mapping and $\alpha(x_0,x_1) \geq 1$, we have $\beta(Tx_0,Tx_1) = \beta(x_1,x_2) \geq 1$ and this implies that $\alpha(x_2,x_3) = \alpha(Tx_1,Tx_2) \geq 1$, continuing the process, we have

$$\alpha(x_{2k},x_{2k+1}) \geq 1 \quad \text{and} \quad \beta(x_{2k+1},x_{2k+2}) \geq 1 \quad \forall \quad k \in \mathbb{N}\cup\{0\}. \tag{2.3}$$

Using similar argument, we have that

$$\beta(x_{2k},x_{2k+1}) \geq 1 \quad \text{and} \quad \alpha(x_{2k+1},x_{2k+2}) \geq 1 \quad \forall \quad k \in \mathbb{N}\cup\{0\}. \tag{2.4}$$

It follows from (2.3) and (2.4) that $\alpha(x_n,x_{n+1}) \geq 1$ and $\beta(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$. Since $\alpha(x_n,x_{n+1})\beta(x_{n+1},x_{n+2}) \geq 1$, we obtain from (2.1)

$$\begin{aligned} d(x_{n+1},x_{n+2}) &= d(Tx_n,Tx_{n+1}) \\ &\leq \varphi(d(x_n,x_{n+1}) + L\min\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}),d(x_n,x_{n+2}),d(x_{n+1},x_{n+1})\} \\ &= \varphi(d(x_n,x_{n+1})) + L.0 \\ &< d(x_n,x_{n+1}), \end{aligned} \tag{2.5}$$

which imples that

$$d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1}).$$

Using similar approach, it is easy to see that

$$d(x_n,x_{n+1}) < d(x_{n-1},x_n).$$

Hence the sequence $\{d(x_n,x_{n+1})\}$ is strictly decreasing and bounded below. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = r$. Using the continuity of φ and the fact that $\varphi(t) < t$ for all $t > 0$ with $\varphi(0) = 0$, we obtain from (2.5) that

$$r = \lim_{n\to\infty} d(x_n,x_{n+1}) \leq \lim_{n\to\infty} \varphi(d(x_{n-1},x_n)) < r,$$

which is a contradiction. Thus, we must have that $r = 0$ and hence

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.6}$$

In what follows, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence, then by Lemma 1.15, there exists an $\epsilon > 0$ and sequences of positive integers $\{x_{n_k}\}$ and $\{x_{m_k}\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k},x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k},x_{n_k}) \geq \epsilon, d(x_{m_k},x_{n_{k-1}}) < \epsilon$ and $(1)-(4)$

of Lemma 1.15 hold. Since $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$, using Lemma 2.6, we obtain that $\alpha(x_{m_k}, x_{m_{k+1}})\beta(x_{n_k}, x_{n_{k+1}}) \geq 1$. Hence, for all $k \geq n_0$, we have

$$\begin{aligned} d(x_{m_{k+1}}, x_{n_{k+1}}) &\leq d(Tx_{m_k}, Tx_{n_k}) \\ &\leq \varphi(d(x_{m_k}, x_{n_k})) + L\min\{d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_k}, x_{n_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\}. \end{aligned} \tag{2.7}$$

Using Lemma 1.15, (2.6), $\varphi(t) < t$ for all $t > 0$ and the continuity of $\varphi(t)$, we have that

$$\begin{aligned} \epsilon &= \lim_{k\to\infty} d(Tx_{m_k}, Tx_{n_k}) \\ &\leq \lim_{k\to\infty} [\varphi(d(x_{m_k}, x_{n_k})) + L\min\{d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_k}, x_{n_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\}]. \\ &< \epsilon, \end{aligned}$$

which is a contradiction. We therefore have that $\{x_n\}$ is Cauchy. Since (X, d) is complete, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Since T is continuous, we have that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\lim_{n\to\infty} x_n = Tx.$$

Thus, T has a fixed point. □

Theorem 2.8. *Let (X, d) be a complete metric space and $T : X \to X$ be an (α, β)-Berinde-φ-contraction mapping. Suppose the following conditions hold:*

1. *T is a (α, β)-cyclic admissible mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,*
3. *if for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, then $\beta(x, Tx) \geq 1$ and $\alpha(x, Tx) \geq 1$.*

Then T has a fixed point.

Proof . We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. In Theorem 2.7, we have establish that $\{x_n\}$ is Cauchy and since (X, d) is complete, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now suppose hypothesis (3) holds, we show that T has a fixed point. Since $\alpha(x_n, x_{n+1}) \geq 1$ and $\beta(x, Tx) \geq 1$, we have that $\alpha(x_n, x_{n+1})\beta(x, Tx) \geq 1$ we obtain from (2.1) that

$$\begin{aligned} d(x_{n+1}, Tx) &= d(Tx_n, Tx) \\ &\leq \varphi(d(x_n, x)) + L\min\{d(x_n, x_{n+1}), d(x, Tx), d(x_n, Tx), d(x, Tx_n)\}. \end{aligned} \tag{2.8}$$

Taking limit as $n \to \infty$ and using (2.6) and the assumptions on φ, it is easy to see that

$$d(x, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = \lim_{n\to\infty} d(Tx_n, Tx) = 0,$$

which implies that $Tx = x$. Hence, T has a fixed point. □ We present an example to support the above results.

Example 2.9. *Let $X = [0, \infty)$ and $d : X \times X \to [0, \infty)$ be defined as $d(x, y) = |x - y|$ for all $x, y \in X$. It is clear that (X, d) is a complete metric space. We define $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x}{9} & if \quad x \in [0, 2) \\ 2x - \frac{35}{4} & if \quad x \in [2, \infty), \end{cases}$$

$\alpha,\beta : X\times X\to[0,\infty)$ *by*

$$\alpha(x,y)=\begin{cases}1.5 & if \quad x,y\in[0,1]\\ 0 & if \quad x,y\in(1,\infty),\end{cases}$$

$$\beta(x,y)=\begin{cases}1 & if \quad x,y\in[0,1]\\ 0 & if \quad x,y\in(1,\infty),\end{cases}$$

and $\alpha,\beta:[0,\infty)\to[0,\infty)$ *by* $\varphi(t)=\frac{t}{5}$.

Proof . For any $x,y\in[0,1]$, we have that $\alpha(x,y)=1.5$ and $\beta(x,y)=1$. Also $Tx=\frac{x}{9}$ and $Ty=\frac{y}{9}$ are in $[0,1]$, as such, we have that $\alpha(x,y)=1.5>1\Rightarrow\beta(Tx,Ty)=1$ and $\beta(x,y)=1\Rightarrow\alpha(Tx,Ty)=1.5>1$. Therefore, T is (α,β)-cyclic admissible mapping. It is easy to see that for any $x_0\in[0,1]$, we have that $\alpha(x_0,T_0)>1$ and $\beta(x_0,Tx_0)=1$. Also all the hypothesis of Theorem 2.7 and Theorem 2.8 are satisfied. We now establish that T is an (α,β)-Berinde-φ-contraction. Since $\alpha(x,Tx)\beta(y,Ty)=1.5>1$ if $x,y\in[0,1]$, we need to show that $d(Tx,Ty)\leq\varphi(d(x,y))+L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$ for any $x,y\in[0,1]$. Observe that, for $L\geq0$, it is easy to see that

$$\begin{aligned}d(Tx,Ty)&=|\frac{x}{9}-\frac{y}{9}|\\&=\frac{1}{9}|x-y|\leq\frac{1}{5}|x-y|+L\min\{|x-\frac{x}{9}|,|y-\frac{y}{9}|,|x-\frac{y}{9}|,|y-\frac{x}{9}|\\&=\varphi(d(x,y))+L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}.\end{aligned}$$

Thus T is an (α,β)-Berinde-φ-contraction with $x=0$ and $x=\frac{35}{4}$ as the two fixed points of T. □

Remark 2.10. *As established in the above example, T has two fixed points $x=0$ and $x=\frac{35}{4}$. Thus, for the uniqueness of fixed point of T, we need additional condition.*

Theorem 2.11. *Suppose that the hypothesis of Theorem 2.8 holds and in addition suppose $\alpha(x,Tx)\geq1$ and $\beta(y,Ty)\geq1$ for all $x,y\in F(T)$, where $F(T)$ is the set of fixed points of T. Then T has a unique fixed point.*

Proof . Let $x,y\in F(T)$, that is $Tx=x$ and $Ty=y$ such that $x\neq y$. Since $\alpha(x,Tx)\geq1$ and $\beta(y,Ty)\geq1$, we have $\alpha(x,Tx)\beta(y,Ty)\geq1$. Thus, we obtain

$$\begin{aligned}d(x,y)&=d(Tx,Ty)\\&\leq\varphi(d(x,y))+L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\},\\&\leq\varphi(d(x,y))<d(x,y),\end{aligned}$$

which is a contradiction. Thus, T has a unique fixed point. □
We present an examples to support Theorem 2.11.

Example 2.12. *Let $X=[0,\infty)$ and $d:X\times X\to[0,\infty)$ be defined as $d(x,y)=|x-y|$ for all $x,y\in X$. We define $T:X\to X$ by*

$$Tx=\begin{cases}\frac{x}{16} & if \quad x\in[0,1]\\ 4x+5 & if \quad x\in(1,\infty),\end{cases}$$

$\alpha, \beta : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1.5 & \text{if} \quad x, y \in [0,1] \\ 0 & \text{if} \quad x, y \in (1,\infty), \end{cases}$$

$$\beta(x,y) = \begin{cases} 2 & \text{if} \quad x, y \in [0,1] \\ 0 & \text{if} \quad x, y \in (1,\infty), \end{cases}$$

and and $\alpha, \beta : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{t}{10}$.

Proof . For any $x, y \in [0,1]$, we have that $\alpha(x,y) = 1.5, \beta(x,y) = 1$. Also $Tx = \frac{x}{9}$ and $Ty = \frac{y}{9}$ are in $[0,1]$, as such we have that $\alpha(x,y) = 1.5 > 1 \Rightarrow \beta(Tx,Ty) = 2$ and $\beta(x,y) = 2 > 1 \Rightarrow \alpha(Tx,Ty) = 1.5 > 1$. Therefore, T is (α,β)-cyclic admissible mapping. It is easy to see that for any $x_0 \in [0,1]$, we have that $\alpha(x_0, T_0) > 1$ and $\beta(x_0, Tx_0) = 1$. Also all the hypothesis of Theorem 2.11 are satisfied. We now establish that T is an (α,β)-Berinde-φ-contraction. Since $\alpha(x,Tx)\beta(y,Ty) = 2 > 1$ if $x, y \in [0,1]$, we need to show that $d(Tx,Ty) \le \varphi(d(x,y)) + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$ for any $x, y \in [0,1]$. Observe that, for $L \ge 0$, it is easy to see that

$$\begin{aligned} d(Tx,Ty) &= |\frac{x}{16} - \frac{y}{16}| \\ &= \frac{1}{16}|x-y| \\ &\le \frac{1}{10}|x-y| + L\min\{d(x,\frac{x}{16}), d(y,\frac{y}{16}), d(x,\frac{y}{16}), d(y,\frac{x}{16})\} \\ &= \varphi(d(x,y)) + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \end{aligned}$$

Thus T is an (α,β)-Berinde-φ-contraction with $x = 0$ the unique fixed points of T. However, we show that Theorem 1.2 and Theorem 1.14 are not applicable. Suppose $x = 0$ and $y = 5$. Observe that

$$d(Tx,Ty) = 25 > \frac{1}{2} = \varphi(d(x,y)).$$

Also, we have

$$\frac{1}{2}d(x,Tx) = 0 < 5 = d(x,y),$$

but

$$d(Tx,Ty) = 25 > 2 = \varphi(d(y,Ty)) = \varphi(m(x,y)),$$

where $m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$. □

Corollary 2.13. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping satisfying the following inequality*

$$\alpha(x,Tx)\beta(y,Ty) \ge 1 \Rightarrow d(Tx,Ty) \le \varphi(d(x,y)),$$

for all $x, y \in X$, with $d(Tx,Ty) > 0, L \ge 0$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function and satisfies $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$. Suppose the following conditions hold:

1. *T is a (α, β)-cyclic admissible mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,*
3. *T is continuous,*
4. *if for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, then $\beta(x, Tx) \geq 1$ and $\alpha(x, Tx) \geq 1$.*

Then T has a fixed point.

Corollary 2.14. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the following inequality*

$$\alpha(x, Tx)\beta(y, Ty) \geq 1 \Rightarrow d(Tx, Ty) \leq kd(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$, where $k \in (0, 1), L \geq 0, \varphi : [0, \infty) \to [0, \infty)$ is a continuous function and satisfies $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$. Suppose the following conditions hold:

1. *T is a (α, β)-cyclic admissible mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,*
3. *T is continuous,*
4. *if for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, then $\beta(x, Tx) \geq 1$ and $\alpha(x, Tx) \geq 1$.*

Then T has a fixed point.

Corollary 2.15. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the following inequality*

$$d(Tx, Ty) > 0 \Rightarrow d(Tx, Ty) \leq \varphi(d(x, y)) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$, with $d(Tx, Ty) > 0, L \geq 0$ where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function and satisfies $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$. Then T has a unique fixed point.

Corollary 2.16. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the following inequality*

$$d(Tx, Ty) \leq \varphi(d(x, y))$$

for all $x, y \in X$, where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function and satisfies $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$. Then T has a unique fixed point.

3. Application

In this section, we apply our main result to the existence of solution of the second order differential equation and integral equation.

3.1. Second order differential equation

In this subsection, we give an application of Corollary 2.13 to the solution of second order differential equation of the form

$$\begin{aligned} x''(t) &= -f(t, x(t)), t \in I \\ x(0) &= x(1) = 0, \end{aligned} \tag{3.1}$$

where $I = [0,1], f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Consider the space $C(I)$ of continuous function defined on I. It is well-known that $C(I)$ with the metric $d(x,y) = sup_{t\in I}|x(t) - y(t)|$ for all $x, y \in C(I)$ is a complete metric space. It is also well-known that the Problem (3.1) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s)f(s,x(s))ds, \tag{3.2}$$

for $t \in I$, where G is the Green function defined by

$$G(t,s) = \begin{cases} (1-t)s & \text{if} \quad 0 \le s \le t \le 1, \\ (1-s)t & \text{if} \quad 0 \le t \le s \le 1. \end{cases}$$

If $x \in C^2(I)$, then $x \in C(I)$ is also a solution of (3.1) if and only if it is a solution (3.2).

Theorem 3.1. *Let $X = C(I)$ and $T : X \to X$ be an operator given by*

$$Tx(t) = \int_0^1 G(t,s)f(s,x(s))ds$$

for all $x \in X$ and $t \in I = [0,1]$. Suppose the following conditions hold:

1. *there exists functions $\alpha, \beta : X \times X \to [0,\infty)$ such that $\alpha(x,y) \ge 1 \Rightarrow \beta(Tx,Ty) \ge 1$ and $\beta(x,y) \ge 1 \Rightarrow \alpha(Tx,Ty) \ge 1$, and for all $x, y \in X$, we have*

$$|f(s,u) - f(s,v)| \le \alpha(x,y)\beta(x,y)|u - v|$$

   for some $s \in I$ and $u, v \in \mathbb{R}$;
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$ and $\beta(x_0, Tx_0) \ge 1$;*
3. *for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, then $\beta(x, Tx) \ge 1$.*

Then the second order differential equation (3.1) has a solution.

Proof . We define

$$\alpha(x,y) = \begin{cases} 2 & \text{if} \quad x \preceq y \\ 0 & \text{otherwise} \end{cases}$$

$$\beta(x,y) = \begin{cases} 1 & \text{if} \quad x \preceq y \\ 0 & \text{otherwise} \end{cases}$$

and $\varphi(t) = \frac{t}{4}$. We define $x, y \in X,\ x \preceq y$ if and only if $x(t) \le y(t)$ for all $t \in [0,1]$. It is clear that if $x \preceq y$, then $\alpha(x,Tx)\beta(y,Ty) > 1$. Thus, we have that

$$\begin{aligned} |Ty(t) - Tx(t)| &\le \int_0^1 G(t,s)|f(s,y(s)) - f(s,x(s))|ds \\ &\le \int_0^1 G(t,s)\alpha(x,y)\beta(x,y)|y(s) - x(s)|ds \qquad (3.3) \\ &= \alpha(x,y)\beta(x,y)\|y - x\| \sup_{t\in[0,1]} \int_0^1 G(t,s)ds \\ &= 2\|x - y\| \sup_{t\in[0,1]} \int_0^1 G(t,s)ds. \end{aligned}$$

It is well-known that for each $t \in I$, we have $\int_0^1 G(t,s)ds = \frac{t(1-t)}{2}$, as such, we obtain that

$$sup_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{8}.$$

Thus, (3.3) becomes

$$\begin{aligned} d(Tx,Ty) &\leq \frac{1}{4} d(x,y) \\ &= \varphi(d(x,y)). \end{aligned}$$

Clearly, all the conditions of Corollary 2.13 are satisfied, and so T has a fixed point. Thus the second order differential equation (3.1) has a solution. □

3.2. Integral Equation

In this subsection, we give an application of Corollary 2.13 to solve the nonlinear integral equation:

$$x(t) = g(t) + \int_a^b M(t,s)K(t,x(s))ds, \tag{3.4}$$

where $M : [a,b] \times [a,b] \to \mathbb{R}^+$, $K : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous functions. Let $X = C([a,b], \mathbb{R})$ be the space of all continuous real valued functions defined on $[a,b]$. We defined $d : X \times X \to [0,\infty)$ by $d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$. Clearly, (X,d) is a complete metric space.

Theorem 3.2. *Let $X = C([a,b], \mathbb{R})$ and $T : X \to X$ be an operator given by*

$$Tx(t) = g(t) + \int_a^b M(t,s)K(t,x(s))ds$$

for all $t, s \in [a,b]$, where $M : [a,b] \times [a,b] \to \mathbb{R}^+$, $K : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous functions. Let $X = C([a,b], \mathbb{R})$ be the space of all continuous real valued functions defined on $[a,b]$. Suppose the following conditions hold:

1. *there exists a continuous mapping $\mu : X \times X \to [0,\infty)$ such that*

$$|K(s,x(s)) - K(s,y(s))| \leq \mu(x,y)|x(s) - y(s)|$$

for all $s \in [a,b]$ and $x, y \in X$;

2. *there exists $\delta \in (0,1)$ and $\alpha, \beta : X \times X \to [0,\infty)$ such that $\alpha(x,y) \geq 1 \Rightarrow \beta(Tx,Ty) \geq 1$ and $\beta(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1$ for all $x, y \in X$ and*

$$\int_a^b M(t,s)\mu(x,y) \leq \delta;$$

3. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;*
4. *for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, then $\beta(x,Tx) \geq 1$ and $\alpha(x,Tx) \geq 1$.*

Then the integral equation (3.4) has a solution.

Proof . We define

$$\alpha(x,y)=\begin{cases}1 & \text{if} \quad x\preceq y\\ 0 & \text{otherwise}\end{cases}$$

$$\beta(x,y)=\begin{cases}2 & \text{if} \quad x\preceq y\\ 0 & \text{otherwise}\end{cases}$$

and $\varphi(t)=\delta t$. We define $x,y\in X,\ x\preceq y$ if and only if $x(t)\leq y(t)$ for all $t\in[0,1]$. It is clear that if $x\preceq y$, we have that $\alpha(x,Tx)\beta(y,Ty)>1$. Thus, we have that

$$\begin{aligned}|Ty(s)-Tx(s)| &\leq \int_a^b |M(t,s)[K(t,y(s))-K(t,x(s))]|ds\\ &\leq \int_a^b M(t,s)\mu(x,y)|y(s)-x(s)|ds\\ &\leq \sup_{s\in[a,b]}|y(s)-x(s)|\int_a^b M(t,s)\mu(x,y)ds\\ &\leq \delta d(y,x)\\ &=\varphi(d(x,y)).\end{aligned}$$

Thus, we have that

$$d(Tx,Ty)\leq\varphi(d(x,y)).$$

Clearly, all the conditions of Corollary 2.13 are satisfied, and so T has a fixed point. Thus the integral equation (3.4) has a solution. □

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The authors declare that they have no competing interests.

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