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Fixed and coincidence points for hybrid rational Geraghty contractive mappings in ordered b-metric spaces

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Abstract

In this paper, we present some fixed and coincidence point theorems for hybrid rational Geraghty contractive mappings in partially ordered b-metric spaces. Also, we derive certain coincidence point results for such contractions. An illustrative example is provided here to highlight our findings.

Keywords: Fixed point; coincidence point; ordered b-metric space.
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1. Introduction and preliminaries

In 2009 Suzuki [17] extended Edelstein's fixed point theorem [19]. Base on Suzuki's paper, many researchers studied different spaces, like complete metric spaces endowed with a partial order, b-metric space (*metric type pace*) and obtained many fixed point results in such spaces (see [7, 12, 16, 20, 21]).

Czerwik [4] introduced the concept of the b-metric space. Several papers dealt with fixed point theory for single-valued and multivalued operators in b-metric spaces are written (see, e.g., [2, 10, 11, 13, 14, 15]).

Definition 1.1. Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

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(b_1) $d(x,y)=0$ iff $x=y$,

(b_2) $d(x,y)=d(y,x)$,

(b_3) $d(x,z)\le s[d(x,y)+d(y,z)]$.

The pair (X,d) is called a b-metric space.

A b-metric is a metric if (and only if) $s=1$. The following example shows that in general a b-metric need not to be a metric.

Example 1.2. [1] Let (X,d) be a metric space and $\rho(x,y)=(d(x,y))^p$, where $p\ge 1$ is a real number. Then ρ is a b-metric with $s=2^{p-1}$. However, (X,ρ) is not necessarily a metric space. For example, if $X=\mathbb{R}$ is the set of real numbers and $d(x,y)=|x-y|$ is the usual Euclidean metric, then $\rho(x,y)=(x-y)^2$ is a b-metric on $\mathbb{R}$ with $s=2$, but it is not a metric on $\mathbb{R}$.

Definition 1.3. [3] Let (X,d) be a $b-$metric space. Then a sequence $\{x_n\}$ in X is called:
(a) $b-$convergent if and only if there exists $x\in X$ such that $d(x_n,x)\to 0$, as $n\to\infty$. In this case, we write $\lim\limits_{n\to\infty} x_n=x$.
(b) $b-$Cauchy if and only if $d(x_n,x_m)\to 0$, as $n,m\to\infty$.

Proposition 1.4. ([3, Remark 2.1]) In a $b-$metric space (X,d) the following assertions hold:
p_1. A $b-$convergent sequence has a unique limit.
p_2. Each $b-$convergent sequence is b-Cauchy.
p_3. In general, a $b-$metric is not continuous.

The b-metric space (X,d) is b-complete if every b-Cauchy sequence in X is b-converges.
Note that a b-metric might not be a continuous function. The following example (see also [7]) illustrates this fact.

Example 1.5. Let $X=\mathbb{N}\cup\{\infty\}$ and let $d:X\times X\to\mathbb{R}$ be defined by

$$d(m,n)=\begin{cases}0, & \text{if } m=n,\\ |\frac{1}{m}-\frac{1}{n}|, & \text{if one of } m,n \text{ is even and the other is even or } \infty,\\ 5, & \text{if one of } m,n \text{ is odd and the other is odd (and } m\ne n) \text{ or } \infty,\\ 2, & \text{otherwise.}\end{cases}$$

Then considering all possible cases, it can be checked that for all $m,n,p\in X$, we have

$$d(m,p)\le \frac{5}{2}(d(m,n)+d(n,p)).$$

Thus (X,d) is a b-metric space (with $s=5/2$). Let $x_n=2n$ for each $n\in\mathbb{N}$. Then

$$d(2n,\infty)=\frac{1}{2n}\to 0 \quad \text{as } n\to\infty,$$

that is $x_n\to\infty$ but $d(x_n,1)=2\not\to 5=d(\infty,1)$ as $n\to\infty$.

Let $\mathfrak{S}$ denote the class of all real functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition

$$\beta(t_n) \to 1 \text{ implies } \ t_n \to 0, \ as\ n \to \infty.$$

In order to generalize the Banach contraction principle, Geraghty proved the following result.

Theorem 1.6. [6] Let (X, d) be a complete metric space, and let $f : X \to X$ be a self-map. Suppose that there exists $\beta \in \mathfrak{S}$ such that

$$d(fx, fy) \leq \beta(d(x, y))d(x, y)$$

holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $f^n x$ converges to z.

In [5] some fixed point theorems for mappings satisfying Geraghty-type contractive conditions are proved in various generalized metric spaces. As in [5] we will consider the class of functions $\mathcal{F}$, where $\beta \in \mathcal{F}$ if $\beta : [0, \infty) \to [0, 1/s)$ and has the property

$$\beta(t_n) \to \frac{1}{s} \text{ implies } \ t_n \to 0, \ as\ n \to \infty.$$

Theorem 1.7. [5] Let $s > 1$ and (X, D, s) be a complete metric type space. Suppose that a mapping $f : X \to X$ satisfies the condition

$$D(fx, fy) \leq \beta(D(x, y))D(x, y)$$

for all $x, y \in X$ and some $\beta \in \mathcal{F}$. Then f has a unique fixed point $z \in X$, and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z in (X, D, s).

In this paper, we present some fixed point and coincidence point theorems for hybrid rational Geraghty contractive mappings in partially ordered b-metric spaces.

2. The main results

Let Ψ be the family of all nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that

$$\lim_{n\to\infty} \psi^n(t) = 0$$

for all $t > 0$.

Lemma 2.1. If $\psi \in \Psi$, then the following are satisfied.
(a) $\psi(t) < t$ for all $t > 0$;
(b) $\psi(0) = 0$.

By the same idea of [9], we now prove following new result.

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and there exists a b-metric d on X such that (X, d) is a b-complete b-metric space. Suppose $s > 1$ and $f : X \to X$ is an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Assume that*

$$s(\frac{1 + sd(x, y)}{1 + \frac{1}{2}d(x, fx)})d(fx, fy) \leq \psi(M(x, y)) + LN(x, y) \tag{2.1}$$

for all comparable elements $x, y \in X$, where $L \geq 0$,

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}\right\}$$

and

$$N(x, y) = \min\{d(x, fx), d(x, fy), d(y, fx), d(y, fy)\}.$$

If f is continuous, then f has a fixed point.

Proof . Since $x_0 \preceq f(x_0)$ and f is an increasing function we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots .$$

Putting $x_n = f^n(x_0)$, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then, $x_{n_0} = fx_{n_0}$ and so we have no thing for prove. Hence, for all $n \in \mathbb{N}$ we assume $d(x_n, x_{n+1}) > 0$.

Step I. We will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Since $\frac{1+sd(x_{n-1},x_n)}{1+\frac{1}{2}d(x_{n-1},fx_{n-1})} = \frac{1+sd(x_{n-1},x_n)}{1+\frac{1}{2}d(x_{n-1},x_n)} \geq \frac{1+d(x_{n-1},x_n)}{1+\frac{1}{2}d(x_{n-1},x_n)} \geq 1$ and using condition (2.1), we obtain

$$d(x_{n+1}, x_n) \leq sd(x_{n+1}, x_n) = sd(fx_n, fx_{n-1}) \leq \psi(M(x_n, x_{n-1})) \leq \psi(d(x_n, x_{n-1})).$$

Because

$$\begin{aligned} M(x_{n-1}, x_n) =& \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1 + d(fx_{n-1}, fx_n)}\} \\ =& \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})}\} \\ =& d(x_{n-1}, x_n) \end{aligned}$$

and

$$\begin{aligned} N(x_{n-1}, x_n) =& \min\{d(x_{n-1}, fx_n), d(x_n, fx_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_{n-1})\} \\ =& \min\{d(x_{n-1}, x_{n+1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n)\} \\ =& 0. \end{aligned}$$

Hence,

$$d(x_n, x_{n+1}) \leq sd(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n). \tag{2.2}$$

By induction, we get that

$$d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n-1})) \leq \psi^2(d(x_{n-1}, x_{n-2})) \leq \cdots \leq \psi^n(d(x_1, x_0)). \tag{2.3}$$

As $\psi \in \Psi$, we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.4}$$

Step II. $\{x_n\}$ is a b-Cauchy sequence, suppose not, i.e. $\{x_m\}$ is not a b-Cauchy sequence. There exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } d(x_{m_i}, x_{n_i}) \geq \varepsilon. \tag{2.5}$$

This means that

$$0 \leq d(x_{m_i}, x_{n_i-1}) < \varepsilon. \tag{2.6}$$

From (2.5) and using the triangular inequality

$$\varepsilon \leq d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_i+1}) + sd(x_{m_i+1}, x_{n_i}).$$

By taking the upper limit as $i \to \infty$

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}). \tag{2.7}$$

By using the triangular inequality

$$d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}).$$

Taking the upper limit as $i \to \infty$ in the above inequality and using (2.6) we get

$$\limsup_{i\to\infty} d(x_{m_i}, x_{n_i}) \leq \varepsilon s. \tag{2.8}$$

From the definition of $M(x, y), N(x, y)$ and the above limits,

$$\begin{aligned} M(x_{m_i}, x_{n_i-1}) =& \max\{d(x_{m_i}, x_{n_{i-1}}), \frac{d(x_{m_i}, fx_{m_i})d(x_{n_{i-1}}, fx_{n_{i-1}})}{1 + d(fx_{m_i}, fx_{n_{i-1}})}\} \\ =& \{d(x_{m_i}, x_{n_{i-1}}), \frac{d(x_{m_i}, x_{m_{i+1}})d(x_{n_{i-1}}, x_{n_i})}{1 + d(x_{m_{i+1}}, x_{n_i})}\} \\ =& d(x_{m_i}, x_{n_{i-1}}) \end{aligned}$$

and

$$\begin{aligned} N(x_{m_i}, x_{n_i-1}) &= \min\{d(x_{m_i}, f(x_{m_i})), d(x_{m_i}, f(x_{n_i-1})), d(x_{n_i-1}, f(x_{m_i})), d(x_{n_i-1}, f(x_{n_i-1}))\} \\ &= \min\{d(x_{m_i}, x_{m_i+1}), d(x_{m_i}, x_{n_i}), d(x_{n_i-1}, x_{m_i+1}), d(x_{n_i-1}, x_{n_i})\}. \end{aligned}$$

If $i \to \infty$, by (2.6)

$$\begin{aligned} \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) &\leq \varepsilon \\ \limsup_{i\to\infty} N(x_{m_i}, x_{n_i-1}) &= 0. \end{aligned} \tag{2.9}$$

Also from (2.1)

$$\begin{aligned} s\Big(\frac{1+sd(x_{m_i},x_{n_{i-1}})}{1+\frac{1}{2}d(x_{m_i},fx_{m_i})}\Big)d(x_{m_{i+1}},x_{n_i}) =& s\frac{1+d(x_{m_i},x_{n_{i-1}})}{1+\frac{1}{2}d(x_{m_i},fx_{m_i})}d(fx_{m_i},fx_{n_{i-1}}) \\ \leq& \psi(M(x_{m_i},x_{n_{i-1}}))+LN(x_{m_i},x_{n_i-1}) \\ =& \psi(d(x_{m_i},x_{n_{i-1}})). \end{aligned}$$

Again if $i\to\infty$ by (2.6), (2.4) and (2.9), we obtain

$$\varepsilon = s(\tfrac{\varepsilon}{s}) \leq (s\limsup_{i\to\infty} d(x_{m_{i+1}},x_{n_i})) \leq \psi(\varepsilon) < \varepsilon \tag{2.10}$$

which is a contradiction. Thus $\{x_n\}$ is a b-Cauchy sequence. Completeness of X yields that $\{x_n\}$ converges to a point $u\in X$.

Step III. Since f is continuous, u is a fixed point of f,

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fu.$$

□

Theorem 2.3. *Under the same hypotheses of Theorem 2.2, instead of the continuity assumption of f, we suppose for any nondecreasing sequence $\{x_n\}$ in X with $x_n\to u\in X$, we have $x_n\preceq u$ for all $n\in\mathbb{N}$. Then f has a fixed point.*

Proof . Repeating the proof of Theorem 2.2, we construct an increasing sequence $\{x_n\}$ in X such that $x_n\to u\in X$. Using the assumption on X we have $x_n\preceq u$. Now we show that $u=fu$.

Suppose that there exists $n_0\in N_1$ such that

$$\frac{1}{2}d(x_{n_0},fx_{n_0}) > sd(x_{n_0},u)$$

and

$$\frac{1}{2}d(x_{n_0+1},fx_{n_0+1}) > sd(x_{n_0+1},u).$$

Then, from (2.2), it follows that

$$\begin{aligned} d(x_{n_0+1},x_{n_0}) &\leq sd(x_{n_0},u)+sd(x_{n_0+1},u) < \frac{1}{2}d(x_{n_0},fx_{n_0})+\frac{1}{2}d(x_{n_0+1},fx_{n_0+1}) \\ &= \frac{1}{2}d(x_{n_0},x_{n_0+1})+\frac{1}{2}d(x_{n_0+1},x_{n_0+2}) \leq \frac{1}{2}d(x_{n_0},x_{n_0+1})+\frac{1}{2}d(x_{n_0},x_{n_0+1}) \\ &= d(x_{n_0+1},x_{n_0}) \end{aligned}$$

which is a contradiction. Hence either

$$\frac{1}{2}d(x_n,fx_n) \leq sd(x_n,u)$$

and

$$\frac{1}{2}d(x_{n+1},fx_{n+1}) \leq sd(x_{n+1},u)$$

for all $n\in N_1$. It is not restrictive to assume that one of these inequalities holds for all $n\in N_1$, for example

$$\frac{1}{2}d(x_n,fx_n) \leq sd(x_n,u). \tag{2.11}$$

By (2.1) and (2.11) we have

$$s\left(\frac{1+d(x_n,u)}{1+\frac{1}{2}d(x_n,fx_n)}\right)d(fu,x_n) = sd(fu,fx_{n-1}) \le \psi(M(u,x_{n-1})) + LN(u,x_{n-1}), \tag{2.12}$$

where

$$\begin{aligned} M(u,x_{n-1}) &= \max\{d(u,x_{n-1}), \frac{d(u,fu)d(x_{n-1},fx_{n-1})}{1+d(fu,fx_{n-1})}\} \\ &= \max\{d(u,x_{n-1}), \frac{d(u,fu)d(x_{n-1},x_n)}{1+d(fu,x_{n-1})}\}. \end{aligned} \tag{2.13}$$

And

$$\begin{aligned} N(u,x_{n-1}) &= \min\{d(x_{n-1},fu), d(u,fx_{n-1}), d(x_{n-1},fx_{n-1}), d(u,fu)\} \\ &= \min\{d(x_{n-1},fu), d(u,x_n), d(x_{n-1},x_n), d(u,fu\}. \end{aligned} \tag{2.14}$$

Letting $n \to \infty$ in (2.13) and (2.14) we get

$$\limsup_{n\to\infty} M(u,x_{n-1}) = \limsup_{n\to\infty} N(u,x_{n-1}) = 0. \tag{2.15}$$

Again, taking the upper limit as $n \to \infty$ in (2.12) and use of (2.15) we have

$$\begin{aligned} d(u,fu) = &\quad s[d(u,x_n) + d(x_n,fu)] \\ \le &\quad [sd(u,x_{n+1}) + (\tfrac{1+\frac{1}{2}d(x_n,fx_n)}{1+d(x_n,u)})\psi(M(x_{n-1},u)) + LN(u,x_{n-1})] \to 0. \end{aligned}$$

So $d(fu,u) = 0$ i.e. $fu = u$. □

Set $\psi(t) = rt$ in Theorem 2.2 and Theorem 2.3, we have the following corollaries.

Corollary 2.4. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X,d) is a b-complete b-metric space. Assume $f : X \to X$ is an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that

$$s\left(\frac{1+sd(x,y)}{1+\frac{1}{2}d(x,fx)}\right)d(fx,fy) \le rM(x,y) + LN(x,y)$$

for all comparable elements $x, y \in X$, where $L \ge 0$,

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\right\}$$

and

$$N(x,y) = \min\{d(x,fx), d(x,fy), d(y,fx), d(y,fy)\}.$$

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in N$, then f has a fixed point.

Corollary 2.5. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space. Assume $f : X \to X$ is an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that

$$s(\frac{1 + sd(x, y)}{1 + \frac{1}{2}d(x, fx)})d(fx, fy) \leq r \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}\right\}$$

for all comparable $x, y \in X$ where $0 \leq r \leq 1$. If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space. Assume $f : X \to X$ is an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$s(\frac{1 + sd(x, y)}{1 + \frac{1}{2}d(x, fx)})d(fx, fy) \leq ad(x, y) + b\frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}$$

for all comparable elements $x, y \in X$, where $a, b \geq 0$ and $0 \leq a + b \leq 1$. If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in N$, then f has a fixed point.

Proof . Since

$$ad(x, y) + b\frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)} \leq (a + b) \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}\right\} \tag{2.16}$$

then from (2.16), we have

$$s(\frac{1 + sd(x, y)}{1 + \frac{1}{2}d(x, fx)})d(fx, fy) \leq r \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}\right\},$$

where $r = a + b$. Hence, all the conditions of Corollary 2.5 hold and f has a fixed point. □

Theorem 2.7. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space. Assume $f : X \to X$ is an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$(\frac{1 + sd(x, y)}{1 + \frac{1}{2}d(x, fx)})d(fx, fy) \leq \beta(d(x, y))M(x, y) + LN(x, y) \tag{2.17}$$

for all comparable elements $x, y \in X$, where $L \geq 0$,

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}\right\}$$

and

$$N(x, y) = \min\{d(x, fx), d(x, fy), d(y, fx), d(y, fy)\}.$$

If f is continuous, then f has a fixed point.

Proof . Put $x_n = f^n(x_0)$. Since $x_0 \preceq f(x_0)$ and f is an increasing function we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq ... \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots .$$

Step I: We will show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Since $x_n \preceq x_{n+1}$, so for each $n \in N$,

$$\frac{1+sd(x_{n-1},x_n)}{1+\frac{1}{2}d(x_{n-1},fx_{n-1})} = \frac{1+sd(x_{n-1},x_n)}{1+\frac{1}{2}d(x_{n-1},x_n)} \geq \frac{1+d(x_{n-1},x_n)}{1+\frac{1}{2}d(x_{n-1},x_n)} \geq 1.$$

Thus by (2.17)

$$\begin{aligned} d(x_n,x_{n+1}) &= d(fx_{n-1},fx_n) \\ &\leq \beta(d(x_{n-1},x_n))M(x_{n-1},x_n) + LN(x_{n-1},x_n) \\ &\leq \beta(d(x_{n-1},x_n))d(x_{n-1},x_n) \\ &\leq \tfrac{1}{s}d(x_{n-1},x_n) \\ &\leq d(x_{n-1},x_n), \end{aligned} \tag{2.18}$$

because

$$\begin{aligned} M(x_{n-1},x_n) &= \max\{d(x_{n-1},x_n), \frac{d(x_{n-1},fx_{n-1})d(x_n,fx_n)}{1+d(fx_{n-1},fx_n)}\} \\ &= \max\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})}\} \\ &= d(x_{n-1},x_n) \end{aligned}$$

and

$$\begin{aligned} N(x_{n-1},x_n) &= \min\{d(x_{n-1},fx_n), d(x_n,fx_n), d(x_{n-1},fx_{n-1}), d(x_n,fx_{n-1})\} \\ &= \min\{d(x_{n-1},x_{n+1}), d(x_n,x_{n+1}), d(x_{n-1},x_n), d(x_n,x_n)\} \\ &= 0. \end{aligned}$$

So $\{d(x_n,x_{n+1})\}$ is decreasing. There exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = r$. Let $r > 0$ and $n \to \infty$ in (2.18), we have

$$\frac{r}{s} \leq r \leq \lim_{n\to\infty} \beta(d(x_{n-1},x_n))r \leq \frac{r}{s}.$$

So $\lim_{n\to\infty} \beta(d(x_{n-1},x_n)) = \frac{1}{s}$ and since $\beta \in \mathcal{F}$ we deduce that $\lim_{n\to\infty} d(x_{n-1},x_n) = 0$ which is a contradiction. Hence $r = 0$, that is,

$$\lim_{n\to\infty} d(x_{n-1},x_n) = 0 \tag{2.19}$$

*Step II:*We will prove that $\{x_n\}$ is a b-Cauchy sequence. Suppose the contrary, i.e., $\{x_n\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } d(x_{m_i},x_{n_i}) \geq \varepsilon. \tag{2.20}$$

This means that

$$0 \leq d(x_{m_i},x_{n_i-1}) < \varepsilon. \tag{2.21}$$

From (2.20) and using the triangular inequality, we get

$$\varepsilon \leq d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_i+1}) + sd(x_{m_i+1}, x_{n_i}).$$

By taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}). \tag{2.22}$$

Using the triangular inequality, we have

$$d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}).$$

Taking the upper limit as $i \to \infty$ in the above inequality and using (2.21) we get

$$\limsup_{i\to\infty} d(x_{m_i}, x_{n_i}) \leq \varepsilon s. \tag{2.23}$$

From the definition of $M(x, y), N(x, y)$ and the above limits,

$$\begin{aligned} M(x_{m_i}, x_{n_i-1}) =& \max\{d(x_{m_i}, x_{n_{i-1}}), \frac{d(x_{m_i}, fx_{m_i})d(x_{n_{i-1}}, fx_{n_{i-1}})}{1 + d(fx_{m_i}, fx_{n_{i-1}})}\} \\ =& \{d(x_{m_i}, x_{n_{i-1}}), \frac{d(x_{m_i}, x_{m_{i+1}})d(x_{n_{i-1}}, x_{n_i})}{1 + d(x_{m_{i+1}}, x_{n_i})}\} \\ =& d(x_{m_i}, x_{n_{i-1}}) \end{aligned}$$

and

$$\begin{aligned} N(x_{m_i}, x_{n_i-1}) &= \min\{d(x_{m_i}, f(x_{m_i})), d(x_{m_i}, f(x_{n_i-1})), d(x_{n_i-1}, f(x_{m_i})), d(x_{n_i-1}, f(x_{n_i-1}))\} \\ &= \min\{d(x_{m_i}, x_{m_i+1}), d(x_{m_i}, x_{n_i}), d(x_{n_i-1}, x_{m_i+1}), d(x_{n_i-1}, x_{n_i})\}. \end{aligned}$$

If $i \to \infty$, by (2.21) and (2.19) we have

$$\begin{aligned} \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) &\leq \varepsilon \\ \limsup_{i\to\infty} N(x_{m_i}, x_{n_i-1}) &= 0. \end{aligned} \tag{2.24}$$

Also from (2.17) we have

$$\begin{aligned} (\frac{1 + sd(x_{m_i}, x_{n_{i-1}})}{1 + \frac{1}{2}d(x_{m_i}, fx_{m_i})})d(x_{m_{i+1}}, x_{n_i}) = (\frac{1 + d(x_{m_i}, x_{n_{i-1}})}{1 + \frac{1}{2}d(x_{m_i}, fx_{m_i})})d(fx_{m_i}, fx_{n_{i-1}}) \\ \leq \psi(M(x_{m_i}, x_{n_{i-1}})) + LN(x_{m_i}, x_{n_i-1}). \end{aligned} \tag{2.25}$$

Again, if $i \to \infty$ by (2.19), (2.22), (2.24) and (2.25) we obtain

$$\begin{aligned} \frac{\varepsilon}{s} &\leq \limsup_{i\to\infty} d(x_{m_{i+1}}, x_{n_i})) \leq \limsup_{i\to\infty}(\frac{1 + sd(x_{m_i}, x_{n_{i-1}})}{1 + \frac{1}{2}d(x_{m_i}, fx_{m_i})}) \limsup_{i\to\infty} d(x_{m_{i+1}}, x_{n_i})) \\ &= \limsup_{i\to\infty}[(\frac{1 + sd(x_{m_i}, x_{n_{i-1}})}{1 + \frac{1}{2}d(x_{m_i}, fx_{m_i})})d(x_{m_{i+1}}, x_{n_i}))] \\ &\leq \limsup_{i\to\infty}[\beta(d(x_{m_i}, x_{n_i-1}))M(x_{m_i}, x_{n_i-1}) + LN(x_{m_i}, x_{n_i-1})] \\ &\leq \limsup_{i\to\infty} \beta(d(x_{m_i}, x_{n_i-1})) \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) + L \limsup_{i\to\infty} N(x_{m_i}, x_{n_i-1})] \\ &= \frac{1}{s} \limsup_{i\to\infty} \beta(d(x_{m_i}, x_{n_i-1})) \leq \frac{\varepsilon}{s} \end{aligned}$$

$$\limsup_{i\to\infty} \beta(d(x_{m_i}, x_{n_i-1})) = \tfrac{1}{s}. \tag{2.26}$$

So

$$\limsup_{i\to\infty} d(x_{m_i}, x_{n_i-1}) = 0,$$

which is a contradiction. Thus $\{x_n\}$ is a b-Cauchy sequence. Completeness of X yields that $\{x_n\}$ converges to a point $u \in X$, that is, $x_n \to u$ as $n \to \infty$.

Step III : Since f is continuous, u is a fixed point of f,

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fu.$$

□

Note that the continuity of f in Theorem 2.7 is not necessary and can be dropped.

Theorem 2.8. *Under the same hypotheses of Theorem* 2.7*, instead of the continuity assumption of* f*, assume that whenever* $\{x_n\}$ *is a nondecreasing sequence in* X *such that* $x_n \to u \in X$*, one has* $x_n \preceq u$ *for all* $n \in \mathbb{N}$*. Then* f *has a fixed point.*

Proof . Repeating the proof of Theorem 2.7, we construct an increasing sequence $\{x_n\}$ in X such that$x_n \to u \in X$. Using the assumption on X we have $x_n \preceq u$. Now, we show that $u = fu$. Suppose that there exists $n_0 \in N_1$such that

$$\frac{1}{2}d(x_{n_0}, fx_{n_0}) > sd(x_{n_0}, u)$$

and

$$\frac{1}{2}d(x_{n_0+1}, fx_{n_0+1}) > sd(x_{n_0+1}, u).$$

Then from (2.2), it follows that

$$\begin{aligned} d(x_{n_0+1}, x_{n_0}) &\leq sd(x_{n_0}, u) + sd(x_{n_0+1}, u) < \frac{1}{2}d(x_{n_0}, fx_{n_0}) + \frac{1}{2}d(x_{n_0+1}, fx_{n_0+1}) \\ &= \frac{1}{2}d(x_{n_0}, x_{n_0+1}) + \frac{1}{2}d(x_{n_0+1}, x_{n_0+2}) \leq \frac{1}{2}d(x_{n_0}, x_{n_0+1}) + \frac{1}{2}d(x_{n_0}, x_{n_0+1}) \\ &= d(x_{n_0+1}, x_{n_0}), \end{aligned}$$

which is a contradiction. Hence either

$$\frac{1}{2}d(x_n, fx_n) \leq sd(x_n, u)$$

and

$$\frac{1}{2}d(x_{n+1}, fx_{n+1}) \leq sd(x_{n+1}, u)$$

for all $n \in N_1$. It is not restrictive to assume that one of these inequalities holds for all $n \in N_1$, for example

$$\frac{1}{2}d(x_n, fx_n) \leq sd(x_n, u). \tag{2.27}$$

By (2.1) and (2.27) we have

$$\begin{aligned} d(u, fu) &= s[d(u, x_{n+1}) + d(x_{n+1}, fu)] \\ &\leq sd(u, x_{n+1}) + \beta(d(x_n, u))M(x_n, u) + LN(x_n, u) \to 0 \end{aligned} \tag{2.28}$$

because

$$\lim_{n\to\infty} M(x_n, u) = \lim_{n\to\infty} \max\{d(x_n, u), \frac{d(x_n, fx_n)d(u, fu)}{1 + d(fx_n, fu)}\} \tag{2.29}$$

$$= \lim_{n\to\infty} \max\{d(x_n, u), \frac{d(x_n, x_{n+1})d(u, fu)}{1 + d(x_{n+1}, fu)}\} \tag{2.30}$$

$$= \max\{0, 0\} \tag{2.31}$$

$$= 0$$

and

$$\lim_{n\to\infty} N(x_n, u) = \lim_{n\to\infty} \min\{d(x_n, fu), d(u, fx_n), d(x_n, fx_n), d(u, fu)\} \tag{2.32}$$

$$= \lim_{n\to\infty} min\{d(x_n, fu), d(u, x_{n+1}), d(x_n, x_{n+1}), d(u, fu)\} \tag{2.33}$$

$$= 0.$$

Therefore (2.28) implies $d(u, fu) = 0$. □

3. Coincidence point results

In this section we study some coincidence point theorem as follows.

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space. Assume $f, T : X \to X$ are such that f is an increasing mapping with respect to T, $fX \subseteq TX$ and there exists an element $x_0 \in X$ with $Tx_0 \preceq f(x_0)$. Suppose that (T, f) satisfy the following condition*

$$(\frac{1 + sd(x, y)}{1 + \frac{1}{2}d(x, fx)})d(fx, fy) \leq \beta(d(Tx, Ty))M^s(x, y) + LN^s(x, y) \tag{3.1}$$

for all comparable elements $x, y \in X$, where $L \geq 0$ and

$$M^s(x, y) = \max\left\{d(Tx, Ty), \frac{d(Tx, fx)d(Ty, fy)}{1 + d(fx, fy)}\right\}$$

and

$$N^s(x, y) = \min\{d(Tx, fx), d(Tx, fy), d(Ty, fx), d(Ty, fy)\}.$$

If f is continuous then (f, T) have a coincidence point.

Proof . Let $x_0 \in X$ and $x_1 \in X$ be such that $x_1 = Tx_0 \preceq fx_0$. Having defined $x_n \in X$, let $x_{n+1} \in X$ be such that $x_{n+1} = Tx_n \preceq fx_n$. By the same argument in the proof of Theorem 2.2, $\{x_n\}$ is a b-Cauchy sequence. Completeness of X yields that $\{x_n\}$ converges to a point $u \in X$ and the continuity of f implies (f, T) have a coincidence point. □

Theorem 3.2. Under the same hypotheses of Theorem 3.1, without the continuity assumption of f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, $x_n \preceq u$ for all $n \in \mathbb{N}$. Then (f, T) have a coincidence point.

Theorem 3.3. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space. Assume $f, T : X \to X$ are such that f is an increasing mapping with respect to T, $fX \subseteq TX$ and there exists an element $x_0 \in X$ with $Tx_0 \preceq f(x_0)$. Suppose that

$$s(\frac{1 + sd(x, y)}{1 + \frac{1}{2}d(x, fx)})d(fx, fy) \le \psi(M(x, y))$$

where

$$M(x, y) = \max\left\{d(Tx, Ty), \frac{d(Tx, fx)d(Ty, fy)}{1 + d(fx, fy)}\right\}$$

for all comparable elements $x, y \in X$. If f is continuous, then (f, T) have a coincidence point.

Theorem 3.4. Under the same hypotheses of Theorem 3.3, without the continuity assumption of f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, $x_n \preceq u$ for all $n \in \mathbb{N}$. Then (f, T) have a fixed point.

Example 3.5. Let $X = \{(0, 0), (4, 0), (0, 4)\}$ and define the partial order $\preceq$ on X by

$$\begin{aligned} \preceq \quad := & \{((0, 0), (0, 0)), ((4, 0), (4, 0)), ((0, 4), (0, 4)) \\ & ((0, 0), (0, 4)), ((0, 4), (4, 0)), ((0, 0), (4, 0))\} \end{aligned}$$

Consider the function $f : X \to X$ given as

$$\mathbf{f} = \begin{pmatrix} (0, 0) & (4, 0) & (0, 4) \\ (0, 4) & (4, 0) & (4, 0) \end{pmatrix}$$

which is increasing with respect to $\preceq$. Let $x_0 = (0, 0)$. Hence $f(x_0) = (0, 4)$, so $x_0 \preceq fx_0$. Define first the b-metric d on X by $d((0, 0), (4, 0)) = 4$, $d((0, 0), (0, 4)) = 6$, $d((0, 4), (4, 0)) = \frac{1}{4}$ and $d(x, x) = 0$. Then (X, d) is a b-complete b-metric space with $s = \frac{24}{17}$. Define the function $\beta \in \mathcal{F}$ given by

$$\beta(t) = \frac{17}{24}e^{\frac{-t}{6}} \ , t > 0$$

and $\beta(0) \in [0, \frac{17}{24})$ and $L = 100000$. Then

$$\begin{aligned} & (\frac{1 + \frac{24}{17}d((0, 0), (0, 4))}{1 + \frac{1}{2}d((0, 0), f(0, 0))})d(f(0, 0), f(0, 4)) \\ = \ & (\frac{1 + \frac{24}{17}d((0, 0), (0, 4))}{1 + \frac{1}{2}d((0, 0), (0, 4))})d((0, 4), (4, 0)) = (\frac{1 + \frac{24}{17}6}{1 + \frac{1}{2}6})\frac{1}{4} = \frac{151}{272} \\ \le \ & \beta(d((0, 0), (0, 4)))M((0, 0), (0, 4)) + 100000N((0, 0), (0, 4)) = \beta(6)6. \end{aligned}$$

Because

$$\begin{aligned} M((0, 0), (0, 4)) &= \max\{d((0, 0), (0, 4)), \frac{d((0, 0), f(0, 0))d((0, 4), f(0, 4))}{1 + d(f(0, 0), f(0, 4))}\} \\ &= \max\{d((0, 0), (0, 4)), \frac{d((0, 0), (0, 4))d((0, 4), (4, 0))}{1 + d((0, 4), (4, 0))}\} \\ &= \max\{6, \frac{6 \times \frac{1}{4}}{1 + \frac{1}{4}}\} = 6 \end{aligned}$$

and

$$\begin{aligned} N((0,0),(0,4)) &= \min\{d((0,0),f(0,0)),d((0,0)),f(0,4)),\\ &\quad d((0,4)),f(0,0)),d((0,4)),f(0,4))\}\\ &= \min\{d((0,0),(0,4)),d((0,0)),(4,0)),\\ &\quad d((0,4)),(0,4)),d((0,4)),(4,0))\}\\ &= 0. \end{aligned}$$

Also

$$sd(f0,f1)=\frac{18}{13}d(3,1)=\frac{18}{13}.\frac{1}{2}\leq\beta(d(0,1))M(0,1)+LN(0,1)\leq\beta(6)M(0,1)+LN(0,1)=\beta(6)6.$$

$$\begin{aligned} &(\frac{1+\frac{24}{17}d((0,0),(4,0))}{1+\frac{1}{2}d((0,0),f(0,0))})d(f(0,0),f(4,0))\\ =\ &(\frac{1+\frac{24}{17}d((0,0),(4,0))}{1+\frac{1}{2}d((0,0),(0,4))})d((0,4),(4,0))=(\frac{1+\frac{24}{17}4}{1+\frac{1}{2}6})\frac{1}{4}=\frac{113}{272}\\ \leq\ &\beta(d((0,0),(4,0)))M((0,0),(4,0))+100000N((0,0),(4,0))=\beta(4)4. \end{aligned}$$

Because

$$\begin{aligned} M((0,0),(4,0)) &= \max\{d((0,0),(4,0)),\frac{d((0,0),f(0,0))d((4,0),f(4,0))}{1+d(f(0,0),f(0,4))}\}\\ &= \max\{d((0,0),(0,4)),\frac{d((0,0),(0,4))d((4,0),(4,0))}{1+d((0,4),(4,0))}\}\\ &= \max\{4,0\}=4. \end{aligned}$$

Also

$$\begin{aligned} &(\frac{1+\frac{24}{17}d((4,0),(0,4))}{1+\frac{1}{2}d((4,0),f(4,0))})d(f(4,0),f(0,4))\\ =\ &(\frac{1+\frac{24}{17}d((4,0),(0,4))}{1+\frac{1}{2}d((4,0),(4,0))})d((4,0),(4,0))=0\\ \leq\ &\beta(d((4,0),(0,4)))M((4,0),(0,4))+100000N((4,0),(0,4)) \end{aligned}$$

Hence f satisfies all the assumptions of Theorem 2.7 and thus it has a fixed point (which is $u=(4,0)$).

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