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Additive Maps Preserving the Essential Points Between Weak Hypervector Spaces

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Abstract

IIn this paper, we introduce the concept of regular linear operators. We prove that the regular linear operators preserve the normality, linear independence and dimension of weak hypervector spaces and prove some important results.

Keywords: Essential point; Regular linear operator; Weak hypervector space.
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1. Introduction

The concept of hyperstructure was first introduced by Marty [3] in 1934 and has attracted attention of many authors in last decades and has constructed some other structures such as hyperrings, hypergroups, hypermodules, hyperfields, and hypervector spaces. These constructions has been applied to many disciplines such as geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, and etc. A wealth of applications of this concepts are given in [1], [2], [14] and [15]. In 1988 the concept of hypervector space was first introduced by Scafati-Tallini. She studied more properties of this new structure in [13]. We considered this generalization of vector space in the viewpoint of analysis and proved important results in this field. See [4, 5, 6, 7, 8, 9, 10, 11, 12].

Note that the hypervector spaces used in this paper are the essential case where there is only one hyperoperation, except for one, the rest are ordinary operations. The general hypervector spaces have all operations multivalued also in the hyperfield (see [15]).

In [4], we showed the existence of essential points in weak hypervector spaces. In this paper, we introduce the concept of regular linear operators. In fact, the regular linear operators are additive operators preserving the essential points. We prove that the regular operators preserve the normality, linear independence and dimension of weak hyper vector spaces and also we prove some important results about them.

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2. Preliminaries

Definition 2.1. *[13] A weak or weakly distributive hypervectorspace over a field F is a quadruple $(X,+,o,F)$ such that $(X,+)$ is an abelian group and $o : F \times X \longrightarrow P_*(X)$ is a multivalued product such that:*

$$\forall a \in F, \forall x, y \in X,\ [ao(x+y)] \cap [aox + aoy] \neq \emptyset, \tag{1}$$

$$\forall a, b \in F, \forall x \in X,\ [(a+b)ox] \cap [aox + box] \neq \emptyset, \tag{2}$$

$$\forall a, b \in F, \forall x \in X,\ ao(box) = (ab)ox, \tag{3}$$

$$\forall a \in F, \forall x \in X,\ ao(-x) = (-a)ox = -(aox), \tag{4}$$

$$\forall x \in X,\ x \in 1ox. \tag{5}$$

We call (1) and (2) weak right and left distributive laws, respectively. Note that the set $ao(box)$ in (3) is of the form $\cup_{y \in box} aoy$.

In throughout of this paper, assume that X and Y are weak hypervector spaces over a field F.

Definition 2.2. *[4] Let $a \in F$ and $x \in X$. Essential point of aox, that we denote it by e_{aox}, for $a \neq 0$ is the element of aox such that $x \in a^{-1}oe_{aox}$. For $a = 0$, we define $e_{aox} = 0$.*

As stated in [4], e_{aox} is not unique, necessarily. So the set of all these elements denoted by E_{aox}. In the mentioned paper we introduced a certain category of weak hypervector spaces. These weak hypervector spaces have been called "normal". In [4], the following lemma stated a criterion for normality of a weak hypervector space.

Lemma 2.3. *[4] If $a \in F$, $0 \neq b \in F$ and $x \in X$, then the following properties hold.*
(a) $x \in E_{1ox}$.
(b) $aoE_{box} = abox$.
(c) $E_{-aox} = -E_{aox}$.
Furthermore, if X is normal, then
(d) E_{aox} is singleton.

Lemma 2.4. *[4] X is normal if and only if*

$$e_{a_1ox} + e_{a_2ox} = e_{(a_1+a_2)ox},\ \forall x \in X,\ \forall a_1, a_2 \in F,$$

$$e_{aox_1} + e_{aox_2} = e_{ao(x_1+x_2)},\ \forall x_1, x_2 \in X,\ \forall a \in F.$$

Definition 2.5. *[4] A subset $M = \{x_1, ..., x_n\}$ of X is said to be linearly independent if the equation $0 = \sum_{i=1}^{n} e_{\alpha_i ox_i}$ implies that $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$, where $\alpha_1, ..., \alpha_n$ are scalars. M is said to be linearly dependent if M isn't linearly independent.*

An arbitrary subset M of X is linearly independent if every nonempty finite subset of M is linearly independent.

Definition 2.6. *[4] A subset M of X is said to be a basis of X if M is linearly independent and spans the elements of X. It means that for any x of X there exists scalars $\alpha_1, ..., \alpha_n$ such that $x = \sum_{i=1}^{n} e_{\alpha_i o x_i}$, where $\{x_1, ..., x_n\}$ is a subspace of M. If there exists a finite basis for X, then X is said to be a finite dimensional weak hypervector space.*

Definition 2.7. *[4] Let X be a weak hypervector space over F. A nonempty subset M of X is called a weak subhypervector space of X, when M satisfies the following properties:*
(i) $x + y \in M, \ \forall x, y \in M,$
(ii) $e_{aox} \in M, \ \forall a \in F, \ \forall x \in M.$

Theorem 2.8. *[4] Let X be a normal weak hypervector space over F and $\emptyset \neq S \subseteq X$. Then the following set is the smallest weak subhypervector space of X containing S:*

$$[S] = \{\sum_{i=1}^{n} e_{a_i o s_i}; a_i \in F, s_i \in S, n \in \mathbb{N}\}.$$

Definition 2.9. *A map $T : X \to Y$ is called weak linear operator if T is additive and satisfies*

$$T(e_{aox}) \subseteq aoTx$$

for all $x \in X$ and $a \in F$.

We denote the set of all weak linear operators by $L_w(X, Y)$.

Theorem 2.10. *Let X and Y be weak hypervector spaces over F. Then $L_w(X, Y)$ with the following sum and product is a weak hypervector space over F.*

$$(T + S)x = Tx + Sx \quad (T, S \in L_w(X, Y), x \in X)$$

$$(aoT)x = aoTx \quad (a \in F, T \in L_w(X, Y), x \in X).$$

3. Main results

Definition 3.1. *Let $T : X \to Y$ be a weak linear operator. T is called a regular linear operator when satisfies*

$$T(E_{aox}) = E_{aoTx}$$

for all $x \in X$ and $a \in F$. We denote the set of all regular linear operators by $R_w(X, Y)$.

Example 3.2. *The following example of weak hypervector space was stated in [4]. We restate it. The set $\mathbb{C}$ with usual sum and the following scalar product is a weak hypervector space on $\mathbb{R}$.*

$$aox = \begin{cases} \{re^{i\theta} : 0 \leq r \leq |a||x|, \theta = \arg(x)\} & x \neq 0 \\ \{0\} & x = 0 \end{cases}$$

Let X be the defined weak hypervector space and $T : X \to X$ be a map such that for every $x \in X$

$$Tx = x + x.$$

We see that $T \in R_w(X, Y)$, because by the normality of $\mathbb{C}$, we have

$$T(e_{aox}) = e_{aox} + e_{aox} = e_{ao(x+x)} = e_{aoTx}.$$

Example 3.3. *Let X be normal, $0 \neq b \in F$ and $T : X \to X$ be a map such that for every $x \in X$*

$$Tx = e_{box}.$$

We show $T \in R_w(X,Y)$. Since X is normal, we have

$$\begin{aligned} T(x_1 + x_2) &= e_{ao(x_1+x_2)} \\ &= e_{aox_1} + e_{aox_2} \\ &= Tx_1 + Tx_2. \end{aligned}$$

So T is additive. By Lemma 2.3, $e_{aoe_{box}} = e_{abox}$. Thus, we have

$$\begin{aligned} T(e_{aox}) &= e_{boe_{aox}} = e_{abox} \\ &= e_{aoe_{box}} = e_{aoTx}. \end{aligned}$$

Proposition 3.4. *Let $T \in R_w(X,Y)$ be a bijective map. Then $T^{-1} \in R_w(X,Y)$.*

Proof . It's clear that T^{-1} is additive. Let $y \in Y$ and $a \in F$. So there exists $x \in X$ such that $x = T^{-1}y$.

$$\begin{aligned} T^{-1}(e_{aoy}) &= T^{-1}(e_{aoTx}) = T^{-1}(T(e_{aox})) \\ &= e_{aox} = e_{aoT^{-1}y}. \end{aligned}$$

The proof is complete. □

In the following propositions, we prove that a regular linear operator preserves the normality of weak hypervector spaces.

Proposition 3.5. *Let $T \in R_w(X,Y)$. Then the following statements are hold.*
(i) If T is surjective and X is normal, then Y is normal.
(ii) If T is injective and Y is normal, then X is normal.

Proof . (i) Suppose that y_1 ,$y_2 \in Y$. Since T is surjective, there exist $x_1, x_2 \in X$ such that $Tx_1 = y_1$ and $Tx_2 = y_2$. Since X is normal, by Lemma 2.4, for all $a \in F$ we have

$$\begin{aligned} e_{aoy_1} + e_{aoy_2} &= e_{aoTx_1} + e_{aoTx_2} = T(e_{aox_1}) + T(e_{aox_2}) \\ &= T(e_{aox_1} + e_{aox_2}) = T(e_{ao(x_1+x_2)}) \\ &= e_{ao(Tx_1+Tx_2)} = e_{ao(y_1+y_2)}. \end{aligned}$$

Similarly, can prove

$$e_{aoy} + e_{boy} = e_{(a+b)oy}, \quad (\forall a, b \in F, \quad y \in Y).$$

Thus by Lemma 2.4, Y is normal.
(ii) Suppose that $x_1, x_2 \in X$. For all $a \in F$, we have

$$\begin{aligned} T(e_{aox_1} + e_{aox_2}) &= T(e_{aox_1}) + T(e_{aox_2}) = e_{aoTx_1} + e_{aoTx_2} \\ &= e_{ao(Tx_1+Tx_2)} = e_{aoT(x_1+x_2)} \\ &= T(e_{ao(x_1+x_2)}). \end{aligned}$$

Since T is injective, we obtain

$$e_{aox_1} + e_{aox_2} = e_{ao(x_1+x_2)},$$

similarly, can prove

$$e_{aox} + e_{box} = e_{(a+b)ox} \qquad (\forall a, b \in F, \quad x \in X)$$

Thus by Lemma 2.4, X is normal. □

Proposition 3.6. *Let $T : X \to Y$ be a weak linear operator. Then the following statements are hold.*
(i) $kerT$ is a weak subhypervector space of X.
(ii) If $T \in R_w(X, Y)$, then ImT is a weak subhypervector space of Y.

Proof . (i) Suppose $x, y \in kerT$ and $a \in F$. By Definition 2.7, it is enough to show that $x+y \in kerT$ and $e_{aox} \in kerT$. For all $x, y \in kerT$, it is easy to check that $x + y \in kerT$. By Definition 2.9, we have $T(E_{aox}) \subseteq aoTx$. Since $Tx = 0$, $aoTx = 0$, $T(e_{aox}) = 0$. So $e_{aox} \in kerT$. This completes the proof.

(ii) Suppose that $y_1, y_2 \in ImT$ and $a \in F$. So there exist $x_1, x_2 \in X$ such that $y_1 = Tx_1$ and $y_2 = Tx_2$. It is easy to check that $y_1 + y_2 \in ImT$. By Definition 3.1, we have

$$E_{aoy_1} = E_{aoTx_1} = T(E_{aox_1})$$

which implies that $e_{aoy_1} \in ImT$ and this completes the proof. □

Lemma 3.7. *Suppose that X is normal and $S \subseteq X$ contains linear independent vectors. If $x \notin [S]$, then $S \cup \{x\}$ contains linear independent vectors.*

Proof . Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta \in F$ and $x_1, x_2, \ldots, x_n \in S$ such that

$$\sum_{i=1}^{n} e_{\alpha_i ox_i} + e_{\beta ox} = 0.$$

We assert that $\beta = 0$. If $\beta \neq 0$, then

$$\beta^{-1} oe_{\beta ox} = -\beta^{-1} o \sum_{i=1}^{n} e_{\alpha_i ox_i}.$$

By Lemma 2.3, we obtain

$$1ox = -\beta^{-1} oy,$$

where $y = \sum_{i=1}^{n} e_{\alpha_i ox_i}$. The above relation yields

$$e_{1ox} = e_{-\beta^{-1}oy}.$$

By Theorem 2.8, $[S]$ is a weak subhypervector space and hence $e_{-\beta^{-1}oy} \in [S]$, because $y \in [S]$. This implies that e_{1ox} that is equal to x, belongs to $[S]$. This is a contradiction. So $\beta = 0$.

By the proved assertion we have

$$\sum_{i=1}^{n} e_{\alpha_i ox_i} = 0,$$

which implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, because $x_1, x_2, \ldots, x_n$ are linear independent. This completes the proof. □

Corollary 3.8. *If X is normal and finite dimensional, then every linear independent of vectors are part of a basis for X.*

Proof . By Lemma 3.7 is clear. □

Theorem 3.9. *Let $T \in R_w(X, Y)$. If X is normal and n-dimensional, then*

$$dim(ImT) + dim(kerT) = dimX.$$

Proof . Let $dim(kerT) = k$. So there exists linear independent set $\{x_1, x_2, \ldots, x_k\} \subseteq kerT$ such that is a basis for $kerT$. It is clear that for all $1 \leq j \leq k$, $Tx_j = 0$.

By Corollary 3.8, there exist $x_{k+1}, x_{k+2}, \ldots, x_n$ in X, such that $\{x_1, \ldots, x_n\}$ is a basis for X. We show that $\{Tx_{k+1}, Tx_{k+2}, \ldots, Tx_n\}$ is a basis for ImT. Let $y \in ImT$. So there exists $x \in X$ such that $y = Tx$. Hence there exist $\alpha_1, \alpha_2, ..., \alpha_n \in F$ such that $x = \sum_{i=1}^{n} e_{\alpha_i o x_i}$ and so

$$y = Tx = T(\sum_{i=1}^{n} e_{\alpha_i o x_i}) = \sum_{i=1}^{n} T(e_{\alpha_i o x_i}) = \sum_{i=1}^{n} e_{\alpha_i o T x_i} = \sum_{i=k+1}^{n} e_{\alpha_i o T x_i}.$$

This implies that $ImT = [Tx_{k+1}, Tx_{k+2}, \ldots, Tx_n]$. Now we show that $\{Tx_{k+1}, Tx_{k+2}, \ldots, Tx_n\}$ is linear independent. Suppose there exist $c_{k+1}, c_{k+2}, \ldots, c_n \in F$ such that $\sum_{i=k+1}^{n} e_{c_i o T x_i} = 0$. This implies

$$T(\sum_{i=k+1}^{n} e_{c_i o x_i}) = 0,$$

and so $\sum_{i=k+1}^{n} e_{c_i o x_i} \in kerT$. Therefore there exist $b_1, b_2, \ldots, b_k \in F$ such that

$$\sum_{i=k+1}^{n} e_{c_i o x_i} = \sum_{i=1}^{k} e_{b_i o x_i}.$$

Since $\{x_1, x_2, \ldots, x_n\}$ is linear independent, $b_1 = b_2 = \cdots = b_k = c_{k+1} = c_{k+2} = \cdots = c_n = 0$. Thus $\{Tx_{k+1}, Tx_{k+2}, \ldots, Tx_n\}$ is linear independent and so by Definition 2.6, the proof is complete. □
In the following lemma we prove that a regular linear operator preserves the linear independence of vectors.

Lemma 3.10. *Let $T \in R_w(X, Y)$. T is injective if and only if for any linear independent set $\{x_1, x_2, \ldots, x_n\}$, $\{Tx_1, Tx_2, \ldots, Tx_n\}$ is linear independent.*

Proof . First let T be injective. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that $\sum_{i=1}^{n} e_{\alpha_i o T x_i} = 0$. Therefore

$$\sum_{i=1}^{n} T(e_{\alpha_i o x_i}) = T(\sum_{i=1}^{n} e_{\alpha_i o x_i}) = 0.$$

Since T is injective, we obtain

$$\sum_{i=1}^{n} e_{\alpha_i o x_i} = 0$$

which implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, because $x_1, x_2, \ldots, x_n$ are linear independent.

Conversely, let $Tx = 0$. If $x \neq 0$, then Since $\{x\}$ is linear independent, by assumption $\{Tx\}$ is linear independent. This is contradiction, because $Tx = 0$. The proof is completed. □

Lemma 3.11. *Let X be normal. If X is m-dimensional, then every linear independent set of X has at most m elements.*

Proof . Let $\{x_1, \ldots, x_m\}$ be a basis for X and $\{y_1, \ldots, y_n\}$ be a linear independent set of X. Thus for any $1 \leq j \leq n$ there exist $\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{mj} \in F$ such that $y_j = \sum_{i=1}^{m} e_{\alpha_{ij} o x_i}$. Let $c_1, \ldots, c_n \in F$.

By Lemmas 2.3 and 2.4, we have

$$\begin{aligned}
e_{c_1oy_1}+\cdots+e_{c_noy_n} &= e_{c_1o\sum_{i=1}^{m}e_{\alpha_{i1}ox_i}}+\cdots+e_{c_no\sum_{i=1}^{m}e_{\alpha_{in}ox_i}}\\
&= \sum_{i=1}^{m}e_{c_1o(e_{\alpha_{i1}ox_i})}+\cdots+\sum_{i=1}^{m}e_{c_no(e_{\alpha_{in}ox_i})}\\
&= \sum_{i=1}^{m}e_{c_1\alpha_{i1}ox_i}+\cdots+\sum_{i=1}^{m}e_{c_1\alpha_{in}ox_i}\\
&= \sum_{j=1}^{n}e_{c_j\alpha_{1j}ox_1}+\cdots+\sum_{j=1}^{n}e_{c_j\alpha_{mj}ox_m}\\
&= e_{(\sum_{j=1}^{n}c_j\alpha_{1j})ox_1}+\cdots+e_{(\sum_{j=1}^{n}c_j\alpha_{mj})ox_m}.
\end{aligned}$$

If $e_{c_1oy_1}+\cdots+e_{c_noy_n}=0$, then $c_1=\cdots=c_n=0$. Since $\{x_1,\ldots,x_m\}$ is linear independent, by above relation we obtain

$$\sum_{j=1}^{n}c_j\alpha_{1j}=\cdots=\sum_{j=1}^{n}c_j\alpha_{mj}=0.$$

Assume on the contrary that $n>m$. The remain of proof is the same proof of this lemma in the classical vector space. With the same reason we can conclude that there exist at least a nonzero c_j and this contradiction completes the proof. $\square$

Lemma 3.12. *Let X be normal, finite dimensional and W be a weak subhypervector space of X. If $dimW=dimX$, then $W=X$.*

Proof . Assume on the contrary that $X\neq W$. Thus there exists $x\in X\setminus W$. If B is a basis for W, then $B\cup\{x\}$ by Lemma 3.7 is a linear independent set with $dimX+1$ elements. By Lemma 3.11, this is a contradiction. Hence $W=X$ and so the proof is completed. $\square$

Theorem 3.13. *Let X be normal and $\{x_1,x_2,\ldots,x_n\}$ be a basis for X. If $y_1,\ldots,y_n$ are arbitrary vectors of Y, then there exists an unique regular linear operator $T:X\to Y$ such that for any $1\leq i\leq n$, $Tx_i=y_i$.*

Proof . Let $x\in X$. Since $\{x_1,x_2,\ldots,x_n\}$ is a basis for X, there exist $c_1,\ldots,c_n\in F$ such that $x=\sum_{i=1}^{n}e_{c_iox_i}$. Now define $T:X\to Y$ with $Tx=\sum_{i=1}^{n}e_{c_ioy_i}$. We show that T is a regular linear operator such that for any $1\leq i\leq n$, $Tx_i=y_i$. Let $x,y\in X$. So there exist $a_1,\ldots,a_n,b_1,\ldots,b_n\in F$ such that $x=\sum_{i=1}^{n}e_{a_iox_i}$ and $y=\sum_{i=1}^{n}e_{b_iox_i}$. Thus $Tx=\sum_{i=1}^{n}e_{a_ioy_i}$ and $Ty=\sum_{i=1}^{n}e_{b_ioy_i}$. By Lemma 2.4, we have $x+y=\sum_{i=1}^{n}e_{(a_i+b_i)ox_i}$ and so

$$\begin{aligned}
T(x+y)=\sum_{i=1}^{n}e_{(a_i+b_i)oy_i} &= \sum_{i=1}^{n}e_{a_ioy_i}+e_{b_ioy_i}\\
&= \sum_{i=1}^{n}e_{a_ioy_i}+\sum_{i=1}^{n}e_{b_ioy_i}\\
&= Tx+Ty,
\end{aligned}$$

which implies that T is additive. Now, let $a \in F$. By Lemmas 2.3 and 2.4, we have

$$\begin{aligned} T(e_{aox}) &= T(e_{ao\sum_{i=1}^{n} e_{a_i ox_i}}) = T(\sum_{i=1}^{n} e_{ao(e_{a_i ox_i})}) \\ &= T(\sum_{i=1}^{n} e_{aa_i ox_i}) = \sum_{i=1}^{n} e_{aa_i oy_i} \\ &= \sum_{i=1}^{n} e_{ao(e_{a_i oy_i})} = e_{ao\sum_{i=1}^{n} e_{a_i oy_i}} \\ &= e_{aoTx} \end{aligned}$$

which implies that T preserves the essential points. So T is a regular linear operator. Since X is normal, by Lemma 2.3, for any $1 \leq i \leq n$, $x_i = e_{1ox_i}$ which implies $Tx_i = e_{1oy_i} = y_i$.

Finally, we show the uniqueness of such T. Suppose U is a regular linear operator such that for any $1 \leq i \leq n$, $Ux_i = y_i$. We have

$$Ux = U(\sum_{i=1}^{n} e_{a_i ox_i}) = \sum_{i=1}^{n} e_{a_i oUx_i} = \sum_{i=1}^{n} e_{a_i oy_i} = Tx$$

which implies $U = T$ and this completes the proof. □

Theorem 3.14. *Let X and Y be finite dimensional, X be normal and $dimX = dimY$. If $T \in R_w(X, Y)$, then the following statements are equivalent.*
(i) T is invertible.
(ii) T is injective.
(iii) T is surjective.
(iv) If $\{x_1, x_2, \dots, x_n\}$ is a basis for X, then $\{Tx_1, Tx_2, \dots, Tx_n\}$ is a basis for Y.
(v) There exists a basis $\{x_1, x_2, \dots, x_n\}$ for X such that $\{Tx_1, Tx_2, \dots, Tx_n\}$ is a basis for Y.

Proof . $(i) \Rightarrow (ii)$: It is clear.

$(ii) \Rightarrow (iii)$: By Theorem 3.9, we have $dim(ImT) + dim(kerT) = dimX$. Since T is injective, $kerT = \{0\}$ and so $dim(kerT) = 0$. This implies that $dim(ImT) = dimX$ and by assumption we obtain

$$dim(ImT) = dimY.$$

From Proposition 3.6, ImT is a weak subhypervector space of Y and so by Lemma 3.12, $ImT = Y$. Therefore, T is surjective.

$(iii) \Rightarrow (iv)$: Let $y \in Y$. Since T is surjective, there exists $x \in X$ such that $Tx = y$. Since $\{x_1, x_2, ..., x_n\}$ is a basis for X, there exist $c_1, ..., c_n \in F$ such that $x = \sum_{i=1}^{n} e_{c_i ox_i}$. Thus we have

$$y = Tx = T(\sum_{i=1}^{n} e_{c_i ox_i}) = \sum_{i=1}^{n} e_{c_i oTx_i}.$$

This implies that $Y = [Tx_1, Tx_2, ..., Tx_n]$. Since by assumption, $dimY = dimX = n$, $Tx_1, Tx_2, ..., Tx_n$ are linear independent. Otherwise, if $Tx_1, Tx_2, ..., Tx_n$ are linear dependent, then $dimY < n$. Therefore, $\{Tx_1, Tx_2, ..., Tx_n\}$ is a basis for Y.

$(iv) \Rightarrow (v)$: It is clear.

$(v) \Rightarrow (1)$: By Theorem 3.13, there exists an unique regular linear operator U such that $U(Tx_i) = x_i$, for any $1 \leq i \leq n$. Since UT is a regular linear operator, it is easy to check that for any $x \in X$,

$UTx = x$. So $UT = I_X$. Now let $y \in Y$. Thus there exist $c_1, ..., c_n \in F$ such that $y = \sum_{i=1}^{n} e_{c_i oTx_i}$ and we obtain $y = T(\sum_{i=1}^{n} e_{c_i ox_i})$ and so

$$TUy = TUT(\sum_{i=1}^{n} e_{c_i ox_i}) = T(\sum_{i=1}^{n} e_{c_i ox_i}) = \sum_{i=1}^{n} e_{c_i oTx_i} = y$$

which implies $TU = I_Y$ and this completes the proof. □

Proposition 3.15. *If Y is normal, then $R_w(X, Y)$ is a weak subhypervector space of $L_w(X, Y)$.*

Proof . It is enough to show that $T + S$ and $E_{aoT} \subseteq R_w(X, Y)$ for all $a \in F$ and $T, S \in R_w(X, Y)$. Since Y is normal, we have

$$\begin{aligned} (S+T)(e_{aox}) &= T(e_{aox}) + S(e_{aox}) \\ &= e_{aoTx} + e_{aoSx} \\ &= e_{ao(Tx+Sx)} \\ &= e_{ao(T+S)x} \end{aligned}$$

so $T + S \in R_w(X, Y)$. It is easy to check that $e_{aoT}(x) = e_{aoTx}$, for all $x \in X$. Let $b \in F$ and set $T_1 = e_{aoT}$. Hence by Lemma 2.3, we have

$$\begin{aligned} T_1(e_{box}) = e_{aoT}(e_{box}) &= e_{aoT(e_{box})} = e_{aoe_{boTx}} \\ &= e_{aboTx} = e_{boe_{aoTx}} \\ &= e_{boe_{aoT}(x)} = e_{boT_1x}. \end{aligned}$$

This completes the proof. □

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