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Covarian mappings and coupled fixed point results in bipolar metric spaces

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Abstract

In this paper, we establish the existence and uniqueness of common coupled fixed point results for three covariant mappings in bipolar metric spaces. Moreover, we give an illustration which presents the applicability of the achieved results also we provided applications to homotopy theory as well as integral equations.

Keywords: Bipolar metric space, ω-compatible mappings, Completeness, Common coupled fixed point.
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1. Introduction and Preliminaries

This work is motivated by the recent work on extension of Banach contraction principle [5] on bipolar metric spaces, which has been done by Mutlu and Gürdal [20]. Also, they investigated some fixed point and coupled fixed point results on this spaces (see [21, 22]). Subsequently, many authors established coupled fixed point theorems in different spaces (see [1, 2, 3, 6, 7, 8, 11, 12, 13, 14, 15, 16, 18, 19, 23, 25, 26]).

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The aim of this paper is to initiate the study of a common coupled fixed point results for three mappings under various contractive conditions in bipolar metric spaces. Finally, we give an example which presents the applicability of our achieved results also we provided applications to homotopy theory as well as integral equations.

First we recall some definitions and examples as follows.

Definition 1.1 ([20]). *Let A and B be a two nonempty sets. Suppose that $d : A \times B \to [0, \infty)$ is a mapping satisfying the following properties:*

(B_0) *If $d(a, b) = 0$ then $a = b$ for all $(a, b) \in A \times B$,*
(B_1) *If $a = b$ then $d(a, b) = 0$, for all $(a, b) \in A \times B$,*
(B_3) *If $d(a, b) = d(b, a)$, for all $a, b \in A \cap B$,*
(B_4) *If $d(a_1, b_2) \leq d(a_1, b_1) + d(a_2, b_1) + d(a_2, b_2)$ for all $a_1, a_2 \in A$, $b_1, b_2 \in B$.*

Then the mapping d is called a bipolar-metric on the pair (A, B) and the triple (A, B, d) is called a bipolar-metric space.

Definition 1.2 ([20]). *Assume (A_1, B_1) and (A_2, B_2) as two pairs of sets.*

The function $F : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a covariant map, if $F(A_1) \subseteq A_2$ and $F(B_1) \subseteq B_2$ and denote this as $F : (A_1, B_1) \rightrightarrows (A_2, B_2)$.

The mapping $F : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a contravariant map, if $F(A_1) \subseteq B_2$ and $F(B_1) \subseteq A_2$ and this as $F : (A_1, B_1) \leftrightarrows (A_2, B_2)$.

In particular, if d_1 and d_2 are bipolar metrics in (A_1, B_1) and (A_2, B_2) respectively. Then in some times we use the notations $F : (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ and $F : (A_1, B_1, d_1) \leftrightarrows (A_2, B_2, d_2)$.

Definition 1.3 ([20]). *Let (A, B, d) be a bipolar metric space. A point $v \in A \cup B$ is said to be left point if $v \in A$, a right point if $v \in B$ and a central point if both.*

Similarly, a sequence $\{a_n\}$ on the set A and a sequence $\{b_n\}$ on the set B are called a left and right sequence respectively.

In a bipolar metric space, sequence is the simple term for a left or right sequence.

A sequence $\{v_n\}$ is convergent to a point v if and only if $\{v_n\}$ is a left sequence, v is a right point and $\lim_{n\to\infty} d(v_n, v) = 0$; or $\{v_n\}$ is a right sequence, v is a left point and $\lim_{n\to\infty} d(v, v_n) = 0$.

A bisequence $(\{a_n\}, \{b_n\})$ on (A, B, d) is sequence on the set $A \times B$. If the sequence $\{a_n\}$ and $\{b_n\}$ are convergent, then the bisequence $(\{a_n\}, \{b_n\})$ is said to be convergent. $(\{a_n\}, \{b_n\})$ is Cauchy sequence , if $\lim_{n,m\to\infty} d(a_n, b_m) = 0$.

A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition 1.4 ([21]). *Let (A, B, d) be a bipolar metric space, $F : (A^2, B^2) \rightrightarrows (A, B)$ be a covariant mapping. If $F(a, b) = a$ and $F(b, a) = b$ for $(a, b) \in A^2 \cup B^2$ then (a, b) is called a coupled fixed point of F.*

2. Main Results

In this section, we give some common coupled fixed point theorems for three covariant mappings in bipolar metric spaces.

Definition 2.1. *Let (A,B,d) be a bipolar metric space, $F:(A^2,B^2)\rightrightarrows(A,B)$ and $f:(A,B)\rightrightarrows(A,B)$ be two covariant mappings. An element (a,b) is said to be a coupled coincident point of F and f. If $F(a,b)=fa$ and $F(b,a)=fb$.*

Definition 2.2. *Let (A,B,d) be a bipolar metric space, $F:(A^2,B^2)\rightrightarrows(A,B)$ and $f:(A,B)\rightrightarrows(A,B)$ be two covariant mappings. An element (a,b) is said to be a common coupled fixed point of F and f. If $F(a,b)=fa=a$ and $F(b,a)=fb=b$.*

Definition 2.3. *Let (A,B,d) be a bipolar metric space, $F:(A^2,B^2)\rightrightarrows(A,B)$ and $f:(A,B)\rightrightarrows(A,B)$ be two covariant mappings are called ω-compatible if $f(F(a,b))=F(fa,fb)$ and $f(F(b,a))=F(fb,fa)$ whenever $F(a,b)=fa$ and $F(b,a)=fb$.*

Theorem 2.4. *Let (A,B,d) be a bipolar metric space and $F,G:(A^2,B^2)\rightrightarrows(A,B)$, $f:(A,B)\rightrightarrows(A,B)$ be covariant mappings satisfying:*

(i) for all $a,b\in A$ and $p,q\in B$ with $\theta\in(0,1)$,

$$d(F(a,b),G(p,q))\leq\theta\max\{d(fa,fp),d(fb,fq)\},$$

(ii) $F(A^2\cup B^2)\cup G(A^2\cup B^2)\subseteq f(A\cup B)$,
(iii) either (F,f) or (G,f) are ω-compatible,
(iv) $f(A\cup B)$ is complete.

Then the mappings F,G and f have a unique common fixed point of the form (u,u).

Proof . Let $a_0,b_0\in A$ and $p_0,q_0\in B$ and from (ii), we construct the bisequence $(\{a_{2n}\},\{p_{2n}\})$, $(\{b_{2n}\},\{q_{2n}\})$, $(\{\omega_{2n}\},\{\chi_{2n}\})$ and $(\{\xi_{2n}\},\{\kappa_{2n}\})$ in (A,B) as

$$\begin{array}{ll}
F(a_{2n},b_{2n})=fa_{2n+1}=\omega_{2n}, & G(p_{2n},q_{2n})=fp_{2n+1}=\chi_{2n},\\
F(b_{2n},a_{2n})=fb_{2n+1}=\xi_{2n}, & G(q_{2n},p_{2n})=fq_{2n+1}=\kappa_{2n},\\
G(a_{2n+1},b_{2n+1})=fa_{2n+2}=\omega_{2n+1}, & F(p_{2n+1},q_{2n+1})=fp_{2n+2}=\chi_{2n+1},\\
G(b_{2n+1},a_{2n+1})=fb_{2n+2}=\xi_{2n+1}, & F(q_{2n+1},p_{2n+1})=fq_{2n+2}=\kappa_{2n+1},
\end{array}$$

for $n=0,1,2,\ldots$.
Now from (i), we have

$$\begin{aligned}
d(\omega_{2n},\chi_{2n+1}) &= d(F(a_{2n},b_{2n}),G(p_{2n+1},q_{2n+1}))\\
&\leq\theta\max\left\{\ d(fa_{2n},fp_{2n+1}),d(fb_{2n},fq_{2n+1})\ \right\}\\
&\leq\theta\max\left\{\ d(\omega_{2n-1},\chi_{2n}),d(\xi_{2n-1},\kappa_{2n})\ \right\}\\
&\leq\theta\max\left\{\ d(\omega_{2n-1},\chi_{2n}),d(\xi_{2n-1},\kappa_{2n})\ \right\},
\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}
d(\xi_{2n},\kappa_{2n+1}) &= d(F(b_{2n},a_{2n}),G(q_{2n+1},p_{2n+1}))\\
&\leq\theta\max\left\{\ d(fb_{2n},fq_{2n+1}),d(fa_{2n},fp_{2n+1})\ \right\}\\
&\leq\theta\max\left\{\ d(\xi_{2n-1},\kappa_{2n}),d(\omega_{2n-1},\chi_{2n})\ \right\}\\
&\leq\theta\max\left\{\ d(\omega_{2n-1},\chi_{2n}),d(\xi_{2n-1},\kappa_{2n})\ \right\}.
\end{aligned}\tag{2.2}$$

Combining (2.1) and (2.2), we get that

$$\begin{aligned}\max\left\{d\left(\omega_{2n},\chi_{2n+1}\right),d\left(\xi_{2n},\kappa_{2n+1}\right)\right\} &\leq \theta\max\left\{d\left(\omega_{2n-1},\chi_{2n}\right),d\left(\xi_{2n-1},\kappa_{2n}\right)\right\}\\ &\leq \theta^{2}\max\left\{d\left(\omega_{2n-2},\chi_{2n-1}\right),d\left(\xi_{2n-2},\kappa_{2n-1}\right)\right\}\\ &\vdots\\ &\leq \theta^{2n}\max\left\{d\left(\omega_{0},\chi_{1}\right),d\left(\xi_{0},\kappa_{1}\right)\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}d\left(\omega_{2n},\chi_{2n+1}\right)&\leq \theta^{2n}\max\left\{d\left(\omega_{0},\chi_{1}\right),d\left(\xi_{0},\kappa_{1}\right)\right\},\\ d\left(\xi_{2n},\kappa_{2n+1}\right)&\leq \theta^{2n}\max\left\{d\left(\omega_{0},\chi_{1}\right),d\left(\xi_{0},\kappa_{1}\right)\right\}.\end{aligned}\tag{2.3}$$

On the other hand, we have

$$\begin{aligned}d\left(\omega_{2n+1},\chi_{2n}\right) &= d\left(F\left(a_{2n+1},b_{2n+1}\right),G\left(p_{2n},q_{2n}\right)\right)\\ &\leq \theta\max\left\{\ d(fa_{2n+1},fp_{2n}),d(fb_{2n+1},fq_{2n})\ \right\}\\ &\leq \theta\max\left\{\ d(\omega_{2n},\chi_{2n-1}),d(\xi_{2n},\kappa_{2n-1})\ \right\},\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}d\left(\xi_{2n+1},\kappa_{2n}\right) &= d\left(F\left(b_{2n+1},a_{2n+1}\right),G\left(q_{2n},p_{2n}\right)\right)\\ &\leq \theta\max\left\{\ d(fb_{2n+1},fq_{2n}),d(fa_{2n+1},fp_{2n})\ \right\}\\ &\leq \theta\max\left\{\ d(\xi_{2n},\kappa_{2n-1}),d(\omega_{2n},\chi_{2n-1})\ \right\}.\end{aligned}\tag{2.5}$$

Combining (2.4) and (2.5), we get that

$$\begin{aligned}\max\left\{d\left(\omega_{2n+1},\chi_{2n}\right),d\left(\xi_{2n+1},\kappa_{2n}\right)\right\} &\leq \theta\max\left\{d\left(\omega_{2n},\chi_{2n-1}\right),d\left(\xi_{2n},\kappa_{2n-1}\right)\right\}\\ &\leq \theta^{2}\max\left\{d\left(\omega_{2n-1},\chi_{2n-2}\right),d\left(\xi_{2n-1},\kappa_{2n-2}\right)\right\}\\ &\vdots\\ &\leq \theta^{2n}\max\left\{d\left(\omega_{1},\chi_{0}\right),d\left(\xi_{1},\kappa_{0}\right)\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}d\left(\omega_{2n+1},\chi_{2n}\right)&\leq \theta^{2n}\max\left\{d\left(\omega_{1},\chi_{0}\right),d\left(\xi_{1},\kappa_{0}\right)\right\},\\ d\left(\xi_{2n+1},\kappa_{2n}\right)&\leq \theta^{2n}\max\left\{d\left(\omega_{1},\chi_{0}\right),d\left(\xi_{1},\kappa_{0}\right)\right\}.\end{aligned}\tag{2.6}$$

Moreover,

$$\begin{aligned}d\left(\omega_{2n},\chi_{2n}\right) &= d\left(F\left(a_{2n},b_{2n}\right),G\left(p_{2n},q_{2n}\right)\right)\\ &\leq \theta\max\left\{\ d(fa_{2n},fp_{2n}),d(fb_{2n},fq_{2n})\ \right\}\\ &\leq \theta\max\left\{\ d(\omega_{2n-1},\chi_{2n-1}),d(\xi_{2n-1},\kappa_{2n-1})\ \right\},\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}d\left(\xi_{2n},\kappa_{2n}\right) &= d\left(F\left(b_{2n},a_{2n}\right),G\left(q_{2n},p_{2n}\right)\right)\\ &\leq \theta\max\left\{\ d(fb_{2n},fq_{2n}),d(fa_{2n},fp_{2n})\ \right\}\\ &\leq \theta\max\left\{\ d(\xi_{2n-1},\kappa_{2n-1}),d(\omega_{2n-1},\chi_{2n-1})\ \right\}.\end{aligned}\tag{2.8}$$

Combining (2.7) and (2.8), we get

$$\begin{aligned}\max\left\{d\left(\omega_{2n},\chi_{2n}\right),d\left(\xi_{2n},\kappa_{2n}\right)\right\} &\leq \theta\max\left\{d\left(\omega_{2n-1},\chi_{2n-1}\right),d\left(\xi_{2n-1},\kappa_{2n-1}\right)\right\}\\ &\leq \theta^{2}\max\left\{d\left(\omega_{2n-2},\chi_{2n-2}\right),d\left(\xi_{2n-2},\kappa_{2n-2}\right)\right\}\\ &\vdots\\ &\leq \theta^{2n}\max\left\{d\left(\omega_{0},\chi_{0}\right),d\left(\xi_{0},\kappa_{0}\right)\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}d\left(\omega_{2n},\chi_{2n}\right)\leq \theta^{2n}\max\left\{d\left(\omega_{0},\chi_{0}\right),d\left(\xi_{0},\kappa_{0}\right)\right\},\\ d\left(\xi_{2n},\kappa_{2n}\right)\leq \theta^{2n}\max\left\{d\left(\omega_{0},\chi_{0}\right),d\left(\xi_{0},\kappa_{0}\right)\right\}.\end{aligned} \tag{2.9}$$

Using the property (B_4), we obtain

$$\begin{aligned}d\left(\omega_{2n},\chi_{2m}\right)\leq d\left(\omega_{2n},\chi_{2n+1}\right)+d\left(\omega_{2n+1},\chi_{2n+1}\right)+\cdots+d\left(\omega_{2m-1},\chi_{2m}\right),\\ d\left(\xi_{2n},\kappa_{2m}\right)\leq d\left(\xi_{2n},\kappa_{2n+1}\right)+d\left(\xi_{2n+1},\kappa_{2n+1}\right)+\cdots+d\left(\xi_{2m-1},\kappa_{2m}\right),\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}d\left(\omega_{2m},\chi_{2n}\right)\leq d\left(\omega_{2m},\chi_{2m-1}\right)+d\left(\omega_{2m-1},\chi_{2m-1}\right)+\cdots+d\left(\omega_{2n+1},\chi_{2n}\right),\\ d\left(\xi_{2m},\kappa_{2n}\right)\leq d\left(\xi_{2m},\kappa_{2m-1}\right)+d\left(\xi_{2m-1},\kappa_{2m-1}\right)+\cdots+d\left(\xi_{2n+1},\kappa_{2n}\right),\end{aligned} \tag{2.11}$$

for each $n, m \in N$ with $n < m$. Then, from (2.3), (2.6), (2.9), (2.10) and (2.11), we have

$$\begin{aligned}&d\left(\omega_{2n},\chi_{2m}\right)+d\left(\xi_{2n},\kappa_{2m}\right)\\ &\leq \left(d\left(\omega_{2n},\chi_{2n+1}\right)+d\left(\xi_{2n},\kappa_{2n+1}\right)\right)+\left(d\left(\omega_{2n+1},\chi_{2n+1}\right)+d\left(\xi_{2n+1},\kappa_{2n+1}\right)\right)\\ &\quad+\cdots+\left(d\left(\omega_{2m-1},\chi_{2m-1}\right)+d\left(\xi_{2m-1},\kappa_{2m-1}\right)\right)+\left(d\left(\omega_{2m-1},\chi_{2m}\right)+d\left(\xi_{2m-1},\kappa_{2m}\right)\right)\\ &\leq 2(\theta^{2n}+\theta^{2n+1}+\cdots+\theta^{2m-1})\max\left\{\ d(\omega_0,\chi_1),d(\xi_0,\kappa_1)\ \right\}\\ &\quad+2(\theta^{2n+1}+\theta^{2n+2}+\cdots+\theta^{2m-1})\max\left\{\ d(\omega_0,\chi_0),d(\xi_0,\kappa_0)\ \right\}\\ &\leq 2\frac{\theta^{2n}}{1-\theta}\max\left\{\ d(\omega_0,\chi_1),d(\xi_0,\kappa_1)\ \right\}\\ &\quad+2\frac{\theta^{2n+1}}{1-\theta}\max\left\{\ d(\omega_0,\chi_0),d(\xi_0,\kappa_0)\ \right\}\to 0 \text{ as } n\to\infty.\end{aligned}$$

Similarly, we can prove $\left(d\left(\omega_{2m},\chi_{2n}\right)+d\left(\xi_{2m},\kappa_{2n}\right)\right)\to 0$ as $n, m \to \infty$. This shows (ω_{2n},χ_{2m}) and (ξ_{2n},κ_{2m}) are Cauchy bisequences in (A, B). Therefore,

$$\lim_{n\to\infty}\left(\omega_{2n},\chi_{2m}\right)=\lim_{n\to\infty}\left(\xi_{2n},\kappa_{2m}\right)=0.$$

Since $f(A\cup B)$ is a complete subspace of (A, B, d), so $\{\omega_{2n+1}\}$, $\{\chi_{2m+1}\}$, $\{\xi_{2n+1}\}$, $\{\kappa_{2m+1}\}\subseteq f(A\cup B)$ are converges in the complete bipolar metric space $(f(A), f(B), d)$. Therefore, there exist $u, v \in f(A)$ and $w, z \in f(B)$ with

$$\lim_{n\to\infty}\omega_{2n+1}=w,\ \lim_{n\to\infty}\xi_{2n+1}=z,\ \lim_{n\to\infty}\chi_{2n+1}=u,\ \lim_{n\to\infty}\kappa_{2n+1}=v. \tag{2.12}$$

Since $f : A\cup B \to A\cup B$ and $u, v \in f(A)$, $w, z \in f(B)$, there exist $l, m \in A$, $r, s \in B$ such that $fl = u$, $fm = v$ and $fr = w$, $fs = z$. From (i) and (B_4), we have

$$\begin{aligned}d\left(F(l,m),w\right)&\leq d\left(F(l,m),\chi_{2n+1}\right)+d\left(\omega_{2n+1},\chi_{2n+1}\right)+d\left(\omega_{2n+1},w\right)\\ &\leq d\left(F(l,m),G(p_{2n+1},q_{2n+1})\right)+d\left(\omega_{2n+1},\chi_{2n+1}\right)+d\left(\omega_{2n+1},w\right)\\ &\leq \theta\max\left\{\ d(fl,fp_{2n+1}),d(fm,fq_{2n+1})\ \right\}\\ &\quad+d\left(\omega_{2n+1},\chi_{2n+1}\right)+d\left(\omega_{2n+1},w\right)\\ &\leq \theta\max\left\{\ d(fl,\chi_{2n}),d(fm,\kappa_{2n})\ \right\}\\ &\quad+d\left(\omega_{2n+1},\chi_{2n+1}\right)+d\left(\omega_{2n+1},w\right)\to 0 \text{ as } n\to\infty.\end{aligned}$$

Therefore, $d\left(F(l,m),w\right)=0$ implies $F(l,m)=w=fr$. Similarly, we can prove that $F(m,l)=z=fs$, $F(r,s)=u=fl$ and $F(s,r)=v=fm$. Since (F,f) are ω-compatible mappings, we have $F(u,v)=fu$, $F(v,u)=fv$ and $F(w,z)=fw$, $F(z,w)=fz$. We prove that $fu=u$, $fv=v$ and $fw=w$, $fz=z$. Now,

$$\begin{aligned} d\left(fu,\chi_{2n}\right) &= d\left(F(u,v),G(p_{2n},q_{2n})\right) \\ &\leq \theta \max\left\{d(fu,fp_{2n}),d(fv,fq_{2n})\right\} \\ &\leq \theta \max\left\{d(fu,\chi_{2n-1}),d(fv,\kappa_{2n-1})\right\}. \end{aligned}$$

Letting $n\to\infty$, we get

$$d\left(fu,u\right)\leq \theta \max\left\{d(fu,u),d(fv,v)\right\}. \tag{2.13}$$

Also,

$$\begin{aligned} d\left(fv,\kappa_{2n}\right) &= d\left(F(v,u),G(q_{2n},p_{2n})\right) \\ &\leq \theta \max\left\{d(fv,fq_{2n}),d(fu,fp_{2n})\right\} \\ &\leq \theta \max\left\{d(fv,\kappa_{2n-1}),d(fu,\chi_{2n-1})\right\}. \end{aligned}$$

Letting $n\to\infty$, we get

$$d\left(fv,v\right)\leq \theta \max\left\{d(fv,v),d(fu,u)\right\}. \tag{2.14}$$

Combining (2.13) and (2.14), we have

$$\begin{aligned} \max\left\{d(fu,u),d(fv,v)\right\} &\leq \theta \max\left\{d(fu,u),d(fv,v)\right\} \\ &< \max\left\{d(fu,u),d(fv,v)\right\}, \end{aligned}$$

which implies that $d(fu,u)=0, d(fv,v)=0$ and so $fu=u, fv=v$. Similarly, we can show $fw=w$ and $fz=z$. Therefore,

$$\begin{aligned} &F(w,z)=fw=w=fr=F(l,m), \qquad F(z,w)=fz=z=fs=F(m,l), \\ &F(u,v)=fu=u=fl=F(r,s), \qquad F(v,u)=fv=v=fm=F(s,r). \end{aligned}$$

On the other hand, from (2.12), we get

$$d\left(fl,fr\right)=d\left(u,w\right)=d\left(\lim_{n\to\infty}\chi_{2n},\lim_{n\to\infty}\omega_{2n}\right)=\lim_{n\to\infty}d(\omega_{2n},\chi_{2n})=0,$$

and

$$d\left(fm,fs\right)=d\left(v,z\right)=d\left(\lim_{n\to\infty}\kappa_{2n},\lim_{n\to\infty}\xi_{2n}\right)=\lim_{n\to\infty}d(\xi_{2n},\kappa_{2n})=0.$$

Since $G(A^2\cup B^2)\subseteq f(A\cup B)$, so there exist $a,b\in A$ and $x,y\in B$ such that $fa=u$, $fb=v$ and $fx=w$, $fy=z$. Therefore, $G(u,v)=fx=w$, $G(v,u)=fy=z$ and $G(w,z)=fa=u$, $G(z,w)=fb=v$. Now, from (B_4) and (i), we have

$$\begin{aligned} &d\left(u,G(x,y)\right) \\ &\leq d\left(u,\chi_{2n+1}\right)+d\left(\omega_{2n+1},\chi_{2n+1}\right)+d\left(\omega_{2n+1},G(x,y)\right) \\ &\leq d\left(u,\chi_{2n+1}\right)+d\left(\omega_{2n+1},\chi_{2n+1}\right)+d\left(F(a_{2n+1},b_{2n+1}),G(x,y)\right) \\ &\leq d\left(u,\chi_{2n+1}\right)+d\left(\omega_{2n+1},\chi_{2n+1}\right)+\theta \max\left\{d(fa_{2n+1},fx),d(fb_{2n+1},fy)\right\} \\ &\leq d\left(u,\chi_{2n+1}\right)+d\left(\omega_{2n+1},\chi_{2n+1}\right)+\theta \max\left\{d(\omega_{2n},fx),d(\chi_{2n},fy)\right\} \\ &\to 0 \text{ as } n\to\infty. \end{aligned}$$

That is, $d\left(u,G(x,y)\right)=0$ implies $G(x,y)=u$, and hence $u=G(x,y)=fa$. Similarly, we prove $v=G(y,x)=fb$ and $w=G(a,b)=fx$, $z=G(b,a)=fy$. Since (G,f) are ω-compatible, so

$G(u,v) = fu$, $G(v,u) = fv$ and $G(w,z) = fw$, $G(z,w) = fz$. But, we have $fu = u$, $fv = v$ and $fw = w$, $fz = z$. Therefore,

$$G(w,z) = fw = w = fx = G(a,b), \qquad G(z,w) = fz = z = fy = G(b,a),$$
$$G(u,v) = fu = u = fa = G(x,y), \qquad G(v,u) = fv = v = fb = G(y,x).$$

On the other hand, from (2.12), we get

$$d\,(fa, fx) = d\,(u, w) = d\left(\lim_{n\to\infty} \chi_{2n}, \lim_{n\to\infty} \omega_{2n}\right) = \lim_{n\to\infty} d(\omega_{2n}, \chi_{2n}) = 0,$$

and

$$d\,(fb, fy) = d\,(v, z) = d\left(\lim_{n\to\infty} \kappa_{2n}, \lim_{n\to\infty} \xi_{2n}\right) = \lim_{n\to\infty} d(\xi_{2n}, \kappa_{2n}) = 0.$$

So $u = w$ and $v = z$. Therefore, $(u,v) \in A^2 \cap B^2$ is coupled fixed point of covariant mappings F, G and f.
Now we prove the uniqueness, we begin by taking $(u^*, v^*) \in A^2 \cup B^2$ be another fixed point of F, G and f. If $(u^*, v^*) \in A^2$, then we have

$$\begin{aligned} d\,(u, u^*) &= d(F(u,v), G(u^*, v^*)) \\ &\le \theta \max\{d(fu, fu^*), d(fv, fv^*)\} \\ &\le \theta \max\{d(u, u^*), d(v, v^*)\}, \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} d\,(v, v^*) &= d(F(v,u), G(v^*, u^*)) \\ &\le \theta \max\{d(fv, fv^*), d(fu, fu^*)\} \\ &\le \theta \max\{d(v, v^*), d(u, u^*)\}. \end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16), we have

$$\begin{aligned} \max\{d(u,u^*), d(v,v^*)\} &\le \theta \max\{d(u,u^*), d(v,v^*)\} \\ &< \max\{d(u,u^*), d(v,v^*)\}. \end{aligned}$$

Therefore, $d\,(u, u^*) = 0, d\,(v, v^*) = 0$ implies $u = u^*, v = v^*$. Similarly, if $(u^*, v^*) \in B^2$, then we have $u = u^*$ and $v = v^*$. Then $(u,v) \in A^2 \cap B^2$ is unique coupled fixed point of covariant mappings F, G and f. Finally we will show that $u = v$.

$$\begin{aligned} d\,(u, v) &= d\,(F(u,v), G(v,u)) \\ &\le \theta \max\{d(fu, fv), d(fv, fu)\} \\ &\le \theta \max\{d(u,v), d(v,u)\}, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} d\,(v, u) &= d\,(F(v,u), G(u,v)) \\ &\le \theta \max\{d(fv, fu), d(fu, fv)\} \\ &\le \theta \max\{d(v,u), d(u,v)\}. \end{aligned} \tag{2.18}$$

Combining (2.17) and (2.18), we get

$$\begin{aligned} \max\{d(u,v), d(v,u)\} &\le \theta \max\{d(u,v), d(v,u)\} \\ &< \max\{d(u,v), d(v,u)\}. \end{aligned}$$

Therefore, $d\,(u,v) = 0, d\,(v,u) = 0 \Rightarrow u = v$. Hence u is the common fixed point of F, G and f. □

Corollary 2.5. *Let (A,B,d) be a bipolar metric space and $F:(A^2,B^2)\rightrightarrows(A,B)$, $f:(A,B)\rightrightarrows(A,B)$ be covariant mappings satisfying:*

(i) *for all $a,b\in A$ and $p,q\in B$ with $\theta\in(0,1)$,*

$$d\left(F(a,b),F(p,q)\right)\leq\theta\max\left\{d(fa,fp),d(fb,fq)\right\},$$

(ii) *$F\left(A^2\cup B^2\right)\subseteq f(A\cup B)$,*

(iii) *(F,f) are ω-compatible,*

(iv) *$f(A\cup B)$ is complete.*

Then the mappings F and f have a unique common fixed point of the form (u,u).

Corollary 2.6. *Let (A,B,d) be a complete bipolar metric space and $F:(A^2,B^2)\rightrightarrows(A,B)$ be a covariant mapping such that*

$$d\left(F(a,b),F(p,q)\right)\leq\theta\max\left\{d(a,p),d(b,q)\right\},$$

for all $a,b\in A$ and $p,q\in B$ with $\theta\in(0,1)$. Then, F has a unique common fixed point of the form (u,u).

Example 2.7. *Let $U_m(R)$ and $L_m(R)$ be the set of all $m\times m$ upper and lower triangular matrices over R. Define $d:U_m(R)\times L_m(R)\to[0,\infty)$ as*

$$d(P,Q)=\sum_{i,j=1}^{m}|p_{ij}-q_{ij}|,$$

for all $P=(p_{ij})_{m\times m}\in U_m(R)$ and $Q=(q_{ij})_{m\times m}\in L_m(R)$. Then obviously $(U_m(R),L_m(R),d)$ is a Bipolar-metric space. Also, define
$F,G:A^2\cup B^2\to A\cup B$ as $F(P,Q)=(\frac{p_{ij}}{8}+\frac{q_{ij}}{4})_{m\times m}$ and $G(P,Q)=(\frac{p_{ij}}{4}+\frac{q_{ij}}{2})_{m\times m}$ where $(P=(p_{ij})_{m\times m},Q=(q_{ij})_{m\times m})\in U_m(R)^2\cup L_m(R)^2$.
Also, $f:A\cup B\to A\cup B$ as $f(P)=(2p_{ij})_{m\times m}$ where $P=(p_{ij})_{m\times m}\in U_m(R)\cup L_m(R)$.
Consider,

$$\begin{aligned}
d(F(P,Q),G(U,V) &= d\left((\tfrac{p_{ij}}{8}+\tfrac{q_{ij}}{4})_{m\times m},(\tfrac{u_{ij}}{4}+\tfrac{v_{ij}}{2})_{m\times m}\right)\\
&= \sum_{i,j=1}^{m}|(\tfrac{p_{ij}}{8}+\tfrac{q_{ij}}{4})-(\tfrac{u_{ij}}{4}+\tfrac{v_{ij}}{2})|\\
&\leq \sum_{i,j=1}^{m}|\tfrac{p_{ij}}{8}-\tfrac{u_{ij}}{4}|+|\tfrac{q_{ij}}{4}-\tfrac{v_{ij}}{2}|\\
&\leq \tfrac{1}{8}\sum_{i,j=1}^{m}|p_{ij}-2u_{ij}|+\tfrac{1}{4}\sum_{i,j=1}^{m}|q_{ij}-2v_{ij}|\\
&\leq \tfrac{1}{8}\sum_{i,j=1}^{m}|2p_{ij}-2u_{ij}|+\tfrac{1}{4}\sum_{i,j=1}^{m}|2q_{ij}-2v_{ij}|\\
&\leq \tfrac{3}{8}\max\left\{\ d(fP,fU),d(fQ,fV)\ \right\}.
\end{aligned}$$

Clearly, F,G and f are satisfies all the conditions of Theorem 2.4 and $(O_{m\times m},O_{m\times m})$ is unique coupled fixed point.

3. Application to the existence of solutions of integral equations

The coupled fixed point theorem proved here pave the way for application on complete bipolar metric spaces to prove the existence and uniqueness of a solution for a Fredholm nonlinear integral equation.

Theorem 3.1. *Let us Consider the integral equation*

$$\alpha(\nu) = \int\limits_{E_1 \cup E_2} \left(K_1(\nu,\vartheta) + K_2(\nu,\vartheta)\right)(H)\, d\vartheta + F(\nu), \tag{3.1}$$

where $H = f(\vartheta,\alpha(\vartheta)) + g(\vartheta,\alpha(\vartheta)))$, $(\nu,\vartheta) \in E_1^2 \cup E_2^2$ *and* $E_1 \cup E_2$ *is a Lebesgue measurable set. Suppose that the following assertions hold:*

(*i*) $K_1 : E_1^2 \cup E_2^2 \to [0,+\infty)$, $K_2 : E_1^2 \cup E_2^2 \to (-\infty, 0]$ *and* $F \in L^\infty(E_1) \cup L^\infty(E_2)$, $f, g : (E_1 \cup E_2) \times R \to R$ *are integrable;*

(*ii*) *there exist* $i, j \in (0, \frac{1}{2})$ *such that*

$$\begin{aligned} 0 &\le f(\nu,\alpha) - f(\nu,\beta) \le i(\alpha - \beta), \\ -j(\alpha-\beta) &\le g(\nu,\alpha) - g(\nu,\beta) \le 0, \end{aligned}$$

for $\nu \in E_1 \cup E_2$ *and* $\alpha, \beta \in R$;

(*iii*) $\|\int\limits_{E_1\cup E_2} (K_1(\nu,\vartheta) - K_2(\nu,\vartheta))\, d\vartheta\| \le 1$,
i.e. $\sup\limits_{\nu \in E_1\cup E_2} \int\limits_{E_1\cup E_2} |K_1(\nu,\vartheta) - K_2(\nu,\vartheta)| d\vartheta \le 1$,
for $(\nu,\vartheta) \in E_1^2 \cup E_2^2$.

Then the equation (3.1) *has a unique solution in* $L^\infty(E_1)^2 \cup L^\infty(E_2)^2$.

Proof . Let $U = L^\infty(E_1)$ and $V = L^\infty(E_2)$ be two normed linear spaces, where E_1, E_2 are Lebesgue measurable sets and $m(E_1 \cup E_2) < \infty$. Let $d : U \times V \to [0,+\infty)$ be defined as $d(\Omega,\Psi) = ||\Omega - \Psi||_\infty$ for all $(\Omega,\Psi) \in U \times V$. Then (U, V, d) is a complete bipolar metric space. Define $S : U^2 \cup V^2 \to U \cup V$ by

$$\begin{aligned} S(\alpha,\beta)(\nu) &= \int\limits_{E_1\cup E_2} K_1(\nu,\vartheta)\left(f(\vartheta,\alpha(\nu)) + g(\vartheta,\beta(\nu))\right) d\vartheta \\ &\quad + \int\limits_{E_1\cup E_2} K_2(\nu,\vartheta)\left(f(\vartheta,\beta(\nu)) + g(\vartheta,\alpha(\nu))\right) d\vartheta + F(\nu), \quad \nu \in E_1 \cup E_2. \end{aligned}$$

Now we have, $d\left(S(\alpha,\beta), S(\kappa,\xi)\right) = ||S(\alpha,\beta) - S(\kappa,\xi)||_\infty$.

Let us first evaluate the following expression:

$$|(S(\alpha,\beta),S(\kappa,\xi))(\nu)| = \left| \begin{array}{l} \int\limits_{E_1\cup E_2} K_1(\nu,\vartheta)\left(f(\vartheta,\alpha(\nu))+g(\vartheta,\beta(\nu))\right)d\vartheta \\ +\int\limits_{E_1\cup E_2} K_2(\nu,\vartheta)\left(f(\vartheta,\beta(\nu))+g(\vartheta,\alpha(\nu))\right)d\vartheta \\ -\int\limits_{E_1\cup E_2} K_1(\nu,\vartheta)\left(f(\vartheta,\kappa(\nu))+g(\vartheta,\xi(\nu))\right)d\vartheta \\ -\int\limits_{E_1\cup E_2} K_2(\nu,\vartheta)\left(f(\vartheta,\xi(\nu))+g(\vartheta,\kappa(\nu))\right)d\vartheta \end{array} \right|$$

$$\begin{aligned} &= \left|\int\limits_{E_1\cup E_2} K_1(\nu,\vartheta)(\psi)d\vartheta\right| + \left|\int\limits_{E_1\cup E_2} K_2(\nu,\nu)(\phi)d\vartheta\right| \\ &\le \int\limits_{E_1\cup E_2} K_1(\nu,\vartheta)\,|\psi|\,d\vartheta - \int\limits_{E_1\cup E_2} K_2(\nu,\vartheta)\,|\phi|\,d\vartheta \\ &\le (i\,\|\alpha-\kappa\|_\infty\, ji\,\|\beta-\xi\|_\infty)\int\limits_{E_1\cup E_2}(K_1(\nu,\vartheta)-K_2(\nu,\vartheta))d\vartheta \end{aligned}$$

where

$$\psi = f(\vartheta,\alpha(\nu)) - f(\vartheta,\kappa(\nu)) + g(\vartheta,\beta(\nu)) - g(\vartheta,\xi(\nu)),$$

$$\phi = f(\vartheta,\beta(\nu)) - f(\vartheta,\xi(\nu)) + g(\vartheta,\alpha(\nu)) - g(\vartheta,\kappa(\nu)).$$

Then,

$$\begin{aligned} & d\left(S(\alpha,\beta),S(\kappa,\xi)\right) \\ & = ||S(\alpha,\beta)-S(\kappa,\xi)||_\infty \\ & \le (i||\alpha-\kappa||_\infty + j||\beta-\xi||_\infty)\,||\int\limits_{E_1\cup E_2}\left(K_1(\nu,\vartheta)-K_2(\nu,\vartheta)\right)d\vartheta|| \\ & \le (i||\alpha-\kappa||_\infty + j||\beta-\xi||_\infty)\sup_{\nu\in E_1\cup E_2}\int\limits_{E_1\cup E_2}|K_1(\nu,\vartheta)-K_2(\nu,\vartheta)|d\vartheta \\ & \le i||\alpha-\kappa||_\infty + j||\beta-\xi||_\infty \\ & \le \theta\max\{||\alpha-\kappa)||_\infty, ||\beta-\xi||_\infty\} \\ & \le \theta\max\{d(\alpha,\kappa),d(\beta,\xi)\}. \end{aligned}$$

Hence, applying Corollary 2.6, we get the desired result. □

4. Application to Homotopy

Theorem 4.1. *Let (A,B,d) be complete bipolar metric space, (U,V) be an open subset of (A,B) and $(\overline{U},\overline{V})$ be closed subset of (A,B) such that $(U,V)\subseteq(\overline{U},\overline{V})$. Suppose $H:(\overline{U}^2\cup\overline{V}^2)\times[0,1]\to A\cup B$ be an operator with following conditions are satisfying:*

(i) $u\neq H(u,v,\kappa)$ *and* $v\neq H(v,u,\kappa)$ *for each* $u,v\in\partial U\cup\partial V$ *and* $\kappa\in[0,1]$,

(ii) *for all* $u,v\in\overline{U}$, $x,y\in\overline{V}$ *and* $\kappa\in[0,1],\theta\in(0,1)$,

$$d\left(H(u,v,\kappa),H(x,y,\kappa)\right)\le\theta\max\{d(u,x),d(v,y)\},$$

(iii) there exists $M \geq 0$ such that

$$d(H(u,v,\kappa), H(x,y,\zeta)) \leq M|\kappa - \zeta|,$$

for every $u, v \in \overline{U}$ and $x, y \in \overline{V}$ and $\kappa, \zeta \in [0,1]$.

Then $H(.,0)$ has a fixed point $\iff$ $H(.,1)$ has a fixed point.

Proof . Let the set

$$X = \{\kappa \in [0,1] : u = H(u,v,\kappa), v = H(v,u,\kappa) \text{ for some } (u,v) \in U^2 \cup V^2\},$$

$$Y = \{\zeta \in [0,1] : x = H(x,y,\zeta), y = H(y,x,\zeta) \text{ for some } (x,y) \in U^2 \cup V^2\}.$$

Since $H(.,0)$ has a fixed point in $U^2 \cup V^2$, so $(0,0) \in X^2 \cap Y^2$.
Now we show that $X^2 \cap Y^2$ is both closed and open in $[0,1]$ and hence by the connectedness $X = Y = [0,1]$.
Let $(\{\kappa_n\}_{n=1}^{\infty}, \{\zeta_n\}_{n=1}^{\infty}) \subseteq (X,Y)$ with $(\kappa_n, \zeta_n) \to (\kappa, \zeta) \in [0,1]$ as $n \to \infty$. We must show that $(\kappa,\zeta) \in X^2 \cap Y^2$. Since $(\kappa_n, \zeta_n) \in (X,Y)$ for $n = 0,1,2,3,\cdots$, there exist bisequences $(u_n, x_n), (v_n, y_n)$ with $u_{n+1} = H(u_n, v_n, \kappa_n)$, $v_{n+1} = H(v_n, u_n, \kappa_n)$ and $x_{n+1} = H(x_n, y_n, \zeta_n)$, $y_{n+1} = H(y_n, x_n, \zeta_n)$. Consider,

$$\begin{aligned} d(u_n, x_{n+1}) &= d(H(u_{n-1}, v_{n-1}, \kappa_{n-1}), H(x_n, y_n, \zeta_n)) \\ &\leq \theta \max\{d(u_{n-1}, x_n), d(v_{n-1}, y_n)\}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} d(v_n, y_{n+1}) &= d(H(v_{n-1}, u_{n-1}, \kappa_{n-1}), H(y_n, x_n, \zeta_n)) \\ &\leq \theta \max\{d(v_{n-1}, y_n), d(u_{n-1}, x_n)\}. \end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), we get

$$\begin{aligned} \max\{d(u_n, x_{n+1}), d(v_n, y_{n+1})\} &\leq \theta \max\{d(u_{n-1}, x_n), d(v_{n-1}, y_n)\} \\ &\leq \theta^2 \max\{d(u_{n-2}, x_{n-1}), d(v_{n-2}, y_{n-1})\} \\ &\leq \theta^n \max\{d(u_0, x_1), d(v_0, y_1)\}. \end{aligned}$$

Thus,

$$\begin{aligned} d(u_n, x_{n+1}) \leq \theta^n \max\{d(u_0, x_1), d(v_0, y_1)\}, \\ d(v_n, y_{n+1}) \leq \theta^n \max\{d(u_0, x_1), d(v_0, y_1)\}. \end{aligned} \tag{4.3}$$

Similarly, we can prove

$$\begin{aligned} d(u_{n+1}, x_n) \leq \theta^n \max\{d(u_1, x_0), d(v_1, y_0)\}, \\ d(v_{n+1}, y_n) \leq \theta^n \max\{d(u_1, x_0), d(v_1, y_0)\}. \end{aligned} \tag{4.4}$$

Also,

$$\begin{aligned} d(u_n, x_n) \leq \theta^n \max\{d(u_0, x_0), d(v_0, y_0)\}, \\ d(v_n, y_n) \leq \theta^n \max\{d(u_0, x_0), d(v_0, y_0)\}, \end{aligned} \tag{4.5}$$

for each $n, m \in N$ with $n < m$. Using the property (B_4) and (4.3), (4.4), (4.5), we have

$$\begin{aligned} & d(u_n, x_m) + d(v_n, y_m) \\ & \leq (d(u_n, x_{n+1}) + d(v_n, y_{n+1})) + (d(u_{n+1}, x_{n+1}) + d(v_{n+1}, y_{n+1})) + \cdots \\ & \quad + (d(u_{m-1}, x_{m-1}) + d(v_{m-1}, y_{m-1})) + (d(u_{m-1}, x_m) + d(v_{m-1}, y_m)) \\ & \leq 2\theta^n \max\{d(u_0, x_1), d(v_0, y_1)\} + M|\kappa_{n+1} - \zeta_{n+1}| + \cdots \\ & \quad + M|\kappa_{m-1} - \zeta_{m-1}| + 2\theta^m \max\{d(u_0, x_1), d(v_0, y_1)\} \to 0 \text{ as } n, m \to \infty. \end{aligned}$$

It follows $\lim\limits_{n\to\infty}(d(u_n,x_m)+d(v_n,y_m))=0$. Similarly, we can show $\lim\limits_{n\to\infty}(d(u_m,x_n)+d(v_m,y_n))=0$. Therefore, (u_n,x_n) and (v_n,y_n) are Cauchy bisequence in (U,V). By completeness, there exist $\xi,\nu\in U$ and $\delta,\eta\in V$ with

$$\lim_{n\to\infty}u_n=\delta,\quad \lim_{n\to\infty}v_n=\eta,\quad \lim_{n\to\infty}x_n=\xi,\quad \lim_{n\to\infty}y_n=\nu. \tag{4.6}$$

Now,

$$\begin{aligned} d(H(\xi,\nu,\kappa),\delta) &\le d(H(\xi,\nu,\kappa),x_{n+1})+d(u_{n+1},x_{n+1})+d(u_{n+1},\delta)\\ &\le d(H(\xi,\nu,\kappa),H(x_n,y_n,\zeta_n))\\ &\quad +d(H(u_n,v_n,\kappa_n),H(x_n,y_n,\zeta_n))+d(u_{n+1},\delta)\\ &\le \theta\max\{d(u_n,x_n),d(v_n,y_n)\}+M|\kappa_n-\zeta_n|+d(u_{n+1},\delta)\\ &< \max\{d(u_n,x_n),d(v_n,y_n)\}\\ &\quad +M|\kappa_n-\zeta_n|+d(u_{n+1},\delta)\to 0 \text{ as } n\to\infty. \end{aligned}$$

It follows $d(H(\xi,\nu,\kappa),\delta)=0$ implies $H(\xi,\nu,\kappa)=\delta$. Similarly we get $H(\nu,\xi,\kappa)=\eta$ and $H(\delta,\eta,\zeta)=\xi$, $H(\eta,\delta,\zeta)=\nu$.
On the other hand, from (4.6), we get

$$d(\xi,\delta)=d(\lim_{n\to\infty}x_n,\lim_{n\to\infty}u_n)=\lim_{n\to\infty}d(u_n,x_n)=0,$$

$$d(\nu,\eta)=d(\lim_{n\to\infty}y_n,\lim_{n\to\infty}v_n)=\lim_{n\to\infty}d(v_n,y_n)=0.$$

Therefore, $\xi=\delta$ and $\nu=\eta$ and hence $\kappa=\zeta$. Thus, $(\kappa,\zeta)\in X^2\cap Y^2$. Clearly $X^2\cap Y^2$ is closed in [0, 1].
Let $(\kappa_0,\zeta_0)\in(X,Y)$, then there exist bisequences $(u_0,x_0),(v_0,y_0)$ with $u_0=H(u_0,v_0,\kappa_0)$, $v_0=H(v_0,u_0,\kappa_0)$ and $x_0=H(x_0,y_0,\zeta_0)$, $y_0=H(y_0,x_0,\zeta_0)$.
Since $U^2\cup V^2$ is open, then there exist $r>0$ such that $X_d(u_0,r)\subseteq U^2\cup V^2$ and $X_d(v_0,r)\subseteq U^2\cup V^2$ and $X_d(x_0,r)\subseteq U^2\cup V^2$ and $X_d(y_0,r)\subseteq U^2\cup V^2$.
Choose $\kappa\in(\zeta_0-\epsilon,\zeta_0+\epsilon)$, $\zeta\in(\kappa_0-\epsilon,\kappa_0+\epsilon)$ such that $|\kappa-\zeta_0|\le\frac{1}{M^n}<\frac{\epsilon}{2}$, $|\zeta-\kappa_0|\le\frac{1}{M^n}<\frac{\epsilon}{2}$ and $|\kappa_0-\zeta_0|\le\frac{1}{M^n}<\frac{\epsilon}{2}$.
Then for $x\in\overline{B_{X\cup Y}(u_0,r)}=\{x,x_0\in V: d(u_0,x)\le r+d(u_0,x_0)\}$,
$y\in\overline{B_{X\cup Y}(v_0,r)}=\{y,y_0\in V: d(v_0,y)\le r+d(v_0,y_0)\}$ and
$u\in\overline{B_{X\cup Y}(r,x_0)}=\{u,u_0\in U: d(u,x_0)\le r+d(u_0,x_0)\}$,
$v\in\overline{B_{X\cup Y}(r,y_0)}=\{v,v_0\in U: d(v,y_0)\le r+d(v_0,y_0)\}$. Also

$$\begin{aligned} d(H(u,v,\kappa),x_0) &= d(H(u,v,\kappa),H(x_0,y_0,\zeta_0))\\ &\le d(H(u,v,\kappa),H(x,y,\zeta_0))+d(H(u_0,v_0,\kappa),H(x,y,\zeta_0))\\ &\quad +d(H(u_0,v_0,\kappa),H(x_0,y_0,\zeta_0))\\ &< \tfrac{2}{M^{n-1}}+\theta\max\{d(u_0,x),d(v_0,y)\}. \end{aligned}$$

Letting $n\to\infty$, we get

$$d(H(u,v,\kappa),x_0)\le\theta\max\{d(u_0,x),d(v_0,y)\}. \tag{4.7}$$

Similarly, we can prove

$$d(H(v,u,\kappa),y_0)\le\theta\max\{d(v_0,y),d(u_0,x)\}. \tag{4.8}$$

Combining (4.7) and (4.8), we get

$$\begin{aligned}\max\{d(H(u,v,\kappa),x_0),d(H(u,v,\kappa),x_0)\} &\leq \theta\max\{d(u_0,x),d(v_0,y)\}\\ &< \max\{d(u_0,x),d(v_0,y)\}\\ &\leq \max\{d(u_0,x_0)+r,d(v_0,y_0)+r\}.\end{aligned}$$

Thus, $d(H(u,v,\kappa),x_0) \leq d(u_0,x_0)+r$, $d(H(u,v,\kappa),x_0) \leq d(v_0,y_0)+r$. Similarly, $d(u_0,H(x,y,\zeta)) \leq d(u,x_0) \leq r+d(u_0,x_0)$ and $d(v_0,H(y,x,\zeta)) \leq d(v,y_0) \leq r+d(v_0,y_0)$. On the other hand,

$$\begin{aligned}d(u_0,x_0) &= d(H(u_0,v_0,\kappa_0),H(x_0,y_0,\zeta_0)) \leq M|\kappa_0-\zeta_0|\\ &\leq M\tfrac{1}{M^n} \leq \tfrac{1}{M^{n-1}} \to 0 \ as\ n\to\infty,\end{aligned}$$

and

$$\begin{aligned}d(v_0,y_0) &= d(H(v_0,u_0,\kappa_0),H(y_0,x_0,\zeta_0)) \leq M|\kappa_0-\zeta_0|\\ &\leq M\tfrac{1}{M^n} \leq \tfrac{1}{M^{n-1}} \to 0 \ as\ n\to\infty.\end{aligned}$$

So $u_0=x_0$ and $v_0=y_0$ and hence $\kappa=\zeta$. Thus for each fixed $\kappa\in(\kappa_0-\epsilon,\kappa_0+\epsilon)$, $H(.,\kappa):\overline{B_{X\cup Y}(u_0,r)}\to\overline{B_{X\cup Y}(u_0,r)}$ and $H(.,\kappa):\overline{B_{X\cup Y}(v_0,r)}\to\overline{B_{X\cup Y}(v_0,r)}$. Thus, we conclude that $H(.,\kappa)$ has a coupled fixed point in $\overline{U}^2\cap\overline{V}^2$. But this must be in $U^2\cap V^2$. Therefore, $(\kappa,\kappa)\in X^2\cap Y^2$ for $\kappa\in(\kappa_0-\epsilon,\kappa_0+\epsilon)$. Hence $(\kappa_0-\epsilon,\kappa_0+\epsilon)\subseteq X^2\cap Y^2$. Clearly $X^2\cap Y^2$ is open in [0, 1]. To prove the reverse, we can use the similar process. □

Theorem 4.2. *Let (A,B,d) be complete bipolar metric space, (U,V) be an open subset of (A,B) and $(\overline{U},\overline{V})$ be closed subset of (A,B) such that $(U,V)\subseteq(\overline{U},\overline{V})$. Suppose $H:(\overline{U}\times\overline{V})\cup(\overline{V}\times\overline{U})\times[0,1]\to A\cup B$ be an operator with following conditions are satisfying:*

(i) $u\neq H(u,v,\kappa)$ *and* $v\neq H(v,u,\kappa)$ *for each* $u,v\in\partial U\cup\partial V$ *and* $\kappa\in[0,1]$,

(ii) *for all* $u,v\in\overline{U}$, $x,y\in\overline{V}$ *and* $\kappa\in[0,1],\theta\in(0,1)$,

$$d\left(H(u,x,\kappa),H(y,v,\kappa)\right)\leq\theta\max\{d(u,y),d(v,x)\},$$

(iii) *there exists* $M\geq 0$ *such that*

$$d(H(u,x,\kappa),H(y,v,\zeta))\leq M|\kappa-\zeta|,$$

for every $u,v\in\overline{U}$ *and* $x,y\in\overline{V}$ *and* $\kappa,\zeta\in[0,1]$.

Then $H(.,0)$ *has a fixed point* $\iff$ $H(.,1)$ *has a fixed point.*

Open Problems:

1) Prove analogue of the results obtained herein on "*Bipolar Orthogonal Metric Spaces*". For more details, see [4, 9, 10, 17].
2) Prove analogue of the results obtained herein on "*Bipolar R-Metric Spaces*". For more details, see [24].

5. Conclusions

In this paper, we obtain the existence and uniqueness solution for two covariant mappings in a complete bipolar metric spaces with an example. Also, we have provided some applications to nonlinear integral equations as well as homotopy theory by using fixed point theorems in bipolar metric spaces.

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