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Stability and convergence theorems of pointwise asymptotically nonexpansive random operator in Banach space

Sabah Hassan Malih[a]

[a] *Department of mathematics, college of Education for pure science (Ibn- AL-Haitham), university of Baghdad, Iraq.*

Abstract

In this paper, we prove the existence of a random fixed point of by using pointwise asymptotically nonexpansive random operator and the stability resultsof two iterative schemes for random operator.

Keywords: Separable Banach space, pointwise asymptotically nonexpansive random operator, random fixed point.

1. Introduction and preliminaries

Random fixed point theorems are stochastic generalization of classical fixed point theorems [1, 2]. Itoh [3, 4], extended several well known fixed point theorems, i.e., for contraction, nonexpansive and condemning, mappings to the random case. Thereafter, various stochastic aspects of Schauder's fixed point theorem have been studied by Sehgal and Singh [5], Papageorgiou [6], Lin [7] and many authors. In a separable metric space, random fixed point theorems for contractive mappings were proved by Spacek [2], Hans [1, 8, 9]. Afterwards, Beg and Shahzad [10, 11].

Definition 1.1. *[12] A measurable space $(\triangle, \sum)$ is a nonempty set $\triangle$ and a σ-algebra $\sum$, that is, a family of subsets of $\triangle$ such that i-$\emptyset \in \sum$. ii-$U^c \in \sum$ if $U \in \sum$. iii-$\cup_{i=1}^{\infty} U_i \in \sum$ if $U_i \in \sum$ for all $i \in N$. The elements of $\sum$ are called measurable sets.*

Definition 1.2. *[13] Let S be a Banach space and $\eta_n : \triangle \to S$ is measurable sequence.*

Definition 1.3. *[13] A mapping $J : \triangle \times S \to S$ is called measurable ($\sum$-measurable) if $\forall$ $D \subseteq S$, where D is open and $J^{-1}(D) = \{d : J(d) \cap D \neq \emptyset\} \in \sum$*

*Corresponding Author: Sabah Hassan Malih
Email address: sabah.h.m@ihcoedu.uobaghdad.edu.iq (Sabah Hassan Malih[a])

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Definition 1.4. *[13] A mapping $J : \triangle \times S \to S$ is random operator, if $\forall\ s \in S$ the mapping $J(0, s) : \triangle \to S$ is measurable.*

Definition 1.5. *[13] A random operator $J : \triangle \times S \to S$ is continuous if $J(s, 0) : \triangle \to S$ is continuous, $\forall\ s \in \triangle$.*

Definition 1.6. *[13] A measurable mapping $\eta : \triangle \to S$ is random fixed point of a random operator $J : \triangle \times S \to S$ if $J(\delta, \eta(\delta)) = \eta(\delta)$ For each $\delta \in \triangle$.*

Definition 1.7. *[14] A Banach space is said to satisfy the Opial's condition if for any sequence $\{s_n\}$ in A, $s_n \to s$ weakly) as $n \to \infty$ and $s \neq c$ implying that*

$$\lim_{n\to\infty} \sup \|s_n - s\| < \lim_{n\to\infty} \sup \|s_n - c\|.$$

Definition 1.8. *[15] An random operator $J : \triangle \times S \to S$ is called compact random operator, if $\overline{A(\gamma, A)}$ is compact subset of V where $\overline{J(\gamma, A)}$ is the closure of $J(\gamma, A)$ in A.*

Definition 1.9. *[] Let $(\triangle, \sum)$ is called a measurable space and S be a nonempty subset of a separable Banach space A. Let $J : \triangle \times S \to S$ be a random operator. $\forall\ \eta_0(\delta) \in S$, the iterative scheme $\{\eta_n(\delta)\}_{n=0}^{\infty} \subset M$ by*

$$\eta_{n+1}(\delta) = h\left(J, \eta_n(\delta)\right), \qquad n = 0, 1, 2, \cdots$$

where g is a measurable function. Let $\gamma(\delta)$ be a random fixed point of A. Let $\{\eta_n(\delta)\}_{n=0}^{\infty} \subset S$ be an arbitrary sequence of a random variable. Set $\epsilon_n = \| \mu_{n+1}(\delta) - h\left(J, \mu_n(\delta)\right) \|$, if $\sum_{n=1}^{\infty} \epsilon_n < \infty$ implies that $\lim_{n\to\infty} \mu_n(\delta) = \gamma(\delta)$, then the iterative scheme $\eta_{n+1}(\delta) = h\left(J, \mu_n(\delta)\right)$, is stable.

Lemma 1.10. *[17] Let be a uniformly convex Banach space, $0 < p \leq \lambda_n \leq q < 1$, $\forall\ n \in N$ and let $\{s_n\}$ and $\{c_n\}$ are two sequences of A such that $\lim_{n\to\infty} \sup \|c_n\| \leq r$ and $\lim_{n\to\infty} \|\beta_n s_n + (1-\beta_n)c_n\| = r$. Then $\lim_{n\to\infty} \|s_n - c_n\| = 0$.*

Lemma 1.11. *[18] Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of nonnegative numbers and $a_{n+1} \leq (1 + b_n)a_n + c_n\ \forall\ n \in N$. If $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

2. Main Results

Through this work we find new random condition, and we find random fixed points in Banach spaces.

Definition 2.1. *Let L be a nonempty a bounded closed convex subset of a Banach space A. A random operator $J : \triangle \times L \to L$ is called pointwise asymptotically nonexpansive random operator if*

$$\left\| J^n\big(\delta, \eta(\delta)\big) - \big(\delta, \mu(\delta)\big) \right\| \leq \lambda_n\big(\eta(\delta)\big) \left\|\eta(\delta) - \mu(\delta)\right\|$$

where η, $\mu : \triangle \to L$, $\forall\ \eta, \mu \in L$, $n \in N$, and $\{\lambda_n\left(\eta(\delta)\right)\}$ is a sequence in $[0, +\infty)$ and $\lambda_n \to 1$ pointwise on L.

Definition 2.2. *A random operator J is pointwise asymptotically contraction random operator if $\lambda_n(\eta(\delta)) \leq 1$ and*

$$\lim_{n\to\infty} \lambda_n(\eta(\delta)) = \lambda \in [0,1)$$

let $\eth_n(\eta(\delta)) = \max\{\lambda_n(\eta(\delta)), 1\}$, denote $\Gamma(L)$ as the class of pointwise asymptotically nonexpansive a random satisfying $\lim_{n\to\infty} \eth_n(\eta(\delta))$. Define $\zeta_n(\eta(\delta)) = \eth_n(\eta(\delta)) - 1$; its clear that $\lim_{n\to\infty} \zeta_n(\eta(\delta)) = 0$.

Definition 2.3. *Define $\Gamma_r(L)$ as a class of all $J \in \Gamma(L)$ such that*

$$\sum_{n=1}^{\infty} \zeta_n(\eta(\delta)) < \infty, \qquad \forall\, \delta \in \triangle, \quad \forall\, \eta(\delta) \in L$$

Theorem 2.4. *Let L a nonempty closed convex subset of a uniformly convex Banach space A which satisfies Opial's condition and let $J \in \Gamma_r(L)$, then $(I-J)(0,\delta)$ is demiclosed at zero, that is, if $\{\eta_n(\delta)\}$ is a sequence in L such that $\eta_n(\delta) \rightharpoonup \eth(\delta)$ and $\lim_{m\to\infty} \lim_{n\to\infty} \sup \left\|\eta_n(\delta) - J^m\big(\delta, \eta_n(\delta)\big)\right\| = 0$, then $(I-J)\big(\delta, \eth(\delta)\big) = 0$.*

Proof .

1) If $J^m\big(\delta, \eta_n(\delta)\big) = \eth(\delta)$ for each $m \in N$, taking $m = 1$ we have $J\big(\delta, \eta_n(\delta)\big) = \eth(\delta)$.
2) If there exists an $N_0 > 0$ and for each $m \geq N_0$, such that $J^m\big(\delta, \eta_n(\delta)\big) \neq \eth(\delta)$, we have can define a function $\emptyset$ on A by $(\eta_n(\delta)) = \lim_{n\to\infty} \sup \|\eta_n(\delta) - \eta(\delta)\|$, $\eta(\delta) \in L$

From $\eta_n(\delta) \rightharpoonup \eth(\delta)$ and we have that

$$\lim_{n\to\infty} \sup \|\eta_n(\delta) - \eta(\delta)\| < \lim_{n\to\infty} \sup \left\|\eta_n(\delta) - J\big(\delta, \eth(\delta)\big)\right\|$$

On other hand

$$\begin{aligned}
&\lim_{n\to\infty} \sup \left\|\eta_n(\delta) - J\big(\delta, \eta_n(\delta)\big)\right\| \\
&= \lim_{n\to\infty} \sup \left\|\eta_n(\delta) - J^m\big(\delta, \eta_n(\delta)\big) + J^m\big(\delta, \eta_n(\delta)\big) - J\big(\delta, \eth(\delta)\big)\right\| \\
&\leq \lim_{n\to\infty} \sup \left(\left\|\eta_n(\delta) - J^m\big(\delta, \eta_n(\delta)\big)\right\| + \left\|J^m\big(\delta, \eta_n(\delta)\big) - J\big(\delta, \eth(\delta)\big)\right\|\right) \\
&\leq \lim_{n\to\infty} \sup \left\|\eta_n(\delta) - J^m\big(\delta, \eta_n(\delta)\big)\right\| + \lim_{n\to\infty} \sup \varrho_m\big(\eth(\delta)\big) \|\eta_n(\delta) - \eth(\delta)\|
\end{aligned}$$

So we have

$$\begin{aligned}
&\lim_{n\to\infty} \sup \left\|\eta_n(\delta) - J^m\big(\delta, \eth(\delta)\big)\right\| \\
&\leq \lim_{n\to\infty} \sup \left\|\eta_n(\delta) - J^m\big(\delta, \eta_n(\delta)\big)\right\| + \varrho_m\big(\eth(\delta)\big) \lim_{n\to\infty} \sup \|\eta_n(\delta) - \eth(\delta)\|
\end{aligned}$$

Since $\varrho_m\big(\eta_n(\delta)\big) = 1$, for each $\eta \in L$ and

$$\lim_{m\to\infty} \lim_{n\to\infty} \sup \left\|\eta_n(\delta) - J^m\big(\delta, \eta_n(\delta)\big)\right\| = 0, \quad \forall\, \epsilon > 0$$

$\exists\, k_1 \in N$, when $m > k_1$, we have

$$\varrho_m\big(\eta_n(\delta)\big) < 1 + \epsilon, \quad \forall\, \eta \in L$$

$$\limsup \big\|\eta_n(\delta) - J^m\big(\delta, \eta_n(\delta)\big)\big\| < \epsilon$$

Now

$$\lim_{m\to\infty} \sup \big\|\eta_n(\delta) - J^m\big(\delta, \eth(\delta)\big)\big\| \leq \epsilon + 1 + \epsilon \lim_{m\to\infty} \sup \|\eta_n(\delta) - \eth(\delta)\|$$

we have

$$\lim_{m\to\infty} \sup \big\|\eta_n(\delta) - J^m\big(\delta, \eth(\delta)\big)\big\| \leq \lim_{n\to\infty} \sup \|\eta_n(\delta) - \eth(\delta)\|$$

Taking $k_2 = \max\{k_0, k_1\}$ and $m > k_2$,

$$\lim_{n\to\infty} \sup \big\|\eta_n(\delta) - J^m\big(\delta, \eth(\delta)\big)\big\| \leq \sup \|\eta_n(\delta) - \eth(\delta)\| < \lim_{n\to\infty} \sup \big\|\eta_n(\delta) - J^m\big(\delta, \eth(\delta)\big)\big\|$$

Which is contradiction.

Then $J^m\big(\delta, \eth(\delta)\big) = \eth(\delta)$ for $m \in N$, hence $J^m\big(\delta, \eth(\delta)\big) = \eth(\delta)$. $\square$

Definition 2.5. *Let L be a nonempty subset of a Banach space A, let $J : \triangle \times L \to L$ be a random operator, and $\{\eta_n(\delta)\}$ be the sequence generated by*

$$\begin{cases} \eta_1(\delta) \in L; \quad \eta_1 : \triangle \to L \\ \eta_{n+1}(\delta) = \beta_n\big(\delta, \eta_n(\delta)\big) + (1-\beta_n) J^n\big(\delta, \eta_n(\delta)\big) \end{cases}$$

where $\beta_n \subset (0,1)$ is a sequence bounded from 1 and 0, and denoted by (RGMI).

Theorem 2.6. *Let L be a nonempty closed convex subset of a uniformly convex separable Banach space A which satisfies Opial's condition.*

Let $J \in \Gamma_r(L)$, and $\{\eta_n(\delta)\}$ be the sequence generated by Definition 2.5. Then $\{\eta_n(\delta)\}$ converges weakly to a random a fixed point of J.

Proof . For each $\eth(\delta) \in RF$, then

$$\begin{aligned} &\|\eta_{n+1}(\delta) - \eth(\delta)\| = \big\|\beta_n\big(\eta_n(\delta) - \eth(\delta)\big) + (1-\beta_n)\left(J^n\big(\delta, \eta_n(\delta)\big) - \eth(\delta)\right) \eta_{n+1}(\delta) - \eth(\delta)\big\| \\ &\leq \beta_n \|\eta_n(\delta) - \eth(\delta)\| + (1-\beta_n)\left\|\left(J^n\big(\delta, \eta_n(\delta)\big) - \eth(\delta)\right)\right\| \\ &\leq \big[\beta_n + (1-\beta_n)\varrho_n\big(\eth(\delta)\big)\big] \|\eta_n(\delta) - \eth(\delta)\| \\ &\leq [1 + \zeta_n(\delta)] \|\eta_n(\delta) - \eth(\delta)\| \end{aligned}$$

Since $\sum_{n=1}^{\infty} \zeta_n(\delta)$, then by Lemma 1.11
We get that $\lim_{n\to\infty} \|\eta_n(\delta) - \eth(\delta)\| = Z$ exists. This implies $\{\eta_n(\delta)\}$ is bounded.
Next, we prove that $\lim_{n\to\infty} \big\|\eta_n(\delta) - J^n\big(\delta, \eta_n(\delta)\big)\big\| = 0$

$$\begin{aligned} &\lim_{n\to\infty} \big\|J^n\big(\delta, \eta_n(\delta)\big) - \eth(\delta)\big\| = \lim_{n\to\infty} \big\|J^n\big(\delta, \eta_n(\delta)\big) - J^n\big(\delta, \eth(\delta)\big)\big\| \\ &\leq \lim_{n\to\infty} \|\eta_n(\delta) - \eth(\delta)\| = a \end{aligned}$$

And

$$\lim \left\|\beta_n\big(\eta_n(\delta)-\eth(\delta)\big)+(1-\beta_n)\left(J^n\big(\delta,\eta_n(\delta)\big)-\eth(\delta)\right)\right\|=\|\eta_{n+1}(\delta)-\eth(\delta)\|=a$$

By Lemma 1.11, we get that $\lim_{n\to\infty}\left\|\eta_n(\delta)-J^n\big(\delta,\eta_n(\delta)\big)\right\|=0$

the sequence $\{\eta_{n_i}(\delta)\}$ of bounded $\{\eta_n(\delta)\}$ such that $n_i\rightharpoonup\eth(\delta)$

for some $\eth(\delta)\in L$. By $\lim_{n\to\infty}\left\|\eta_n(\delta)-J^n\big(\delta,\eta_n(\delta)\big)\right\|=0$ and Theorem 2.4, we get that $\eth(\delta)\in RF$.

Now, we show that $\{\eta_n(\delta)\}$ converges weakly to $\eth(\delta)$. Take another subsequence $\{\eta_{m_i}(\delta)\}$ of $\{\eta_n(\delta)\}$ such that $\eta_{m_i}(\delta)\rightharpoonup\gamma(\delta)$, $\gamma(\delta)\in L$, then $\gamma(\delta)\in RF$. Now we show that $\gamma(\delta)=\eth(\delta)$.

Assume $\gamma(\delta)\neq\eth(\delta)$, we have

$$\begin{aligned}\lim_{n\to\infty}\|\eta_n(\delta)-\eth(\delta)\|&=\lim_{i\to\infty}\|\eta_{n_i}(\delta)-\eth(\delta)\|<\lim_{i\to\infty}\|\eta_{n_i}(\delta)-\gamma(\delta)\|=\lim_{i\to\infty}\|\eta_{m_i}(\delta)-\gamma(\delta)\|\\&<\lim_{i\to\infty}\|\eta_{m_i}(\delta)-\eth(\delta)\|=\lim_{n\to\infty}\|\eta_n(\delta)-\eth(\delta)\|\end{aligned}$$

This means $\lim_{n\to\infty}\|\eta_n(\delta)-\eth(\delta)\|<\lim_{n\to\infty}\|\eta_n(\delta)-\eth(\delta)\|$ which is a contraction, so we get that $\gamma(\delta)=\eth(\delta)$. □

Definition 2.7. *Let L be a nonempty subset of a Banach space A, let $J:\triangle\times L\to L$ be a random operator, and $\{\eta_n(\delta)\}$ be the sequence generated by*

$$\begin{cases}\eta_1(\delta)\in L;\quad \eta_1:\triangle\to L\\ \mu_n(\delta)=\beta_n\eta_n(\delta)+(1-\beta_n)J^n\big(\delta,\eta_n(\delta)\big)\\ \eta_{n+1}(\delta)=\rho_n\eta_n(\delta)+(1-\rho_n)J^n\big(\delta,\mu_n(\delta)\big)\end{cases}$$

where $\{\rho_n\}$, $\{\beta_n\}\subset(0,1)$ are sequences bounded away from 1 and 0, and denoted by (RGII).

Theorem 2.8. *Let L and A satisfies condition theorems 2.6. Let $J\in\Gamma_r(c)$, and $\{\eta_n(\delta)\}$ generated by definition 2.7. Then $\{\eta_n(\delta)\}$ converges weakly to a random fixed point of J.*

Proof . For each $\mu(\delta)\in RF$, we have

$$\begin{aligned}\|\eta_{n+1}(\delta)-\eth(\delta)\|&=\left\|\rho_n\big(\eta_n(\delta)-\eth(\delta)\big)+(1-\rho_n)\left(J^n\big(\delta,\mu_n(\delta)\big)-\eth(\delta)\right)\right\|\\&\le\rho_n\|\eta_n(\delta)-\eth(\delta)\|+(1-\rho_n)\left\|J^n\big(\delta,\mu_n(\delta)\big)-\eth(\delta)\right\|\\&\le\rho_n\|\eta_n(\delta)-\eth(\delta)\|+(1-\rho_n)\varrho_n\big(\eth(\delta)\big)\|\mu_n(\delta)-\eth(\delta)\|\end{aligned}$$

$$\begin{aligned}\|\mu_n(\delta)-\eth(\delta)\|&=\left\|\beta_n\big(\eta_n(\delta)-\eth(\delta)\big)+(1-\beta_n)\left(J^n\big(\delta,\eta_n(\delta)\big)-\eth(\delta)\right)\right\|\\&\le\beta_n\|\eta_n(\delta)-\eth(\delta)\|+(1-\beta_n)\left\|J^n\big(\delta,\eta_n(\delta)\big)-\eth(\delta)\right\|\\&\le\beta_n\|\eta_n(\delta)-\eth(\delta)\|+(1-\beta_n)\varrho_n\big(\eth(\delta)\big)\|\eta_n(\delta)-\eth(\delta)\|\\&\le\varrho_n\big(\eth(\delta)\big)\|\eta_n(\delta)-\eth(\delta)\|\end{aligned}$$

$$\begin{aligned}\|\eta_{n+1}(\delta)-\eth(\delta)\|&\le\rho_n\|\eta_n(\delta)-\eth(\delta)\|+(1-\rho_n)\varrho_n^2\big(\eth(\delta)\big)\|\eta_n(\delta)-\eth(\delta)\|\\&=\left[1+(1-\rho_n)\left(\varrho_n^2\big(\eth(\delta)\big)-1\right)\right]\|\eta_n(\delta)-\eth(\delta)\|\\&=\left[1+\left(\varrho_n^2\big(\eth(\delta)\big)-1\right)\right]\|\eta_n(\delta)-\eth(\delta)\|\end{aligned}$$

Since $\varrho_n^2\big(\eth(\delta)\big)-1=\zeta_n^2\big(\eth(\delta)\big)+2\zeta_n^2\big(\eth(\delta)\big)$ and $\sum_{n=1}^{\infty}\zeta_n^2\big(\eth(\delta)\big)<\infty$, then $\sum_{n=1}^{\infty}\big(\varrho_n^2\big(\eth(\delta)\big)-1\big)<\infty$ it follows from Lemma 1.11 that $\lim_{n\to\infty}\|\eta_n(\delta)-\eth(\delta)\|=Z$ exists. This implies $\{\eta_n(\delta)\}$ is bounded.

Now

$$\lim_{n\to\infty}\|\eta_{n+1}(\delta)-\eth(\delta)\|\le\lim_{n\to\infty}\rho_n\|\eta_n(\delta)-\eth(\delta)\|+\lim_{n\to\infty}(1-\rho_n)\varrho_n\big(\eth(\delta)\big)\|\mu_n(\delta)-\eth(\delta)\|$$

And $\lim_{n\to\infty}\varrho_n\big(\eth(\delta)\big)=1$, where $\{\rho_n\}$ is bounded away from 0 and 1, we have $\lim_{n\to\infty}\|\mu_n(\delta)-\eth(\delta)\|\ge Z$.

On other hand $\lim_{n\to\infty}\|\mu_n(\delta)-\eth(\delta)\|\le Z$

Then $\lim_{n\to\infty}\|\mu_n(\delta)-\eth(\delta)\|=Z$

Since

$$\begin{aligned}\lim_{n\to\infty}\|J^n(\delta,\eta_n(\delta))-\eth(\delta)\|&=\lim_{n\to\infty}\|J^n(\delta,\eta_n(\delta))-J^n(\delta,\eth(\delta))\|\\&\le\lim_{n\to\infty}\varrho_n\big(\eth(\delta)\big)\|\eta_n(\delta)-\eth(\delta)\|=Z\end{aligned}$$

And

$$\lim_{n\to\infty}\|\beta_n(\eta_n(\delta)-\eth(\delta))+(1-\beta_n)(J^n(\delta,\eta_n(\delta))-\eth(\delta))\|=\lim_{n\to\infty}\|\mu_n(\delta)-\eth(\delta)\|=Z$$

By Lemma 1.10, we get that $\lim_{n\to\infty}\Big\|\eta_n(\delta)-J^n\big(\delta,\eta_n(\delta)\big)\Big\|=0$. Theorem 2.4, we get that $\eth(\delta)\in RF$.

Now, we prove that $\{\eta_n(\delta)\}$ converges weakly to $\eth(\delta)$. Take another subsequence $\{\eta_{m_i}(\delta)\}$ of $\{\eta_n(\delta)\}$ such that $\eta_{m_i}(\delta)\rightharpoonup\gamma(\delta)$, $\gamma(\delta)\in L$, then $\gamma(\delta)\in RF$. Now we show that $\gamma(\delta)=\eth(\delta)$.

Assume $\gamma(\delta)\ne\eth(\delta)$, we have

$$\begin{aligned}\lim_{n\to\infty}\|\eta_n(\delta)-\eth(\delta)\|&=\lim_{i\to\infty}\|\eta_{n_i}(\delta)-\eth(\delta)\|<\lim_{i\to\infty}\|\eta_{n_i}(\delta)-\gamma(\delta)\|=\lim_{i\to\infty}\|\eta_{m_i}(\delta)-\gamma(\delta)\|\\&<\lim_{i\to\infty}\|\eta_{m_i}(\delta)-\eth(\delta)\|=\lim_{i\to\infty}\|\eta_n(\delta)-\eth(\delta)\|\end{aligned}$$

This means $\lim_{n\to\infty}\|\eta_n(\delta)-\eth(\delta)\|<\lim_{n\to\infty}\|\eta_n(\delta)-\eth(\delta)\|$ which is a contraction, so we get that $\gamma(\delta)=\eth(\delta)$. □

Theorem 2.9. *Let L, A and $\eta_{n+1}(\delta)$ satisfies conditions Theorem 2.6, let $J\in\Gamma_r(L)$ and J^n be a compact random operator. Then the iterative scheme on $(RGMI)$ is stable.*

Proof . Let $\{\alpha_n(\delta)\}$ be an arbitrary sequence such that $\sum_{n=1}^{\infty}\epsilon_n<\infty$. For each $\gamma(\delta)\in RF$, since J is a pointwise, we have

$$\begin{aligned}&\|\alpha_{n+1}(\delta)-\eth(\delta)\|\\&=\Big\|\alpha_{n+1}(\delta)-\big(\beta_n\alpha_n(\delta)+(1-\beta_n)J^n\big(\delta,\alpha_n(\delta)\big)\big)+\beta_n\alpha_n(\delta)+(1-\beta_n)J^n\big(\delta,\alpha_n(\delta)\big)-\eth(\delta)\Big\|\\&\le\epsilon_n+\Big\|\beta_n\alpha_n(\delta)+(1-\beta_n)\Big\|J^n\big(\delta,\alpha_n(\delta)\big)-\eth(\delta)\Big\|\\&\le\epsilon_n+\beta_n\Big\|\alpha_n(\delta)+(1-\beta_n)\Big\|J^n\big(\delta,\alpha_n(\delta)\big)-\eth(\delta)\Big\|\\&\le\epsilon_n+\big[1+\zeta_n(\delta)\big]\big\|\alpha_n(\delta)-\eth(\delta)\big\|\end{aligned}$$

Since $\sum_{n=1}^{\infty}\zeta_n\big(\eth(\delta)\big)<\infty$ and $\sum_{n=1}^{\infty}\epsilon_n<\infty$, so with Lemma 1.11, we have $\{\alpha_n(\delta)\}$ is bounded and $\lim_{n\to\infty}\Big\|\alpha_n(\delta)-\eth(\delta)\Big\|=Z$.

On the other hand,

$$\lim_{n\to\infty}\Big\|J^n\big(\delta,\alpha_n(\delta)\big)-\eth(\delta)\Big\| \le \lim_{n\to\infty}\varrho_n\big(\gamma(\delta)\big)\Big\|\alpha_n(\delta)-\eth(\delta)\Big\| \le Z$$

And

$$\lim_{n\to\infty}\Big\|\beta_n\left(\alpha_n(\delta)-\eth(\delta)\right)+(1-\beta_n)\left(J^n\big(\delta,\alpha_n(\delta)\big)-\eth(\delta)\right)\Big\| = Z$$

By Lemma 1.10, we have $\lim_{n\to\infty}\Big\|\alpha_n(\delta)-J^n\left(\gamma,\alpha_n(\delta)\right)\Big\|=0$ because of J^n being a compact random operator, $\exists$ $\{\alpha_{n_i}(\delta)\}$ of $\{\alpha_n(\delta)\}$ and a $\eth(\delta)\in A$ such that $\lim_{n\to\infty}\Big\|J^n\left(\delta,\alpha_{n_i}(\delta)\right)-\eth(\delta)\Big\|=0$

$$\lim_{n\to\infty}\Big\|\alpha_{n_i}(\delta)-\eth(\delta)\Big\| \le \lim_{n\to\infty}\Big\|\alpha_{n_i}(\delta)-J^{n_i}\left(\delta,\alpha_{n_i}(\delta)\right)\Big\|+\lim_{n\to\infty}\Big\|J^{n_i}\alpha_{n_i}(\delta)-\eth(\delta)\Big\|=0$$

By Theorem 2.10, we have that $\eth(\delta)\in RF$.

Since $\lim_{n\to\infty}\big\|\alpha_n(\delta)-\eth(\delta)\big\|=Z$, thus $\lim_{n\to\infty}\alpha_n(\delta)=\eth(\delta)$. □

Theorem 2.10. *Let L, A and $\eta_{n+1}(\delta)$ satisfies conditions Theorem 2.6, let $J\in\Gamma_r(L)$ and J^n be a compact random operator. Then the iterative scheme on $(RGII)$ is stable.*

Proof . Let $\{\alpha_n(\delta)\}$ be an arbitrary sequence such that $\sum_{n=1}^{\infty}\epsilon_n<\infty$. For each $\gamma(\delta)\in RF$. It follows that

$$\Big\|\alpha_{n+1}(\delta)-\eth(\delta)\Big\| = \Big\|\alpha_{n+1}(\delta)-g\big(J,\alpha_n(\delta)\big)+g\big(J,\alpha_n(\delta)\big)-\eth(\delta)\Big\| \le \epsilon_n+\Big\|h\big(J,\alpha_n(\delta)\big)-\eth(\delta)\Big\|$$

Now

$$\begin{aligned}\Big\|h\big(J,\alpha_n(\delta)\big)-\eth(\delta)\Big\| &= \Big\|\rho_n\left(\alpha_n(\delta)-\eth(\delta)\right)+(1-\rho_n)J^n\left(\delta,\mu_n(\delta)\right)-\eth(\delta)\Big\| \\ &\le \rho_n\Big\|\alpha_n(\delta)-\eth(\delta)\Big\|+(1-\rho_n)\Big\|J^n\left(\delta,\mu_n(\delta)\right)-\eth(\delta)\Big\| \\ &\le \rho_n\Big\|\alpha_n(\delta)-\eth(\delta)\Big\|+(1-\rho_n)\varrho_n\big(\eth(\delta)\big)\Big\|\mu_n(\delta)-\eth(\delta)\Big\|\end{aligned}$$

$$\begin{aligned}\Big\|\mu_n(\delta)-\eth(\delta)\Big\| &= \Big\|\beta_n\left(\alpha_n(\delta)-\eth(\delta)\right)+(1-\beta_n)\left(J^n\left(\delta,\alpha_n(\delta)\right)-\eth(\delta)\right)\Big\| \\ &\le \beta_n\Big\|\alpha_n(\delta)-\eth(\delta)\Big\|+(1-\beta_n)\Big\|J^n\left(\delta,\alpha_n(\delta)\right)-\eth(\delta)\Big\| \\ &\le \beta_n\Big\|\alpha_n(\delta)-\eth(\delta)\Big\|+(1-\beta_n)\varrho_n\Big(\eth(\delta)\Big)\Big\|\alpha_n(\delta)-\eth(\delta)\Big\| \\ &\le \varrho_n\Big(\eth(\delta)\Big)\Big\|\alpha_n(\delta)-\eth(\delta)\Big\|\end{aligned}$$

Now

$$\begin{aligned}\Big\|\alpha_{n+1}(\delta)-\eth(\delta)\Big\| &\le \epsilon_n+\Big[1+\varrho_n\big(\eth(\delta)\big)^2-1\Big]\ \Big\|\alpha_{n+1}(\delta)-\eth(\delta)\Big\| \\ &\le \epsilon_n+\Big[1+\big(\zeta_n\big(\eth(\delta)\big)\big)^2+2\zeta_n\big(\eth(\delta)\big)\Big]\ \Big\|\alpha_n(\delta)-\eth(\delta)\Big\|\end{aligned}$$

Since $\sum_{n=1}^{\infty}\zeta_n\big(\eth(\delta)\big)<\infty$ and $\sum_{n=1}^{\infty}\epsilon_n<\infty$, so by Lemma 1.11 we get that $\{\alpha_n(\delta)\}$ is bounded and there exists $\lim_{n\to\infty}\Big\|\alpha_n(\delta)-\eth(\delta)\Big\|=Z$ and $\lim_{n\to\infty}\varrho_n\big(\eth(\delta)\big)=1$ we have $\lim_{n\to\infty}\Big\|\mu_n(\delta)-\eth(\delta)\Big\|\le Z$, $\lim_{n\to\infty}\Big\|\mu_n(\delta)-\eth(\delta)\Big\|=Z$. If $\lim_{n\to\infty}\Big\|\mu_n(\delta)-\eth(\delta)\Big\|<Z$ then $\lim_{n\to\infty}\Big\|g\big(J,\alpha_n(\delta)\big)-\eth(\delta)\Big\|<Z$

$$Z=\lim_{n\to\infty}\Big\|\alpha_{n+1}(\delta)-\eth(\delta)\Big\|\le\lim_{n\to\infty}\epsilon+\lim_{n\to\infty}\Big\|g\big(J,\alpha_n(\delta)\big)-\eth(\delta)\Big\|<Z$$

Which is contradiction, so $\lim_{n\to\infty}\Big\|\mu_n(\delta)-\gamma(\delta)\Big\|=Z$

Since

$$\lim_{n\to\infty}\Big\|J^n\big(\delta,\alpha_n(\delta)\big)-\eth(\delta)\Big\|\le\lim_{n\to\infty}\varrho_n\big(\eth(\delta)\big)\Big\|\alpha_n(\delta)-\eth(\delta)\Big\|\le Z$$

and

$$\lim_{n\to\infty}\Big\|\beta_n\big(\alpha_n(\delta)-\eth(\delta)\big)+(1-\beta_n)J^n\big(\delta,\alpha_n(\delta)\big)-\eth(\delta)\Big\|=\lim_{n\to\infty}\Big\|\mu_n(\delta)-\eth(\delta)\Big\|=Z$$

By Lemma 1.10, we get that $\lim_{n\to\infty}\Big\|\alpha_n(\delta)-J^n\big(\delta,\alpha_n(\delta)\big)\Big\|=0$

Since J^n is compact random operator, $\exists\ \{\alpha_{n_i}(\delta)\}$ of $\{\alpha_n(\delta)\}$ and a $\eth(\delta)\in A$ such that

$$\lim_{n\to\infty}\Big\|J^n\Big(\delta,\alpha_{n_i}(\delta)\Big)-\eth(\delta)\Big\|=0$$

$$\lim_{n\to\infty}\Big\|\alpha_{n_i}(\delta)-\eth(\delta)\Big\|\le\lim_{n\to\infty}\Big\|\alpha_{n_i}(\delta)-J^{n_i}\left(\delta,\alpha_{n_i}(\delta)\right)\Big\|+\lim_{n\to\infty}\Big\|J^{n_i}\alpha_{n_i}(\delta)-\eth(\delta)\Big\|=0$$

By Theorem 2.10, we have that $\eth(\delta)\in RF$.

Since $\lim_{n\to\infty}\Big\|\alpha_n(\delta)-\eth(\delta)\Big\|=Z$, thus $\lim_{n\to\infty}\alpha_n(\delta)=\eth(\delta)$. □

3. Open Problem

Study the convergence fixed point of modified the results papers [19, 20] under random operator

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