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Numerical Solution of Second Order IVP by Fuzzy Transform Method

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Abstract

In this paper, we employed fuzzy transforms to present a new method for solving the problem through a second-order fuzzy initial value. The advantage of the fuzzy transform method is that, unlike other methods (e.g. high-order fuzzy Taylor series), it does not require any higher-order derivative calculation, thus reducing the computational cost. In two examples, the results of the newly proposed method were examined against several conventional methods, indicating the more desirable performance of the new method.

Keywords: Differential equations, Second-order initial value problem (IVP), Fuzzy transform method.

1. Introduction

Fuzzy differential equations play a crucial role in various fields including mathematics, physics, statistics, and engineering. The fuzzy differential equation was first expressed by Kandel and Bayat [6]. There are various techniques to define fuzzy derivatives and therefore to explore fuzzy differential equations [1, 3, 4, 5]. The first and foremost method involves the Hokuhara derivative concept for fuzzy functions. Relying on this method, studies have concentrated on whether there is any uniformity of solutions to fuzzy differential equations [2, 8]. Since it is impossible to express many problems accurately, it is critical to use numerical methods. A few numerical methods have been investigated for solving fuzzy differential equations through the concept of Hokuhara derivative, such as the fuzzy Euler method by Friedman [5], Pederson [9], Abbasbandy's Taylor method [1],

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and Allahviranloo's predictive-corrective method [3]. In classical mathematics, various types of transforms such as integral, Laplace, Fourier, wavelet, etc. have been adopted as powerful tools to develop approximate models and solve differential equations. Their main idea is to transform a space into a special space of functions in which the desired calculations are conducted easier. Then, another transform to the initial space can be applied to produce the initial or approximate function, which is called the inverse transform. One of such transforms is known as a fuzzy transform. The majority of popular classic transforms, some of which were mentioned earlier, use a kernel for the entire domain. However, fuzzy transforms use kernels or local characteristics to build approximate models. Functions are described locally by these characteristics. The obtained local descriptions join together to provide a general description of the function. Fuzzy inverse transform specifies the initial or approximate function as a linear combination of local characteristics. Since fuzzy transforms are considered a special type of Takagi-Sugeno fuzzy system with several additional properties. Such transforms are strongly associated with fuzzy systems. Fuzzy transforms are employed to solve fuzzy differential equations. This paper primarily attempted to find a new solution for differential equations with second-order fuzzy initial value through fuzzy transforms.

In this paper, we use fuzzy transform for the second-order initial value problem (IVP). This paper is organized as follows:
In Section 2, we recall some basic definitions, symbols as well as theorems which used throughout the paper. In Section 3, reviews basic definition briefly and extends the Taylor formula to obtaining the numerical solution of IVP fuzzy differential equation of second kind, we also represent the numerical solution for second-order IVP in three state as well. Some numerical results have been presented in Section 4 to show accuracy and advantage of the proposed method. Finally, some concluding remarks are given in Section 5.

2. Preliminaries

We now recall some basic definitions, symbols as well as a couple of theorems needed through the paper.

Definition 2.1. *A fuzzy number is a fuzzy set $u : \mathbb{R} \to [0,1]$ which satisfies the following properties:*

a) u is upper semi-continuous on $\mathbb{R}$,

b) $u(x) = 0$ outside of some interval [c,d],

c) there are the real numbers a and b with $c \leq a \leq b \leq d$, such that u is increasing on $[c,a]$, decreasing on $[b,d]$ and $u(x) = 1$ for each $x \in [a,b]$,

d) u is fuzzy convex set (that is $u(\lambda x + (1-\lambda)y) \geq min\,\{u(x), u(y)\}$, $\forall x, y \in \mathbb{R}, \lambda \in [0,1]$).

The set of all fuzzy numbers is denoted by E.

Definition 2.2. *Let $n > 2$, $a = x_0 = x_1 < \cdots < x_n = x_{n+1} = b$ be fixed nodes within $[a,b] \subseteq \mathbb{R}$. Fuzzy sets $A_1, \cdots, A_n : [a,b] \to [0,1]$, identified with their membership functions defined on $[a,b]$, establish a fuzzy partition of $[a,b]$, if they fulfill the following conditions for $k = 1, \cdots, n$:*

(1) $A_k(x_k) = 1$;

(2) $A_k(x) = 0$, if $x \in [a,b] \setminus (x_{k-1}, x_{k+1})$;

(3) $A_k(x)$ is continuous on $[x_{k-1}, x_{k+1}]$;

(4) $A_k(x)$ *for* $k = 2, \cdots, n$ *strictly increases on* $[x_{k-1}, x_{k+1}]$ *and for* $k = 1, \cdots, n-1$ *strictly decreases on* $[x_k, x_{k+1}]$*;*

(5) for all $x \in [a, b]$ *the Ruspini condition holds*

$$\sum_{k=1}^{n} A_k(x) = 1.$$

We say that n *is a size of fuzzy partition* $\{A_1, \cdots, A_n\}$ *with the elements to which we refer as to basic functions.*

Definition 2.3. *For any* $u \in E$ *the* α*-cut set of* u *is denoted by* $[u]^\alpha$ *and define by* $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$*, where* $0 \leq \alpha \leq 1$*. The notation* $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha], 0 \leq \alpha \leq 1$ *refers to the lower and upper branches on* u*.*

A fuzzy number u in parametric form is a pair $(\underline{\mathrm{u}}, \bar{u})$, of functions $\underline{\mathrm{u}}(r), \bar{u}(r)$, $0 \leq \alpha \leq 1$, which satisfy the following requirements:

1. $\underline{\mathrm{u}}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$, and right continuous at 0,
2. $\bar{u}(r)$ is a bounded non-increasing left continuous function in $(0, 1]$, and right continuous at 0,
3. $\underline{\mathrm{u}}(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$.
   For arbitrary $u = (\underline{\mathrm{u}}, \bar{u})$, $\nu = (\underline{\mathrm{v}}, \bar{\nu})$ and $k \geq 0$, addition $(u + \nu)$ and multiplication by k as $(\underline{u + \nu}) = \underline{u}(r) + \underline{\nu}(r)$, $(\overline{u + \nu}) = \bar{u}(r) + \bar{\nu}(r)$, $(\underline{ku}(r)) = k\underline{u}(r)$, $\overline{ku}(r) = k\bar{u}(r)$ are defined.

Since each $y \in \mathbb{R}$ can be regarded as a fuzzy number y defined by

$$\tilde{y}(k) = \begin{cases} 1 & t = y \\ 0 & t \neq y. \end{cases}$$

Definition 2.4. *The Hausdorff distance between fuzzy numbers given by* $D : E \times E \to R_+ U\{0\}$*,*

$$D(u, v) = sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)|\}.$$

It is easy to see that D *is a metric in* E *and has the following properties (see [10]).*

(i). $D(u \oplus w, \nu \oplus w) = D(u, \nu), \forall u, \nu, w \in E$,

(ii). $D(k \odot u, k \odot \nu) = |k| D(u, \nu), \forall \in R, u, \nu \in E$,

(iii). $D(u \oplus \nu, w \oplus e) \leq D(u, w) + D(\nu, e), \forall u, \nu, w, e \in E$,

(iv). (D, E) *is a complete metric space.*

Definition 2.5. *The function* $f : T \to E$ *is called a fuzzy function and the* α*-cut set of* f *is represented by* $f(t, \alpha) = \left[\underline{f}(t, \alpha), \overline{f}(t, \alpha)\right], \quad \forall \alpha \in [0, 1]$*. A fuzzy function may have domain and fuzzy range. so the function* $f : E \to E$ *is also a fuzzy function.*

Definition 2.6. *Let $f : \mathbb{R} \to E$ be a fuzzy valued function. If for arbitrary fixed $t_0 \in R$ and $\varepsilon > 0$, $\ni \delta > 0$ such that*

$$| t - t_0 | < \delta \Rightarrow D(f(t), f(t_0)) < \epsilon,$$

f is said to be continuous.

It is well-known that the H-derivative (differentiability in the sense of Hukuhara) for fuzzy mappings was initially introduced by Puri and Ralescu [11] it is based on the H-difference of sets, as follows:

Definition 2.7. *Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y, and it is denoted by $x \ominus y$.*

Definition 2.8. *Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y, and it is denoted by $x \ominus y$.*

Let $A_1, \cdots, A_n$ be basic functions, which form a fuzzy partition of $[a, b]$, and f be any continuous function on $[a, b]$. We say that the n-tuple of real numbers $[F_1, \cdots, F_n]$ given by

$$F_k = \frac{\int_a^b f(x)A_k(x)dx}{\int_a^b A_k(x)dx}, \quad k = 1, 2, \cdots, n.$$

Theorem 2.9. *(See [12].) Let polynomial F_k^m be the orthogonal projection of $f \in L_2(A_k)$ on $L_n^m(Ak)$. Then we have $F_k^m = c_{k,0}P_k^0 + c_{k,1}P_k^1 + \cdots + c_{k,m}P_k^m$, where for all $i = 0, 1, \ldots, m$,*

$$c_{k,j} = \frac{(f, P_k^i)_k}{(P_k^i, P_k^i)_k} = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)P_k^i(x)A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} P_k^i(x)P_k^i(x)A_k(x)dx}. \tag{2.1}$$

Definition 2.10. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function from $L_2(A_1, \cdots, A_n)$, and let $m \geq 0$ be a fix integer.*
Denote F_k^m as the $k-th$ orthogonal projection of $f|_{[x_{k-1}, x_{k+1}]}$ on $L_2^m(A_k), k = 1, \cdots, n$. We say that $n-$tuple of orthogonal polynomials $[F_1^m, \cdots, F_n^m]$ is an F^m- transform of f with respect to $A_1, \cdots, A_n$, or formally, $F^m[f] = [F_1^m, \cdots, F_n^m]$.
Then F_k^m is called the $k-$ F^m-transform component of f.
In special case for $m = 2$, we denote F_k^2 as the 2-th orthogonal projection of $f|_{[}x_{k-1}, x_{k+1}]$ on $L_2^2(A_k), k = 1, \cdots, n$.
We say that $n-$tuple of orthogonal polynomials $[F_1^2, \ldots, F_n^2]$ is an F^2-transform of f with respect $A_1, \cdots, A_n$, or formally

$$F^2[f] = [F_1^2, \ldots, F_n^2].$$

Then F_k^2 is called the $k-th$ F^2-transform component of f, and for $k = 1, 2, \ldots, n$, we have $F_k^2 = c_{k,0}P_k^0 + c_{k,1}P_k^1 + c_{k,2}P_k^2$, where $c_{k,i}$ defined (2.1).

Theorem 2.11. *Polynomials*

$$P_k^0 = 1, \quad P_k^1(x) = x - x_k, \quad P_k^2(x) = (x - x_k)^2 - \frac{I_2}{h}$$

where

$$I_2 = \int_{k-1}^{k+1} (x - x_k)^2 A_k(x)dx,$$

are orthogonal in $L_2(A_k), k = 1, \ldots, n$.

Proof . See Lemma 4 and Lemma 5 in [12]. □

Theorem 2.12. *(see [12].) By applying Simpson's formula, we get*

$$I_2 = \int_{k-1}^{k+1} (x - x_k)^2 A_k(x)dx = \frac{8h_1^3}{3} A_k(x_k + h_1) + R_1,$$

where

$$R_1 = -\frac{(x_{k+1} - x_{k-1})((\eta - x_k)^2 A_k(\eta))^{(4)}}{180} h_1^4 = -\frac{h_5}{45}((\eta - x_k)^2 A_k(\eta))^{(4)}, \quad \eta \in (x_{k-1}, x_{k+1}).$$

Theorem 2.13. *The following vector of quadratic polynomials:*

$$F^2[f] = [c_{1,0} + c_{1,1}P_1^1(x) + c_{1,2}P_1^2(x), \ldots, c_{k,1} + c_{k,2}P_k^1 + c_{k,2}P_k^2],$$

is the F^2transform of f with respect to $A_1, \cdots, A_n$, where for every $k = 1, 2, \ldots, n$,

$$c_{k,0} = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx}$$

$$c_{k,1} = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)(x - x_k)A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} (x - x_k)^2 A_k(x)dx}$$

$$c_{k,2} = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)((x - x_k)^2 - \frac{I_2}{h})A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} ((x - x_k)^2 - \frac{I_2}{h})A_k(x)dx}.$$

Theorem 2.14. *(see [7].) Let $A_1, \cdots, A_n$ be an $h-$uniform partion of $[a, b]$. Let function f and A_k $k = 1, 2, \ldots, n$, the following estimation holds true:*

$$c_{k,2} = \frac{f''(x_k)}{2} + O(h^2).$$

Definition 2.15. *Let $f : [a, b] \longrightarrow R$ be a function from $L_2(A_1, \cdots, A_n)$, and let $m \geq 0$ be a fix integer.*
Denote F_k^m as the $k-th$ orthogonal projection of $f[x_{k-1}, x_{k+1}]$ on $L_2^m(A_k), k = 1, \cdots, n$. We say that $n-$tuple of orthogonal polynomials $[F_1^m, \cdots, F_n^m]$ is an F^m-transform of f with respect to $A_1, \cdots, A_n$, or formally, $F^m[f] = [F_1^m, \cdots, F_n^m]$.
Then F_k^m is called the $k-$ F^m-transform component of f.
In special case for $m = 2$, we denote F_k^2 as the 2-th orthogonal projection of $f|_{[}x_{k-1}, x_{k+1}]$ on $L_2^2(A_k), k = 1, \cdots, n$.
We say that $n-$tuple of orthogonal polynomials $[F_1^2, \ldots, F_n^2]$ is an F^2-transform of f with respect $A_1, \cdots, A_n$, or formally

$$F^2[f] = [F_1^2, \ldots, F_n^2].$$

Then F_k^2 is called the $k-th$ F^2-transform component of f, and for $k = 1, 2, \ldots, n$, we have $F_k^2 = c_{k,0}P_k^0 + c_{k,1}P_k^1 + c_{k,2}P_k^2$,
where $c_{k,i}$ defined as (2.1).

3. Construction of the solution of second order IVP

In this section, we recall some definition for obtaining the new results firstly, after which we extends the Taylor formula to obtaining the numerical solution of IVP fuzzy differential equation of second kind.

Let us remind the brief properties of fuzzy number as well as differential of which.
A fuzzy number $u : \mathbb{R} \longrightarrow [0,1]$ is a fuzzy subset of the real line, satisfy the following properties:

(i) u is normal, i.e., there exists $s_0 \in \mathbb{R}$ such that $u(s_0) = 1$,

(ii) u is convex fuzzy set (i.e. $u(ts + (1-tr) \geq min\{u(s), u(r)\}, \forall t \in [0,1], s, r \in \mathbb{R})$,

(iii) u is upper semi-continuous on $\mathbb{R}$,

(iv) $cls \in \mathbb{R}|u(s) > 0$ is compact where cl denotes the closure of subset. The set of fuzzy numbers is denoted by $\mathbb{R}_F$.

For $0 < \alpha \leq$ and $u \in R_F$, denote $[u]_\alpha = \{s \in \mathbb{R}|u(s) \geq \alpha\}$ and $[u]_0 = cl\{s \in \mathbb{R}|u(s) > 0\}$. It is well-known that for any $\alpha \in [0,1]$, $[u]_\alpha$ is a bounded close interval.
The notation

$$[u]_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha].$$

denotes explicitly the α-level set of u, For $u \in \mathbb{R}_F$ and sum $u + v$ and the scaler product λ, u are defined by $[u+v]_\alpha = [u]_\alpha + [v]_\alpha, [\lambda, u]_\alpha = \lambda[u]_\alpha, \forall \alpha \in [0,1]$ where $[u]_\alpha + [v]_\alpha$ means the usual addition of two intervals (subsets) od R and $\lambda[u]_\alpha$ means the usual product between a scaler and a subset of R.
The metric structure is given by the Hausdorff distance

$$D_\infty : \mathbb{R}_F \times \mathbb{R}_F \longrightarrow \mathbb{R}_+ \cup \{0\},$$

by

$$D_\infty(u, v) = \sup_{\alpha \in [0,1]} max\{|\underline{u}_\alpha - \underline{u}_\alpha|, |\overline{u}_\alpha - \overline{u}_\alpha|\}.$$

Definition 3.1. *Let $f : (a, b) \to \mathbb{R}_F$ and $x_0 \in (a, b)$. We say that f is differential at x_0, If there exists an element $\acute{f}(x_0) \in \mathbb{R}_F$, such that*

(1). for all $h > 0$ sufficiently near to 0, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h} = \acute{f}(x_0)$$

or

(2). for all $h < 0$ sufficiently near to 0, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \to 0^-} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0^-} \frac{f(x_0) \ominus f(x_0 - h)}{h} = \acute{f}(x_0)$$

Now according to the Taylor formula, if we take 4-th differentiable from $y(x)$ on interval I, we have

$$y(x+h) = y(x-h) + 2hy'(x) + O(h^2) \tag{3.1}$$

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + O(h^4) \tag{3.2}$$

From (3.1), (3.2) we can write:

$$h^2 y''(x) = 2y(x+h) - 2y(x) + (y(x+h) - y(x-h))$$

$$y''(x) = \frac{2y(x+h) - 2y(x) + (y(x+h) - y(x-h))}{h^2} + O(h^2) \tag{3.3}$$

and

$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h) \tag{3.4}$$

Now we set $y_1(x) = y(x+h)$ and $y_2 = y(x-h)$, then we apply the Fuzzy transform on the expression (3.3) and (3.4) and we have

$$F_n[y'] = \frac{1}{h}(F_n[y_1] - F_n[y]) + O(h) \tag{3.5}$$

$$F_n[y''] = \frac{1}{h^2}(F_n[y_1] - 2F_n[y] + F_n[y_2]) + O(h^2) \tag{3.6}$$

where

$$\begin{aligned}
F_n[y_1] &= [Y_{1_2}, Y_{1_3}, \cdots, Y_{1_{n-1}}] = [Y_{1_1}, Y_{1_2}, \cdots, Y_{1_{n-2}}],\\
F_n[y_2] &= [Y_{2_2}, Y_{2_3}, \cdots, Y_{2_{n-1}}] = [Y_{2_1}, Y_{2_2}, \cdots, Y_{2_{n-2}}],\\
F_n[y'] &= [Y'_2, Y'_3, \cdots, Y'_{n-1}],\\
F_n[y''] &= [Y''_2, Y''_3, \cdots, Y''_{n-1}],\\
F_n[y] &= [Y_2, Y_3, \cdots, Y_{n-1}] = [Y_2, Y_3, \cdots, Y_{n-2}],
\end{aligned}$$

From the properties and definition of fuzzy transform for $k = 2, 3, \ldots, n-2$ we can write:

$$Y_{1,k} = \frac{\int_{x_{k-1}}^{x_{k+1}} y(x+h)A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx} \tag{3.7}$$

Furtheremore, we know from Definition 2.2 that $\int_{x_{k-1}}^{x_{k+1}} A_k(x)dx = h$ so we have:

$$Y_{1,k} = \frac{\int_{x_{k-1}}^{x_{k+1}} y(x+h)A_k(x)dx}{h}$$

Now we set $x + h = t$ and due to the fact that $A_k(t-h) = A_{k+1}(t)$, we can conlcule that

$$\begin{aligned}
Y_{1,k} &= \frac{1}{h}\int_{x_k}^{x_{k+2}} y(t+h)A_{k+1}(t)dt\\
&= \frac{1}{h}\int_{x_k}^{x_{k+2}} y(t)A_{k+1}(t)dt = Y_{k+1}.
\end{aligned} \tag{3.8}$$

Now for special case of $k = n-1$ and by the fact that $A_{n-1}(t-h) = A_n(t)$ in Equation (3.7) we have:

$$\begin{aligned} Y_{1,k} &= \frac{\int_{x_{n-2}}^{x_n} y(t+h)A_{n-1}(t)dt}{\int_{x_{n-2}}^{x_n} A_{n-1}(t)dt} = \frac{\int_{x_{n-2}}^{x_n} y(t)A_n(t)dt}{\int_{x_{n-2}}^{x_n} A_n(t)dt} \\ &= \frac{2}{h}\int_{x_{n-2}}^{x_n} y(t)A_n(t)dt = \frac{2}{h}\bigg(\frac{h}{2}\Big(y(x_n)A_n(x_n) + y(x_{n-1})A_n(x_{n-1})\Big)\bigg) + O(h^3) \\ &= y(x_n) + O(h^2) = Y_n + O(h^2), \end{aligned}$$

So we obtain the the equation as fallows.

$$\begin{cases} Y_{1_k} = Y_{k+1}, \\ Y_{1_{n-1}} = Y_n + O(h^2), \end{cases}$$

Similarly, for $n = 3, 4, \ldots, n-1$ we have

$$\begin{cases} Y_{2_l} = Y_{l-1}, \\ Y_{2_2} = Y_1 + O(h^2), \end{cases}$$

Now according to the (3.4) we have

$$Y'_k = \frac{1}{h}(Y_{1_k} - Y_k + O(h) = \frac{1}{h}(y_{k+h} - Y_k) + O(h), \tag{3.9}$$

we can also rewrite the (3.3) as fallows.

$$\begin{aligned} Y''_k &= \frac{1}{h^2}(Y_{1_k} - 2Y_k + Y_{2_k}) + O(h^2) \\ &= \frac{1}{h^2}(Y_{k+1} - 2Y_k + Y_{k-1}) + O(h^2) \end{aligned} \tag{3.10}$$

By using (**??**) we can construct the coefficients matrix of order $(n-2)\times(n-1)$ such as:

$$D = \frac{1}{h}\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & 0 & \ldots & 0 & & -1 & 1 \end{bmatrix}_{(n-2)\times(n-1)}$$

We also can using (3.10) and derive the coefficient matrix of the order $(n-2)\times n$ as fallows.

$$T = \frac{1}{h^2}\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 & -2 & 1 \end{bmatrix}_{(n-2)\times(n)}$$

Now we can rewrite the Fuzzy transform by using above coefficients matrices

$$F_n[y'] = DF_n^c[y], \quad F_n[y''] = T\tilde{F}_n^c[y],$$

where

$$\tilde{F}_n[y] = [Y_1, Y_2, \cdots, Y_n]^T, \quad , F_n^c[y] = [Y_2, Y_3, \cdots, Y_{n-1}, Y_n]^T,$$

Now for converting matrix D to an square $(n \times n)$ matrix we add two rows and a single column like as:

$$D^c = \frac{1}{h} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & 0 & \ldots & 0 & & -1 & 1 \end{bmatrix}_{n\times n} \tag{3.11}$$

we also add two rows to matrix T like:

$$T = \frac{1}{h^2} \begin{bmatrix} h & 0 & 0 & 0 & 0 & \cdots & 0 \\ -h & h & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 & -2 & 1 \end{bmatrix}_{n\times n}$$

Both of T^c and D^c are non singular of order $(n \times n)$. Now we can represent a pattern to solution of IVP differential equation in three states.

i) The IVP differential equations without y':

The IVP differential equation in this state as

$$\begin{cases} y'' = f(x, y), \\ y(x_1) = y_1, \quad y(x_2) = y_2, \end{cases} \tag{3.12}$$

Now for approximate the solution of the above IVP we use the Fuzzy Transform for both side of (3.12) such as:

$$F_n[y''] = F_n[f(x, y)] \Longrightarrow T\tilde{F}_n[y] = F_n[f],$$

where in which $F_n[f] = [F_n, F_3, \cdots, F_{n-1}]^T$ is a fuzzy transform $f(x, y)$ respect to x such that we eliminate the first and end point of domain as boundary point. Then we impose two components $\frac{y_1}{h^2}$ and $\frac{y_2}{h^2}$ for completeness of $\tilde{F}_n[f]$ as fallows.

$$\tilde{F}_n[f] = \left[\frac{y_2}{h^1}, \frac{y_2}{h^2}, F_2, F_3, \cdots, F_{n-1}\right]^T,$$

therefore by applying the T^c and (**??**) one can write :

$$T^c\tilde{F}_n[y] = \tilde{F}_n[f] \Longrightarrow \tilde{F}_n[y] = (T^c)^{-1}\tilde{F}_n[f], \tag{3.13}$$

For obtaining the solution we need to find the T^c inverse as :

$$T^{c^{-1}} = h^2 \begin{bmatrix} 1/h & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1/h & 1/h & 0 & 0 & 0 & \cdots & 0 \\ 1/h & 2/h & 1 & 0 & 0 & \cdots & 0 \\ 1/h & 3/h & 2 & 1 & 0 & \cdots & 0 \\ 1/h & 4/h & 3 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & 0 \\ 1/h & n-1/h & n-2 & \dots & 3 & 2 & 1 \end{bmatrix}_{n\times n}$$

By substitution $T^{c^{-1}}$ into (3.13) we have:

$$\tilde{F}_n[y] = h^2 \begin{bmatrix} 1/h & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1/h & 1/h & 0 & 0 & 0 & \cdots & 0 \\ 1/h & 2/h & 1 & 0 & 0 & \cdots & 0 \\ 1/h & 3/h & 2 & 1 & 0 & \cdots & 0 \\ 1/h & 4/h & 3 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & 0 \\ 1/h & n-1/h & n-2 & \dots & 3 & 2 & 1 \end{bmatrix}_{n\times n} \times \left[\frac{y_2}{h^1}, \frac{y_2}{h^2}, F_2, F_3, \cdots, F_{n-1}\right]^T,$$

with expanding the above system we obtain:

$$Y_1 = h^2\left(\frac{1}{h} \times \frac{1}{h}\right) = \frac{1}{h}y_1,$$

$$Y_2 = h^2\left(\frac{1}{h} \times \frac{y_1}{h^2} + \frac{1}{h}\frac{y_2}{h^2}\right) = \frac{1}{h}y_1 + \frac{1}{h}y_2 = \frac{1}{h}(y_1 + y_2),$$

$$Y_3 = h^2\left(\frac{1}{h} \times \frac{y_1}{h^2} + \frac{2}{h}\frac{y_2}{h^2} + 1 \times F_2\right) = \frac{1}{h}(y_1 + 2y_2) + h^2 F_2,$$

similarlly,

$$Y_k = \frac{1}{h}\big(y_+(k+1)y_2\big) + h^2(k-2)F_2 + (k-3)F_3 + (k-4)f_4 + \cdots + F_{k-1},$$

$$Y_k = \frac{1}{h}\big(y_+(k+1)y_2\big) + h^2 \sum_{j=2}^{k-1}(k-j)F_j,$$

Now we can evaluate the $Y_3, Y_4, \cdots, Y_n$, from above system and (3.13). These components could not be evaluate in direct maner, so we use the following approximation pattern such as :

$$\tilde{F}_k = \frac{\int_a^b f(x, Y_k)A_k(x)dx}{\int_a^b A_k(x)dx}, \quad k = 2, 3, \cdots, n-1,$$

where $|F_k - \tilde{F}_k| = O(h^2)$.

ii) The IVP differential equations with y':

Let IVP differential equation as

$$\begin{cases} y'' + p(x)y' = f(x,y), \\ y(x_1) = y_1, \quad y(x_2) = y_2, \end{cases} \tag{3.14}$$

In order to find the solution of the above IVP we use the Fuzzy Transform for both side of (3.14) such as:

$$F_n[y'' + p(x)y'] = F_n[f(x,y)] \Longrightarrow \tilde{F}_n[y''] + p(x)F_n[y'] = F_n[f(x,y)],$$

where in which $F_n[f] = [F_n, F_3, \cdots, F_{n-1}]^T$ is a fuzzy transform $f(x,y)$ respect to x such that we eliminate the first and end point of domain as boundary point. We have also

$$\tilde{F}_n[y'] = D\overline{F}_n[y], \quad \tilde{F}_n[y''] = T\tilde{F}_n[y]$$

where

$$\overline{F}_n[y] = [y_2, y_3, \dots, y_n]^T, \quad \tilde{F}_n[y] = T\tilde{F}_n[y_2, y_3, \dots, y_n]^T,$$

therefore by (**??**) we can write the fuzzy transform as:

$$T^c\tilde{F}_n[y] + p(x)D\overline{F}_n[y] = F_n[f(x,y)],$$

Then we impose two components $\dfrac{y_1}{h^2}$ and $y' = \dfrac{y_2}{h^2}$ for completeness of $F_n[f]$ as fallows.

$$\tilde{F}_n[f] = \left[\frac{y_1}{h^2}, y_1', F_2, F_3, \dots, F_{n-1}\right]^T$$

Now we can rewrite the system of IVP differential equation by using D^c, T^c and $\tilde{F}_n[f]$ as fallows.

$$(T^c + p(x)D^c)\tilde{F}_n[y] = \tilde{F}_n[f] \Longrightarrow, \tilde{F}_n[y] = (T^c + p(x)D^c)^{-1}\tilde{F}_n[f]. \tag{3.15}$$

Both of T^c and D^c are non singular therefore any linear combination of which such as $T^c + p(x)D^c$ is non sigular if $\dfrac{1}{h} \neq -p(x)$. So its enough to find the inverse of $T^c + p(x)D^c$ as

$$T^{c^{-1}} = h^2 \begin{bmatrix} 1/h & 0 & 0 & 0 \\ 1/h & 1/h & 0 & 0 \\ 1/h & \dfrac{(1+p(x)h)^2 - 1}{p(x)h^2 + (1+p(x)h)} & \dfrac{1}{1+p(x)h} & 0 \\ 1/h & \dfrac{(1+p(x)h)^3 - 1}{p(x)h^2 + (1+p(x)h)} & \dfrac{(1+p(x)h)^2 - 1}{p(x)h(1+p(x)h)^2} & 0 \\ 1/h & \vdots & \vdots & 0 \\ 1/h & \dfrac{(1+p(x)h)^{n-1} - 1}{p(x)h^2 + (1+p(x)h)} & \dfrac{(1+p(x)h)^{n-2} - 1}{p(x)h(1+p(x)h)^2} & \dfrac{1}{1+p(x)} \end{bmatrix}_{n\times n}$$

In the similar way we have

$$Y_1 = y_1$$

$$Y_2 = (y_1 + hy_2)$$

$$Y_k = y_1 + \frac{1 + p(x)h)^{k-1} - 1}{p(x)(1+p(x))^{k-2}}y_1' + h\sum_{j=2}^{k-2}\frac{1 + p(x)h)^{k-j} - 1}{p(x)(1+p(x))^{k-j}}F_j + \frac{h^2}{1+p(x)h}F_{k-1}.$$

iii) The IVP differential equations with $g(y')$:

Let IVP differential equation as

$$\begin{cases} y'' + g(y') = f(x, y), \\ y(x_1) = y_1, \quad y(x_2) = y_2, \end{cases} \tag{3.16}$$

In order to find the solution of the above IVP we assume that g is linear i.e. $g(y') = ay' + b$ then we rewritr the (3.16) such as:

$$\begin{cases} y'' + ay' + b = f(x, y), \\ y(x_1) = y_1, \quad y(x_2) = y_2, \end{cases} \tag{3.17}$$

Now we can apply the Fuzzy Transform method for both sides of (3.17) and we have

$$F_n[Y'' + ay' + b] = F_n[f(x, y)],$$

we can see that this state is a special case of previous section by $p(x) = a$ and we could rewrite the right hand of equation as $f(x, y) - b$ and using the method which is

$$Y_k = y_1 + \frac{1 + ah)^{k-1} - 1}{a(1+a)^{k-2}} y_1' + h \sum_{j=2}^{k-2} \frac{1 + ah)^{k-j} - 1}{a(1+a)^{k-j}} F_j + \frac{h^2}{1 + ah} F_{k-1}.$$

4. Numerical results

In this section, we examine the proposed method to indicate the efficiency of which to approximate the solution of the second order of IVP differential equations.

Example 4.1. *Consider the following second-order fuzzy differential equation with fuzzy initial value*

$$y^{''}(t) = -y(t), \quad (t > 0), \tag{4.1}$$

$$y(0) = 0, \quad y^{''} = (0.9 + 0.1r, 1.1 - 0.1r).$$

The exact solution is as follows $Y(t, r) = ((0.9 + 0.1r)\sin(t), (1.1 - 0.1r)\sin(t))$.

Comparing the results of our suggested solution with the results of Fuzzy Improved Runge-Kutta Nystrom (FIRKN) method [13] can be seen in Table **??**.

Example 4.2. *Consider the following second order differential equation of IVP as*

$$\begin{aligned} &y''(t) = -y(t) + t, \quad (t > 0), \\ &y(0) = [0.9 + 0.1r, 1.1 - 0.1r], \quad y(0) = [1.8 + 0.2r, 2.2 - 0.2r] \end{aligned} \tag{4.2}$$

The exact solution under (1)-differentiability:

$$y(t) = [y_1(t, r), y_1(t, r)] \tag{4.3}$$

where

$$\begin{aligned} y_1(t, r) &= \left(\frac{4}{5} + \frac{1}{5}r\right)\sin(t) + \left(\frac{9}{10} + \frac{1}{10}r\right)\cos(t) + t, \\ y_2(t, r) &= \left(\frac{6}{5} - \frac{1}{5}r\right)\sin(t) + \left(\frac{11}{10} - \frac{1}{10}r\right)\cos(t) + t \end{aligned} \tag{4.4}$$

Table 1: Comparing the results of our suggested solution with the results of (FIRKN) method for Example 4.1

	$y_1(t,r)$	$y_1(t,r)$	$y_2(t,r)$	$y_2(t,r)$
r	FIRKN5 (s = 4)	present method	FIRKN5 (s = 4)	present method
0.0	2.18E-10	1.01E-12	2.66E-10	1.11E-12
0.1	2.20E-10	1.02E-12	2.64E-10	1.10E-12
0.2	2.23E-10	1.03E-12	2.61E-10	1.09E-12
0.3	2.25E-10	1.04E-12	2.59E-10	1.08E-12
0.4	2.27E-10	1.05E-12	2.57E-10	1.07E-12
0.5	2.30E-10	1.06E-12	2.54E-10	1.06E-12
0.6	2.32E-10	1.07E-12	2.52E-10	1.05E-12
0.7	2.37E-10	1.08E-12	2.49E-10	1.04E-12
0.8	2.40E-10	1.09E-12	2.47E-10	1.03E-12
0.9	2.42E-10	1.10E-12	2.44E-10	1.02E-12
1	2.36E-10	1.11E-12	2.42E-10	1.01E-12

Comparing the results of our suggested solution with the results of Fuzzy Improved Runge-Kutta Nystrom (FIRKN) method [13] can be seen in Table 2.

Table 2: Comparing the results of our suggested solution with the results of (FIRKN) method for Example 4.2.

	$y_1(t,r)$	$y_1(t,r)$	$y_2(t,r)$	$y_2(t,r)$
r	FIRKN5 (s = 4)	present method	FIRKN5 (s = 4)	present method
0.0	3.13E-8	2.18E-10	4.32E-8	3.06E-10
0.1	3.19E-8	2.19E-10	4.26E-8	3.04E-10
0.2	3.25E-8	2.20E-10	4.20E-8	3.06E-10
0.3	3.31E-8	2.21E-10	4.15E-8	3.06E-10
0.4	3.37E-8	2.22E-10	4.09E-8	3.01E-10
0.5	3.43E-8	2.23E-10	4.03E-8	3.07E-10
0.6	3.49E-8	2.24E-10	3.97E-8	3.05E-10
0.7	3.55E-8	2.25E-10	3.91E-8	3.09E-10
0.8	3.61E-8	2.26E-10	3.85E-8	3.03E-10
0.9	3.67E-8	2.27E-10	3.79E-8	3.02E-10
1	3.73E-8	2.29E-10	3.73E-8	3.07E-10

5. Conclusion

Since the most physical phenomena in mathematics, physics, mechanics, and scientific computation are modeled by IVP differential equations, many of such equations are defined inside an ambiguous environment. Analytically calculation of such problems in a limited and semi-limited space is not an easy task, lacking the necessary accuracy and speed. Therefore, certain numerical methods can be useful for solving such problems. Meanwhile, the second-order initial fuzzy value problems have different states that should be considered. In this paper, we solved three models of second-order initial fuzzy value problems, presented in the third section. Our newly proposed method is based on fuzzy transforms. Fuzzy transforms can reduce computational cost since they do not require any high-order derivative calculations.

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