Int. J. Nonlinear Anal. Appl.
Volume 11, Special Issue, Winter and Spring 2020, 543-546
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/IJNAA.2020.4751

Two Efficient Methods For Solving Non-linear Fourth-Order PDEs

Huda Omran Altaie [a*]

[a] *University of Baghdad / Department of Mathematics College of Education For Pure Scineces/ Ibn Al-Haitham Iraq*

Abstract

This paper studies a novel technique based on the use of two effective methods like modified Laplace-variational method (MLVIM) and a new Variational method (MVIM)to solve PDEs with variable coefficients. The current modification for the (MLVIM) is based on coupling of the Variational method (VIM) and Laplace- method (LT). In our proposal there is no need to calculate Lagrange multiplier. We applied Laplace method to the problem .Furthermore, the nonlinear terms for this problem is solved using homotopy method (HPM). Some examples are taken to compare results between two methods and to verify the reliability of our present methods.

Keywords: Homotopy method, He's polynomials, PDEs, Laplace-Variational iteration method, Lagrange multipliers. Introduction

1. Introduction

In recent years, applications of partial differential equations have greatly attracted several researchers [1,2,3].Some types of differential equations such as PDEs played a fluent role in engineering and sciences. Several methods suggested like an Explicit scheme , Spline- scheme, and decomposition method [4,5,6] to find solutions of 4^{th} order Partial differential equations. An evolution of the VIM is presented [7]. Thus, homotopy-perturbation method [8]was developed to solve PDEs with nonlinear terms. Recently, some authors have used Laplace variational method to solve several types of PDEs [9].

*Corresponding Author: Huda Omran Altaie
Email address: (Huda Omran Altaie [a*])

Received: February 2020 *Revised:* August 2020

2. Modified -Variational Iteration Method for PDEs

Consider a non-homogeneous PDEs with variable coefficients of the form:

$$\frac{\partial^2 u}{\partial t^2} + \gamma(x)\frac{\partial^4 u}{\partial x^4} = \varphi(x,t), \quad u(x,0) = g(x), \quad \frac{\partial u}{\partial t}(x,0) = M(x)\cdots \quad (2.1)$$

Where, $\gamma(x)x$ is a variable coefficient and the boundary conditions are

$$u(a,t) = \mu_1(x), \quad u(b,t) = \mu_2(x)$$
$$\frac{\partial^4 u}{\partial x^4}(a,t) = \mu_3(x), \quad \frac{\partial^4 u}{\partial x^4}(b,t) = \mu_4(x)$$

Apply modified variational iteration method of Eq. (2.1)

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \lambda(\mathfrak{z})[\frac{\partial^2 u_k(x,\eta)}{\partial \eta^2} + \gamma(x)\frac{\partial^4 \tilde{u_k}}{\partial x^4}(x,\mathfrak{z}) - \Phi(x,\mathfrak{z})]\partial_{\mathfrak{z}}$$

where λ is a Lagrange multiplier ($\lambda = \xi - t$), and k denote the $k-th$ approximation,$\partial\tilde{u_k} = 0$ and the solution is given by $u = \lim_{n\to\infty} u_n$.

3. Modified Laplace variational iteration method (MLVIM)

First, use Laplace transform of PDEs and multiply it with an unknown parameter. After taking homotopy perturbation method [8] to calculate the nonlinear terms. Consider the following differential equation and take the Laplace transform , that yields: $\mathfrak{L}[Mu - Nu - c] = 0\cdots \quad (3.1)$
Then $\lambda(s)\{\mathfrak{L}[Mu] - \mathfrak{L}[Nu + c]\} = 0$. Thus, the Laplace transform of Eq(3.1) can be written $u_{(k+1)}(s) = u_k(s) + \lambda(s)\{\mathfrak{L}[Mu_k] - \mathfrak{L}[Nu_k + c]\}$.
Using the inverse Laplace transform

$$u_{(k+1)}(s) = u_k(s) + \mathfrak{L}^{(-1)}[\lambda(s)\{\mathfrak{L}[Mu_k] - \mathfrak{L}[Nu_k + c]\}]$$

Then, use homotopy method (HPM) to get the solution.

4. Comparison of Modified Variational Iteration Method (MVIM) Combined with Modified Laplace and Variations Iteration Method (MLTVIM)for solving PDEs

Here, the major goal is to present a comparative study for solving nonlinear PDEs with variable coefficients using (MVIM) and (MLTVIM).

Example (4.1): By using (MVIM)

Consider homogenous PDEs with variable coefficients

$$\frac{\partial^2 w}{\partial t^2} + (\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 w}{\partial t^4} = 0 \quad 1 < x < 2, \quad t > 0 \quad (4.1)$$

With initial condition $w(x,0) = 0, \frac{\partial w}{\partial t}(x,0) = 1 + \frac{x^5}{120}$, and B.C

$$w((0,50),t) = (1 + \frac{(0.50)^5}{120})sin(t), \quad w(1,t) = (\frac{121}{120})sin(t)$$
$$\frac{\partial^2 w}{\partial x^2}(\frac{1}{2},t) = \frac{1}{6}(0.125)sin(t), \quad \frac{\partial^2 w}{\partial x^2}(1,t) = \frac{1}{6}sin(t)$$

By Applying Modified Variational Iteration Method to Eq.(2.3), we obtain

$$w_{(k+1)}(x,t) = w_{(k)}(x,t) + \int_0^t \lambda(\mathfrak{z})[\frac{\partial^2 w_k(x,\mathfrak{z})}{\partial \mathfrak{z}^2} + (\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 \tilde{w}}{\partial x^4}(x,\mathfrak{z})]\partial \mathfrak{z}, k \geqslant 0]$$

Suppose ($\lambda = \xi - t$), that implies

$$w_{(k+1)}(x,t) = w_{(k)}(x,t) + \int_0^t (\xi - t)[\frac{\partial^2 w_k(x,\mathfrak{z})}{\partial \mathfrak{z}^2} + (\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 \tilde{w}}{\partial x^4}(x,\mathfrak{z})]\partial \mathfrak{z}$$

Now take

$$w_0(x,t) = w(x,t) = (1 + \frac{x^5}{120})t,$$

$$w_1(x,t) = (1 + \frac{x^5}{120})t + \int_0^t (\xi - t)\left[(\frac{1}{x} - \frac{x^4}{120})x_{\mathfrak{z}}\right]\partial \mathfrak{z} = (1 + \frac{x^5}{120})t - (1 + \frac{x^5}{120})(\frac{t^3}{3!})$$

$$w_2(x,t) = (1 + \frac{x^5}{120})t - (1 + \frac{x^5}{120})(\frac{t^3}{3!}) + \int_0^t (\xi - t)\left[-(1 + \frac{x^5}{120}) + (\frac{1}{x} - \frac{x^4}{120})(x^3 - x\frac{3^3}{3!})\right]\partial \mathfrak{z} =$$

$$(1 + \frac{x^5}{120})(t) - (1 + \frac{x^5}{120})(\frac{t^3}{3!}) + (1 + \frac{x^5}{120})(\frac{t^5}{5!})$$

$$w_3(x,t) = (1 + \frac{x^5}{120})(t) - (1 + \frac{x^5}{120})(\frac{t^3}{3!}) + (1 + \frac{x^5}{120})(\frac{t^5}{5!}) - (1 + \frac{x^5}{120})(\frac{t^7}{7!})$$

$$w_k(x,t) = (1 + \frac{x^5}{120})(t) - (1 + \frac{x^5}{120})(\frac{t^3}{3!}) + (1 + \frac{x^5}{120})(\frac{t^5}{5!}) - (1 + \frac{x^5}{120})(\frac{t^7}{7!}) + \ldots$$

$$= (1 + \frac{x^5}{120})(t - (\frac{t^3}{3!}) + (\frac{t^5}{5!}) - (\frac{t^7}{7!}) + \ldots)$$

Then the exact solution $w(x,t) = (1 + \frac{x^5}{120})sin(t)$ **Example (4.1) :By using (MLVIM)** By using the same previous example, and using Laplace transform in eq.(4.1) $\mathfrak{L}\left[\frac{\partial^2 w}{\partial t^2} + (1 + \frac{x^4}{120})\frac{\partial^4 w}{\partial t^4}\right] = 0$ and by multiplying the above equation with $\lambda\lambda\mathfrak{L}\left[\frac{\partial^2 w}{\partial t^2} + (\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 w}{\partial t^4}\right] = 0$ Then ,we get $w_{(k+1)}(x,s) = w_{(k)}(x,s) + \lambda(s)\mathfrak{L}\left[\frac{\partial^2 w}{\partial t^2} + (1 + \frac{x^4}{120})\frac{\partial^4 w}{\partial t^4}\right]$.

Thus, $\delta w_{(k+1)}(x,s) = \delta w_{(k)}(x,s) + s^2\lambda(s)w_{(k)}(x,s)$. That gives $\lambda(s) = -\frac{1}{s^2}$.

After putting this value of $\lambda(s)$ and applying Laplace inverse, we obtain

$$w_{(k+1)}(x,t) = w_{(k)}(x,t) - \frac{1}{s^2}\mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}\left\{\frac{\partial^2 w_k}{\partial t^2} + (\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 w_k}{\partial t^4}\right\}\right]$$

When $\frac{\partial^2 w_k}{\partial t^2} = 0 \quad k = 0, 1, 2,$ .Applying HPM, we get

$$p^0 : w_0(x,t) = w_0(x,0) + t\frac{\partial w}{\partial t}(x,0),$$

$$p^1 : w_1(x,t) = -\mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}\left\{(\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 w_0}{\partial x^4}\right\}\right],$$

$$p^2 : w_2(x,t) = -\mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}\left\{(\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 w_1}{\partial x^4}\right\}\right],$$

That yields:

$$w_0(x,t) = (1 + \frac{x^5}{120})t,$$
$$w_1(x,t) = (1 + \frac{x^5}{120})t - (1 + \frac{x^5}{120})(\frac{t^3}{3!}),$$
$$w_2(x,t) = (1 + \frac{x^5}{120})t - (1 + \frac{x^5}{120})(\frac{t^3}{3!}) + (1 + \frac{x^5}{120})(\frac{t^5}{5!}),$$

Therefore, we can get the result as follows:

$$w(x,t) = (1 + \frac{x^5}{120})(t - (\frac{t^3}{3!}) + (\frac{t^5}{5!}) - (\frac{t^7}{7!}) + \dots) = (1 + \frac{x^5}{120})sin(t)$$

5. Conclusions

A novel technique was given to build a modified method to solve PDEs with variable coefficients . This technique is reliable, and efficient. Also, we presented a comparative study between two effective methods like (MVIM) and (MLTVIM), that were recently developed. These techniques allow for fast convergent series . The computations are easy and straightforward.

References

[1] M.A. Noor, K.I. Noor, S.T. Mohyud-Din, Modified variational iteration technique for solving singular fourth-order parabolic partial differential equations, Nonlinear Anal. 71 (12) (2009) 630–640.
[2] A. Khaliq, E. Twizell, A family of second order methods for variable coefficient fourth order parabolic partial differential equations, Int. J. Comput. Math. 23 (1) (1987) 63–76.
[3] M. Dehghan, J. Manafian, The solution of the variable coefficients fourth-order parabolic partial differential equations by the homotopy perturbation method, Z. Naturf. a 64 (7–8) (2009) 420–430.
[4] D. Evans, A stable explicit method for the finite-difference solution of a fourth-order parabolic partial differential equation, Comput. J. 8 (3) (1965) 280–287.
[5] T. Aziz, A. Khan, J. Rashidinia, Spline methods for the solution of fourth-order parabolic partial differential equations, Appl. Math. Comput. 167(1) (2005) 153–166.
[6] A.-M. Wazwaz, Analytic treatment for variable coefficient fourth-order parabolic partial differential equations, Appl. Math. Comput. 123 (2) (2001) 219–227.
[7] J.-H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput. 114 (2–3) (2000) 115–123.
[8] Y. Khan, Q. Wu, Homotopy perturbation transform method for nonlinear equations using He's polynomials, Comput. Math. Appl. 61 (8) (2011) 1963–1967.
[9] S.A. Khuri, A. Sayfy, A Laplace variational iteration strategy for the solution of differential equations, Appl. Math. Lett. 25 (12) (2012) 2298–2305.
[10] H. Kumar Mishra, A.K. Nagar, He-Laplace method for linear and nonlinear partial differential equations, J. Appl. Math. 2012 (2012) 16.