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A note on some new Hermite–Hadamard type inequalities for functions whose nth derivatives are strongly η-convex

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Abstract

In this paper, we establish some new variants of the Hermite–Hadamard integral type inequalities for functions whose nth derivatives in absolute value at certain powers are strongly η-convex.

Keywords: Hermite–Hadamard type inequality, strongly η-convex functions, Hölder's inequality, Power mean inequality

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1. Introduction

Let I denote an interval in $\mathbb{R}$ and I° the interior of I. A function $f : I \to \mathbb{R}$ is said to be convex on I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \qquad \forall x, y \in I, t \in [0,1].$$

The following double inequality, for convex functions, is known in the literature as the Hermite–Hadamard inequality.

Theorem 1.1. *If $f : [a,b] \to \mathbb{R}$ is convex on $[a,b]$ with $a < b$, then*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

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With the introduction of different kinds of convexity over the years, many authors have provided several generalizations of the Hermite–Hadamard inequality corresponding to these new classes of functions. For some recent results related to the Hermite–Hadamard inequality, we refer the interested reader to the papers [1, 2, 3, 4, 6, 8, 9, 20, 10, 16, 14, 15, 17, 18, 19, 21, 7].

Recently Gordji et al. [12] introduced the concept of η-convexity as follows:

Definition 1.2 ([12]). *A function $f : I \to \mathbb{R}$ is said to be η-convex with respect to the bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)), \qquad \forall x, y \in I, t \in [0,1].$$

Remark 1.3. *If we put $\eta(x,y) = x - y$ in Definition 1.2, then we recover the classical definition of convex functions.*

In 2017, Awan et al. [5] extended the class of η-convex functions to the class of strongly η-convex functions as follows:

Definition 1.4 ([5]). *A function $f : I \to \mathbb{R}$ is said to be strongly η-convex with respect to the bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with modulus $\mu \geq 0$, if*

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)) - \mu t(1-t)(x-y)^2, \qquad \forall x, y \in I, t \in [0,1].$$

Remark 1.5. *If $\eta(x,y) = x - y$ in Definition 1.4, then we have the class of strongly convex functions.*

In [12], the authors proved the following Hermite–Hadamard type inequality for η-convex functions as follows:

Theorem 1.6. *Suppose that $f : I \to \mathbb{R}$ is an η-convex function such that η is bounded from above on $f(I) \times f(I)$. Then for any $a, b \in I$ with $a < b$,*

$$2f\left(\frac{a+b}{2}\right) - M_\eta \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(b) + \frac{\eta(f(a), f(b))}{2}, \tag{1.2}$$

where M_η is an upper bound of η on $f([a,b]) \times f([a,b])$.

Awan et al. [5], proved the following extension of Theorem 1.1 for strongly η-convex functions as follows:

Theorem 1.7. *Let $f : [a,b] \to \mathbb{R}$ be a strongly η-convex function with modulus $\mu \geq 0$. If η is bounded from above on $f([a,b]) \times f([a,b])$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} + \frac{\mu}{12}(b-a)^2 &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{f(a)+f(b)}{2} + \frac{\eta(f(a), f(b)) + \eta(f(b), f(a))}{4} \\ &- \frac{\mu}{6}(b-a)^2 \\ &\leq \frac{f(a)+f(b)}{2} + \frac{M_\eta}{2} - \frac{\mu}{6}(b-a)^2, \end{aligned} \tag{1.3}$$

where M_η is an upper bound of η on $f([a,b]) \times f([a,b])$.

The authors in [5] also proved the following Hermite–Hadamard type inequality for functions whose nth derivatives are strongly η-convex.

Theorem 1.8. *Let $f : I \to \mathbb{R}$ be an n-times differentiable strongly η-convex function on I° where $a, b \in I^\circ$ with $a < b$ and $f^{(n)} \in L_1([a, b])$. If $|f^{(n)}|^p$ is strongly η-convex function with $\mu \geq 1$, then for $n \geq 2$ and $p \geq 1$, we have*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \leq \frac{(b-a)^n}{2n!} \psi_1^{1-\frac{1}{p}}(n) \Big[\psi_1(n)|f^{(n)}(b)|^p + \psi_2(n)\eta(|f^{(n)}(a)|^p, |f^{(n)}(b)|^p) - \psi_3(n)\mu(b-a)^2 \Big], \quad (1.4)$$

where $\psi_1(n) := \frac{n-1}{n+1}$, $\psi_2(n) := \frac{n^2-2}{(n+1)(n+2)}$ and $\psi_3(n) := \frac{n-1}{(n+1)(n+2)}$.

For more information about the class of η-convex functions and some recent results, we refer the interested reader to the papers [5, 11, 12, 13, 14, 15, 17]. Motivated by the above results, our goal is to introduce some new Hermite–Hadamard integral inequalities for functions whose nth derivatives in absolute value at some powers are strongly η-convex.

2. Main results

To prove our main results, we need the following lemmas by Zhang et al. [22].

Lemma 2.1. *Let $f : I \to \mathbb{R}$ be n-times differentiable on $I^\circ, a, b \in I$ with $a < b$ and $n \in \mathbb{N}$. If $f^{(n)} L_1([a, b])$, then*

$$\frac{1}{b-a} \int_a^b f(t)dt - \sum_{k=1}^{n} \frac{(b-a)^{k-1}[f^{(k-1)}(a) + (-1)^{k-1} f^{(k-1)}(b)]}{2(k!)} = \frac{(b-a)^n}{2(n!)} \int_0^1 [t^n + (t-1)^n] f^{(n)}(ta + (1-t)b)dt. \quad (2.1)$$

Lemma 2.2. *Let $f : I \to \mathbb{R}$ be n-times differentiable on $I^\circ, a, b \in I$ with $a < b$ and $n \in \mathbb{N}$. If $f^{(n)} \in L_1([a, b])$, then*

$$\frac{1}{b-a} \int_a^b f(t)dt - \sum_{k=1}^{n} \frac{[1 + (-1)^{k-1}](b-a)^{k-1}}{2^{k-1}(k!)} f^{(k-1)} \left(\frac{a+b}{2} \right) = \frac{(b-a)^n}{n!} \left[\int_0^{\frac{1}{2}} (-t)^n f^{(n)}((1-t)a + tb)dt + \int_{\frac{1}{2}}^1 (1-t)^n f^{(n)}(ta + (1-t)b)dt \right]. \quad (2.2)$$

Theorem 2.3. *Let $f : I \to \mathbb{R}$ be n-times differentiable on $I^\circ, a, b \in I$ with $a < b$ and $n \in \mathbb{N}$. If $f^{(n)} \in L_1([a, b])$, and $|f^{(n)}|^q$ for $q \geq 1$ is strongly η-convex with modulus $\mu \geq 0$, then we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \sum_{k=1}^{n} \frac{(b-a)^{k-1}[f^{(k-1)}(a) + (-1)^{k-1} f^{(k-1)}(b)]}{2(k!)} \right|$$

$$\leq \frac{(b-a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-\frac{1}{q}}\left(\frac{2}{n+1}|f^{(n)}(b)|^q+\frac{1}{n+1}\eta\Big(|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\Big)\right.$$
$$\left.-\frac{2\mu(b-a)^2}{(n+2)(n+3)}\right)^{\frac{1}{q}}. \tag{2.3}$$

Proof . By using Lemma 2.1, the Power mean inequality and the strong η-convexity of $|f^{(n)}|^q$, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\sum_{k=1}^{n}\frac{(b-a)^{k-1}[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)]}{2(k!)}\right|$$
$$\leq \frac{(b-a)^n}{2(n!)}\int_0^1 (t^n+(1-t)^n)\left|f^{(n)}(ta+(1-t)b)\right|dt \tag{2.4}$$
$$\leq \frac{(b-a)^n}{2(n!)}\left(\int_0^1 t^n+(1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (t^n+(1-t)^n)\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^n}{2(n!)}\left(\int_0^1 t^n+(1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (t^n+(1-t)^n)\Big(|f^{(n)}(b)|^q\right.$$
$$\left.+t\eta\Big(|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\Big)-\mu t(1-t)(b-a)^2\Big)dt\right)^{\frac{1}{q}}$$
$$=\frac{(b-a)^n}{2(n!)}\left(\int_0^1 t^n+(1-t)^n dt\right)^{1-\frac{1}{q}}\left(|f^{(n)}(b)|^q\int_0^1 (t^n+(1-t)^n)\,dt\right.$$
$$\left.+\eta\Big(|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\Big)\int_0^1 t\,(t^n+(1-t)^n)\,dt-\mu(b-a)^2\int_0^1 t(1-t)\,(t^n+(1-t)^n)\,dt\right)^{\frac{1}{q}}$$
$$=\frac{(b-a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{1-\frac{1}{q}}\left(\frac{2}{n+1}|f^{(n)}(b)|^q+\frac{1}{n+1}\eta\Big(|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\Big)\right.$$
$$\left.-\frac{2\mu(b-a)^2}{(n+2)(n+3)}\right)^{\frac{1}{q}},$$

where

$$\int_0^1 t^n+(1-t)^n dt=\frac{2}{n+1},$$

$$\int_0^1 t\Big(t^n+(1-t)^n\Big)dt=\frac{1}{n+1}$$

and

$$\int_0^1 t(1-t)\Big(t^n+(1-t)^n\Big)dt=\frac{2}{(n+2)(n+3)}.$$

This completes the proof of the theorem.

□

Theorem 2.4. *Let $f : I \to \mathbb{R}$ be n-times differentiable on $I^\circ, a, b \in I$ with $a < b$ and $n \in \mathbb{N}$. If $f^{(n)} \in L_1([a,b])$, and $|f^{(n)}|^q$ for $q > 1$ is strongly η-convex with modulus $\mu \geq 0$, then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \sum_{k=1}^{n}\frac{(b-a)^{k-1}[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)]}{2(k!)}\right|$$
$$\leq \frac{(b-a)^n}{2(n!)}\left(\frac{2}{n+1}\right)^{\frac{1}{p}}\left(|f^{(n)}(b)|^q + \frac{1}{2}\eta\Big(|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\Big) - \frac{\mu(b-a)^2}{6}\right)^{\frac{1}{q}}, \tag{2.5}$$

where $\dfrac{1}{p}+\dfrac{1}{q} = 1$.

Proof . By using Lemma 2.1, the Hölder's inequality and the strong η-convexity of $|f^{(n)}|^q$, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \sum_{k=1}^{n}\frac{(b-a)^{k-1}[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)]}{2(k!)}\right|$$
$$\leq \frac{(b-a)^n}{2(n!)}\int_0^1 (t^n + (1-t)^n)\left|f^{(n)}(ta+(1-t)b)\right| dt$$
$$\leq \frac{(b-a)^n}{2(n!)}\left(\int_0^1 \Big[t^n+(1-t)^n\Big]^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^n}{2(n!)}\left(\int_0^1 \Big[t^n+(1-t)^n\Big]^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \Big(|f^{(n)}(b)|^q + t\eta\Big(|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\Big) - \mu t(1-t)(b-a)^2\Big)dt\right)^{\frac{1}{q}}$$
$$= \frac{(b-a)^n}{2(n!)}\left(\int_0^1 \Big[t^n+(1-t)^n\Big]^p dt\right)^{\frac{1}{p}}\left(|f^{(n)}(b)|^q\int_0^1 1\cdot dt + \eta\Big(|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\Big)\int_0^1 t\, dt - \mu(b-a)^2\int_0^1 t(1-t)dt\right)^{\frac{1}{q}}$$
$$= \frac{(b-a)^n}{2(n!)}\left(\int_0^1 \Big[t^n+(1-t)^n\Big]^p dt\right)^{\frac{1}{p}}\left(|f^{(n)}(b)|^q + \frac{1}{2}\eta\Big(|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\Big) - \frac{\mu(b-a)^2}{6}\right)^{\frac{1}{q}}. \tag{2.6}$$

It can easily be verified that $t^n + (1-t)^n \leq 1$ for $t \in [0,1]$. So, it follows that

$$\int_0^1 \Big[t^n+(1-t)^n\Big]^p dt \leq \int_0^1 t^n + (1-t)^n dt = \frac{2}{n+1}. \tag{2.7}$$

Hence, the desired inequality follows from (2.6) and (2.7). This completes the proof of the theorem. □

Theorem 2.5. *Let $f : I \to \mathbb{R}$ be n-times differentiable on $I^\circ, a, b \in I$ with $a < b$ and $n \in \mathbb{N}$. If $f^{(n)} \in L_1([a,b])$, and $|f^{(n)}|^q$ for $q \geq 1$ is strongly η-convex with modulus $\mu \geq 0$, then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \sum_{k=1}^{n}\frac{[1+(-1)^{k-1}](b-a)^{k-1}}{2^{k-1}(k!)}f^{(k-1)}\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^n}{n!}\left(\frac{1}{2^{n+1}(n+1)}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2^{n+1}(n+1)}|f^{(n)}(a)|^q + \frac{1}{2^{n+2}(n+2)}\eta\Big(|f^{(n)}(b)|^q,|f^{(n)}(a)|^q\Big) - \frac{\mu(b-a)^2(n+4)}{2^{n+3}(n+2)(n+3)}\right)^{\frac{1}{q}} + \left(\frac{1}{2^{n+1}(n+1)}|f^{(n)}(b)|^q + \frac{n+3}{2^{n+2}(n+1)(n+2)}\eta\Big(|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\Big) - \frac{\mu(b-a)^2(n+4)}{2^{n+3}(n+2)(n+3)}\right)^{\frac{1}{q}}\right]. \tag{2.8}$$

Proof . By using Lemma 2.2, the Power mean inequality and the strong η-convexity of $|f^{(n)}|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - \sum_{k=1}^{n}\frac{[1+(-1)^{k-1}](b-a)^{k-1}}{2^{k-1}(k!)}f^{(k-1)}\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^n}{n!}\left[\int_0^{\frac{1}{2}} t^n\left|f^{(n)}((1-t)a+tb)\right|dt + \int_{\frac{1}{2}}^1 (1-t)^n\left|f^{(n)}(ta+(1-t)b)\right|dt\right] \\
&\leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^n\left|f^{(n)}((1-t)a+tb)\right|^q dt\right)^{\frac{1}{q}} \right.\\
&\qquad\left. + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (1-t)^n\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^n\Big(|f^{(n)}(a)|^q + t\eta\Big(|f^{(n)}(b)|^q,|f^{(n)}(a)|^q\Big) - \mu t(1-t)(b-a)^2\Big)dt\right)^{\frac{1}{q}}\right. \\
&\quad\left. + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (1-t)^n\Big(|f^{(n)}(b)|^q + t\eta\Big(|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\Big) - \mu t(1-t)(b-a)^2\Big)dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1-\frac{1}{q}}\left(|f^{(n)}(a)|^q\int_0^{\frac{1}{2}} t^n dt + \eta\Big(|f^{(n)}(b)|^q,|f^{(n)}(a)|^q\Big)\int_0^{\frac{1}{2}} t^{n+1}dt - \mu(b-a)^2\int_0^{\frac{1}{2}} t^{n+1}(1-t)dt\right)^{\frac{1}{q}}\right. \\
&\quad\left. + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1-\frac{1}{q}}\left(|f^{(n)}(b)|^q\int_{\frac{1}{2}}^1 (1-t)^n dt + \eta\Big(|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\Big)\int_{\frac{1}{2}}^1 t(1-t)^n dt - \mu(b-a)^2\int_{\frac{1}{2}}^1 t(1-t)^{n+1}dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(b-a)^n}{n!}\left(\frac{1}{2^{n+1}(n+1)}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2^{n+1}(n+1)}|f^{(n)}(a)|^q\right.\right.
\end{aligned}$$

$$+\frac{1}{2^{n+2}(n+2)}\eta\Big(|f^{(n)}(b)|^q,|f^{(n)}(a)|^q\Big)-\frac{\mu(b-a)^2(n+4)}{2^{n+3}(n+2)(n+3)}\Big)^{\frac{1}{q}}$$
$$+\Bigg(\frac{1}{2^{n+1}(n+1)}|f^{(n)}(b)|^q+\frac{n+3}{2^{n+2}(n+1)(n+2)}\eta\Big(|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\Big)$$
$$-\frac{\mu(b-a)^2(n+4)}{2^{n+3}(n+2)(n+3)}\Bigg)^{\frac{1}{q}}\Bigg].$$

This completes the proof of the theorem.
□

Theorem 2.6. *Let $f:I\to\mathbb{R}$ be n-times differentiable on $I^\circ, a,b\in I$ with $a<b$ and $n\in\mathbb{N}$. If $f^{(n)}\in L_1([a,b])$, and $|f^{(n)}|^q$ for $q>1$ is strongly η-convex with modulus $\mu\geq 0$, then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\sum_{k=1}^{n}\frac{[1+(-1)^{k-1}](b-a)^{k-1}}{2^{k-1}(k!)}f^{(k-1)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^n}{2(n!)}\left(\frac{1}{2^{np}(np+1)}\right)^{\frac{1}{p}}\Bigg[\Bigg(|f^{(n)}(a)|^q+\frac{1}{4}\eta\Big(|f^{(n)}(b)|^q,|f^{(n)}(a)|^q\Big)\tag{2.9}$$
$$-\frac{\mu(b-a)^2}{6}\Bigg)^{\frac{1}{q}}+\Bigg(|f^{(n)}(b)|^q+\frac{3}{4}\eta\Big(|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\Big)-\frac{\mu(b-a)^2}{6}\Bigg)^{\frac{1}{q}}\Bigg],\tag{2.10}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof . Again, using Lemma 2.2, the Hölder's inequality and the strong η-convexity of $|f^{(n)}|^q$, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\sum_{k=1}^{n}\frac{[1+(-1)^{k-1}](b-a)^{k-1}}{2^{k-1}(k!)}f^{(k-1)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^n}{n!}\left[\int_0^{\frac{1}{2}}t^n\left|f^{(n)}((1-t)a+tb)\right|dt+\int_{\frac{1}{2}}^1(1-t)^n\left|f^{(n)}(ta+(1-t)b)\right|dt\right]$$
$$\leq\frac{(b-a)^n}{n!}\Bigg[\left(\int_0^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f^{(n)}((1-t)a+tb)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^1(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\Bigg]$$
$$\leq\frac{(b-a)^n}{n!}\Bigg[\left(\int_0^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\Bigg(\int_0^{\frac{1}{2}}\Big(|f^{(n)}(a)|^q+t\eta\Big(|f^{(n)}(b)|^q,|f^{(n)}(a)|^q\Big)$$
$$-\mu t(1-t)(b-a)^2\Big)dt\Bigg)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^1(1-t)^{np}dt\right)^{\frac{1}{p}}\Bigg(\int_{\frac{1}{2}}^1\Big(|f^{(n)}(b)|^q+t\eta\Big(|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\Big)$$

$$
\begin{aligned}
& \left. - \mu t(1-t)(b-a)^2\Big) dt\right)^{\frac{1}{q}}\Bigg] \\
\leq \frac{(b-a)^n}{n!}\Bigg[& \left(\int_0^{\frac{1}{2}} t^{np} dt\right)^{\frac{1}{p}} \Bigg(|f^{(n)}(a)|^q \int_0^{\frac{1}{2}} 1\, dt + \eta\Big(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q\Big)\int_0^{\frac{1}{2}} t\, dt \\
& - \mu(b-a)^2 \int_0^{\frac{1}{2}} t(1-t)dt\Bigg)^{\frac{1}{q}} \\
+ & \left(\int_{\frac{1}{2}}^1 (1-t)^{np} dt\right)^{\frac{1}{p}} \Bigg(|f^{(n)}(b)|^q \int_{\frac{1}{2}}^1 1\, dt + \eta\Big(|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\Big)\int_{\frac{1}{2}}^1 t\, dt \\
& - \mu(b-a)^2 \int_{\frac{1}{2}}^1 t(1-t)dt\Bigg)^{\frac{1}{q}}\Bigg] \\
\leq \frac{(b-a)^n}{n!} & \left(\frac{1}{2^{np+1}(np+1)}\right)^{\frac{1}{p}} \Bigg[\Bigg(\frac{1}{2}|f^{(n)}(a)|^q + \frac{1}{8}\eta\Big(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q\Big) \\
- \frac{\mu(b-a)^2}{12}\Bigg)^{\frac{1}{q}} & + \Bigg(\frac{1}{2}|f^{(n)}(b)|^q + \frac{3}{8}\eta\Big(|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\Big) - \frac{\mu(b-a)^2}{12}\Bigg)^{\frac{1}{q}}\Bigg] \\
= \frac{(b-a)^n}{2(n!)} & \left(\frac{1}{2^{np}(np+1)}\right)^{\frac{1}{p}} \Bigg[\Bigg(|f^{(n)}(a)|^q + \frac{1}{4}\eta\Big(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q\Big) \\
- \frac{\mu(b-a)^2}{6}\Bigg)^{\frac{1}{q}} & + \Bigg(|f^{(n)}(b)|^q + \frac{3}{4}\eta\Big(|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\Big) - \frac{\mu(b-a)^2}{6}\Bigg)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

This completes the proof of the theorem.
□

Remark 2.7. *By substituting $\mu = 0$ in the above theorems, we obtain results for the η-convex functions.*

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