Four Step Hybrid Block Method for the Direct Solution of Fourth Order Ordinary Differential Equations

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Abstract

This paper proposes a direct four-step implicit hybrid block method for directly solving general fourth-order initial value problems of ordinary differential equations. In deriving this method, the approximate solution in the form of power series is interpolated at four points, i.e. \( x_n, x_{n+1}, x_{n+2}, x_{n+3} \) while its forth derivative is collocated at all grid points, i.e. \( x_n, x_{n+\frac{1}{4}}, x_{n+1}, x_{n+\frac{3}{4}}, x_{n+2}, x_{n+\frac{3}{2}}, x_{n+3}, x_{n+\frac{5}{2}} \) and \( x_{n+4} \) to produce the main continuous schemes. In order to verify the applicability of the new method, the properties of the new method such as local truncation error, zero stability, order and convergence are also established. The performance of the newly developed method is then compared with the existing methods in terms of error by solving the same test problems. The numerical results reveal that the proposed method produces better accuracy than several existing methods when solving the same initial value problems (IVPs) of second order ODEs.

Keywords: Hybrid block method, Fourth initial value problem, Collocation and Interpolation, Four step

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1. Introduction

This paper focuses on solving the following initial value problem (IVPs) of fourth order ordinary differential equations (ODEs)

\[
y^{(4)} = f(x, y, y', y'', y'''), \quad x \in [a, b]
\]  

(1.1)
with initial conditions
\[ y(a) = \delta_0, \quad y'(a) = \delta_1, \quad y''(a) = \delta_2, \quad y'''(a) = \delta_3 \]

Numerous physical problems such as elasticity, deformation of structures, or soil settlement can be modelled in the form of Equation (1). However, the solutions to these problems may not be obtained analytically. Therefore, it is essential to develop numerical methods to cater this issue (see Twizell [13] and Henrici [9]).

Although Equation (1) can be solved by reducing it into the equivalent first order system of ODEs, this approach will increase the number of equations and as a result, more work is required. Consequently, more computational time is needed. To avoid this drawback, scholars have opted to solve (1) using direct methods instead. Direct methods have also been proven to produce more accurate numerical results compared to reduction methods (see Abdelrahim and Omar [1, 2]).

One of the direct methods available is block method. Block method is capable of finding numerical solution at more than one point simultaneously. Nonetheless, block method has zero-stability barriers. To overcome this setback, hybrid block methods were introduced (see Adessanya [4], Yap [11], Awoyemi [5], Awoyemi and Kayode [6]). In literature, four-step implicit method to solve IVPs of fourth order ODEs was proposed by [12]. The basic properties of the method was investigated and found to be zero stable, consistent and convergent. Subsequently, a four-step hybrid method with three hybrid points was developed by Yap and Ismail [13]. The numerical results revealed that the accuracy of this method was better than the previous non-hybrid methods. Not only this method has good properties of numerical method, the results obtained were also accurate. The method was tested on some numerical examples and the accuracy of the method can still be improved.

In this Four-step hybrid block model, new strategy of selecting the hybrid values within the grid points was considered. Consequently, this new approach has given rise to a better results when compared with the previous method developed by Abdelrahim and Omar [3]. One of the major advantages of this novel developed numerical method over current method is its ability to solving fourth order ordinary differential equations effectively. In order to further examine the capability of the new method, the same number of block used in the previous method was also considered. The new method gave better result at every point and this demonstrated the superiority of the method over existing methods in terms of accuracy.

2. Methodology

In this section, a four-step block method with a specific three off-step point \( x_{n+\frac{1}{4}}, x_{n+\frac{3}{4}} \) and \( x_{n+\frac{5}{4}} \), for solving (1.1) is developed.

Let the approximate solution of equation (1.1) be the polynomial of the form:
\[
y(x) = \sum_{i=0}^{s+v-1} a_i \left( \frac{x - x_n}{h} \right)^i.
\]

(2.1)

to be the approximate solution of (1.1) where \( x \in [x_n, x_{n+1}] \) for \( n = 0, 1, 2, ..., N - 1 \), \( v \) represents the number of collocation points, \( s \) denotes of the number of interpolation points which is equal to the order of differential equation and \( h = x_n - x_{n-1} \) is constant step size of partition of the interval.
\[ [a, b] \text{ which is given by } a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b. \]

The fourth derivative of equation (2.1) is

\[ y^{(iv)}(x) = f(x, y, y', y'', y''') = \sum_{i=4}^{s+1-1} \frac{i(i-1)(i-2)(i-3)}{h^4} a_i \left( \frac{x - x_n}{h} \right)^{i-4}. \]  

Equation (2.1) is interpolated at \( x_{n+i}, i = 0(1)3 \) while Equation (2.2) is collocated at all points in the selected interval. The producing equations are reformed in matrix mode as below:

\[ KX = L \]  

where

\[
K = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 & 512 \\
1 & 3 & 9 & 27 & 81 & 243 & 729 & 2187 & 6561 & 19683 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
X = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}]^T
\]

\[
L = \begin{pmatrix}
y_n, y_{n+1}, y_{n+2}, y_{n+3}, f_n, f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+1}, f_{n+2}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+3}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+4}\end{pmatrix}^T
\]

The unknown values of \( a'_i, i = 0(1)11 \) can be obtained by using Gaussian elimination method and then substituted these values back into Equation (2.1) to produce a continuous implicit scheme with its derivatives of the form:

\[ y^{(j)}(x) = \sum_{i=0}^{3} \alpha_i^{(j)}(x) y_{n+i} + \sum_{i=0}^{4} \beta_i^{(j)}(x) f_{n+i} + \sum_{s = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}} \beta_s^{(j)}(x) f_{n+s}, \quad j = 0(1)3. \]  

where the values of \( \alpha_i, \beta_i, i = 1(1)4 \) and \( \beta_s, s \in \left\{ \frac{1}{4}, \frac{5}{2}, \frac{7}{2} \right\} \) in equation (5), are

\[
\alpha_0 = \frac{(x - x_n)^2}{h^2} - \frac{(x - x_n)^3}{(6h^2)} - \frac{11(x - x_n)}{6h} + 1
\]

\[
\alpha_1 = \frac{(x - x_n)^3}{(2h^3)} - \frac{5(x - x_n)^2}{(2h^2)} + \frac{3(x - x_n)}{h}
\]

\[
\alpha_2 = \frac{2(x - x_n)^2}{h^2} - \frac{(x - x_n)^3}{(2h^2)} - \frac{3(x - x_n)}{(2h)}
\]

\[
\alpha_3 = \frac{(x - x_n)^3}{(6h^3)} - \frac{(x - x_n)^2}{(2h^2)} + \frac{(xx_n)}{(3h)}
\]
\[
\beta_0 = \frac{(x - x_n)^4}{24} - \frac{(521h^2(x - x_n)^2)}{56448} - \frac{(2843(x - x_n)^5)}{(50400h)} + \frac{(473(x - x_n)^6)}{(12096h^2)} \\
- \frac{(5737(x - x_n)^7)}{(352800h^3)} + \frac{(239(x - x_n)^8)}{(56448h^4)} - \frac{(431(x - x_n)^9)}{(635040h^5)} + \frac{(13(x - x_n)^{10})}{(211680h^6)} \\
- \frac{(x - x_n)^{11}}{(415800h^7)} - \frac{(37321h(x - x_n)^3)}{6350400} + \frac{(3761h^3(x - x_n))}{1108800} \\
\beta_1 = \frac{18067h^2(x - x_n)^2}{64800} - \frac{(7(x - x_n)^5 + 2423(x - x_n)^6)}{(270h)} - \frac{(997(x - x_n)^7)}{(48600h^2)} \\
+ \frac{(899(x - x_n)^8)}{(90720h^4)} + \frac{(37(x - x_n)^9)}{(20412h^5)} + \frac{(61(x - x_n)^{10})}{(340200h^6)} - \frac{(x - x_n)^{11}}{(133650h^7)} \\
- \frac{(128293h(x - x_n)^3)}{1020600} - \frac{(77107h^3(x - x_n))}{498960} \\
\beta_2 = \frac{(5836h^2(x - x_n)^2)}{423360} + \frac{(x - x_n)^5}{(24h)} - \frac{(2633(x - x_n)^6)}{(30240h^2)} + \frac{(383(x - x_n)^7)}{(5880h^3)} \\
- \frac{(3439(x - x_n)^8)}{(141120h^4)} + \frac{(317(x - x_n)^9)}{(63504h^5)} - \frac{(19(x - x_n)^{10})}{(35280h^6)} + \frac{(x - x_n)^{11}}{(41580h^7)} \\
- \frac{(20039h(x - x_n)^3)}{653040} - \frac{(3923h^3(x - x_n))}{36960} \\
\beta_3 = \frac{(713h^2(x - x_n)^2)}{36960} + \frac{(7(x - x_n)^5)}{(198h)} - \frac{(901(x - x_n)^6)}{(11880h^2)} + \frac{(3341(x - x_n)^7)}{(55440h^3)} \\
- \frac{(2711(x - x_n)^8)}{(110880h^4)} + \frac{(34((x - x_n)^9)}{(6237h^5)} - \frac{(53(x - x_n)^{10})}{(83160h^6)} + \frac{(x - x_n)^{11}}{(32670h^7)} \\
+ \frac{(17h(x - x_n)^3)}{3564} - \frac{(2111h^3(x - x_n))}{87120} \\
\beta_4 = \frac{(689h^2(x - x_n)^2)}{1814400} + \frac{(7(x - x_n)^5)}{(4320h)} - \frac{(1369(x - x_n)^6)}{(388800h^2)} + \frac{(1307(x - x_n)^7)}{(453600h^3)} \\
- \frac{(443(x - x_n)^8)}{(362880h^4)} + \frac{(47(x - x_n)^9)}{(163296h^5)} - \frac{(7(x - x_n)^{10})}{(194400h^6)} + \frac{(x - x_n)^{11}}{(534600h^7)} \\
+ \frac{(2921h(x - x_n)^3)}{8164800} - \frac{(1499h^3(x - x_n))}{1995840} \\
\beta_1^i = \frac{(3554816h^2(x - x_n)^2)}{42567525} + \frac{(4096(x - x_n)^5)}{(57915h)} - \frac{(1190912(x - x_n)^6)}{(18243225h^2)} \\
+ \frac{(1290752(x - x_n)^7)}{(42567525h^3)} - \frac{71168(x - x_n)^8)}{(851350h^4)} + \frac{21248(x - x_n)^9)}{(15324309h^5)} - \frac{16384(x - x_n)^{10}}{(127702575h^6)} \\
+ \frac{1024(x - x_n)^{11}}{(200675475h^7)} - \frac{33535232h(x - x_n)^3}{383107725} - \frac{(47104h^3(x - x_n))}{1911195} \\
\beta_2^i = \frac{(428(x - x_n)^6)}{(3645h^2)} - \frac{(112(x - x_n)^5)}{(2025h)} - \frac{(82h^2(x - x_n)^2)}{(1701)} - \frac{(3874(x - x_n)^7)}{(42525h^3)} \\
+ \frac{(304(x - x_n)^8)}{(8505h^4)} - \frac{(587(x - x_n)^9)}{(76545h^5)} + \frac{(22(x - x_n)^{10})}{(25515h^6)} - \frac{(8(x - x_n)^{11})}{(200475h^7)} \\
- \frac{(851h(x - x_n)^3)}{38275} + \frac{(3376h^3(x - x_n))}{66825}
\[
\beta_2 \equiv -\frac{(1556(x - x_n)^6)}{(61425h^2)} - \frac{(16(x - x_n)^5)}{(1365h)} - \frac{(1754h^2(x - x_n)^2)}{429975} - \frac{(326(x - x_n)^7)}{(15925h^3)} + \frac{(244(x - x_n)^8)}{(28665h^4)} - \frac{(101(x - x_n)^9)}{(51597h^5)} + \frac{(34(x - x_n)^{10})}{(143325h^6)} - \frac{(8(x - x_n)^{11})}{(675675h^7)} - \frac{(2881h(x - x_n)^{3})}{1289925} + \frac{(32h^3(x - x_n))}{5005}
\]

For \(j = 0\), Equation (2.4) is evaluated at the non-interpolating point \(x_{n+\frac{1}{2}}, x_{n+\frac{3}{2}}, x_{n+\frac{5}{2}}, x_{n+4}\) while for \(j = 1(1)^3\) Equation (2.4) are evaluated at all points in the selected interval to produce the discrete schemes with its derivatives. Discrete schemes and its derivatives are combined on a block of the form

\[
AY_m = BR_1 + h^4[DR_2 + ER_3]
\]  
(2.5)
where,

$$A =
\begin{pmatrix}
1 & \frac{-7}{128} & 0 & \frac{33}{128} & 0 & \frac{-7}{128} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{5}{16} & 1 & \frac{-15}{16} & 0 & \frac{-5}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{16} & 0 & \frac{15}{16} & 1 & \frac{-15}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{16} & 0 & 6 & 0 & \frac{-4}{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{16} & 0 & \frac{3}{(2\pi)} & 0 & \frac{-1}{(3\pi)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{16} & 0 & \frac{19}{(2\pi)} & 0 & \frac{-11}{(3\pi)} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{(2\pi)} & 0 & \frac{-1}{k} & 0 & \frac{-1}{k} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{(2\pi)} & 0 & \frac{7}{(8\pi)} & 0 & \frac{-23}{(24\pi)} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{k} & 0 & \frac{-1}{(2\pi)} & 0 & \frac{-1}{(3\pi)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{(8\pi)} & 0 & \frac{47}{(8\pi)} & 0 & \frac{-71}{(24\pi)} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{(2\pi)} & 0 & \frac{3}{k} & 0 & \frac{-11}{(6\pi)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{k} & 0 & \frac{19}{(2\pi)} & 0 & \frac{-13}{(3\pi)} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{5}{k^2} & 0 & \frac{-1}{k^2} & 0 & \frac{1}{k^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{17}{(4\pi^2)} & 0 & \frac{-13}{(4\pi^2)} & 0 & \frac{-3}{(4\pi^2)} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{k^2} & 0 & \frac{-1}{k^2} & 0 & \frac{-1}{k^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{(2\pi^2)} & 0 & \frac{7}{(2\pi^2)} & 0 & \frac{-3}{(2\pi^2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{k^2} & 0 & \frac{2}{k^2} & 0 & \frac{-1}{k^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{(2\pi^2)} & 0 & \frac{13}{(2\pi^2)} & 0 & \frac{-5}{(2\pi^2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{k^2} & 0 & \frac{5}{k^2} & 0 & \frac{-2}{k^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{k^2} & 0 & \frac{8}{k^2} & 0 & \frac{-3}{k^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{k^2} & 0 & \frac{3}{k^2} & 0 & \frac{-1}{k^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{k^2} & 0 & \frac{3}{k^2} & 0 & \frac{-1}{k^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{k^2} & 0 & \frac{3}{k^2} & 0 & \frac{-1}{k^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{k^2} & 0 & \frac{3}{k^2} & 0 & \frac{-1}{k^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}

\[ y_m = \begin{pmatrix} y_n + \frac{1}{4} \\ y_{n+1} \\ y_{n+\frac{5}{2}} \\ y_{n+2} \\ y_{n+\frac{7}{2}} \\ y_{n+3} \\ y_{n+4} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{27}{128} & 0 & 0 & 0 \\ \frac{15}{16} & 0 & 0 & 0 \\ -\frac{15}{16} & 0 & 0 & 0 \\ -\frac{11}{24} & -1 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 \\ \end{pmatrix}, \quad R_1 = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+\frac{5}{2}} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{pmatrix}, \quad D = \begin{pmatrix} 781079h^4 \\ 2641241120h^4 \\ -185290h^4 \\ 36125720h^4 \\ 4036h^4 \\ 147456h^4 \\ 3114h^4 \\ 35280h^4 \\ 30716h^4 \\ 110880h^4 \\ -755789156h^4 \\ 7220794412224h^4 \\ 4981h^4 \\ 4569960h^4 \\ -1115376h^4 \\ 2980454400h^4 \\ -399538h^4 \\ 278218800h^4 \\ 7198276h^4 \\ 85158540h^4 \\ 31676h^4 \\ 1108800h^4 \\ 759974h^4 \\ 1656960h^4 \\ -521h^2 \\ 38227h \\ -23083140h^2 \\ 24172909000h^2 \\ 3471h^2 \\ 3024000h^2 \\ 113671h^2 \\ 27095030h^2 \\ -209h^2 \\ 43200h^2 \\ 1890756h^2 \\ 135475200h^2 \\ 62290h^2 \\ 705060h^2 \\ 39701h^2 \\ 211680h^2 \\ -13521h^2 \\ 108340h^2 \\ 55919200h^2 \\ 80103120000h \\ -20971h \\ 1058400h \\ 2948536h \\ 33868800h \\ 301120h \\ 1058400h \\ 272856h \\ 33868800h \\ 104699h \\ 1058400h \\ -1 \\ 15479h \\ 1058400h \end{pmatrix}, \quad R_2 = \begin{pmatrix} f_{n+\frac{1}{4}} \\ f_{n+1} \\ f_{n+\frac{5}{2}} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \quad \text{and} \quad R_3 = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+\frac{5}{2}} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{pmatrix} \]
Multiplying Equation (2.5) by inverse of $A$ to have a hybrid block method of the form

$$A^{(0)}Y_n = A^{-1}BR_1 + h^4[A^{-1}DR_2 + A^{-1}ER_3]$$

(2.6)

Equation (2.6) can be clearly written as following:

$$y_{n+1} = y_n + \frac{h}{4}y_n' + \frac{h^2}{32}y_n'' + \frac{h^3}{384}y_n''' + \frac{h^4}{10967}f_n - \frac{4379}{15872}f_{n+1}$$

(2.7a)

$$+ \frac{24218}{183119}f_{n+2} + \frac{139098}{123489}f_{n+3} + \frac{1037172}{1437543}f_{n+4}$$

(2.7b)

$$+ \frac{13758}{94691}f_{n+5} + \frac{1177194}{1726130}f_{n+6} - \frac{2324754}{72904921600}$$

(2.7c)
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\[ y_{n+1} = y_n + h y_n' + \frac{h^2 y_n''}{2} + \frac{h^3 y_n'''}{6} + h^4 \frac{1636651 f_n}{13970880} + \frac{64091 f_{n+1}}{44906400} - \frac{2227 f_{n+2}}{13970880} \\
+ \frac{373 f_{n+3}}{2195424} + \frac{2437 f_{n+4}}{179625600} + \frac{120832508 f_{n+\frac{1}{2}}}{4214184975} + \frac{377 f_{n+\frac{1}{2}}}{4209975} - \frac{1147 f_{n+\frac{3}{2}}}{14181975}, \]

\[ y_{n+2} = y_{n+1} + 2 h y_n' + +\frac{h^2 y_n''}{2} + \frac{4 h^3 y_n'''}{3} + \frac{82408 f_n}{1091475} + \frac{279896 f_{n+1}}{1403325} - \frac{18926 f_{n+2}}{218295} \\
+ \frac{29936 f_{n+3}}{343035} - \frac{808 f_{n+4}}{280665} + \frac{350224384 f_{n+\frac{1}{2}}}{842836995} + \frac{461312 f_{n+\frac{3}{2}}}{4209975} + \frac{303616 f_{n+\frac{3}{2}}}{14181975}, \]

\[ h^4 \left[ \frac{22936 f_{n+3}}{343035} - \frac{808 f_{n+4}}{280665} + \frac{350224384 f_{n+\frac{1}{2}}}{842836995} + \frac{461312 f_{n+\frac{3}{2}}}{4209975} + \frac{303616 f_{n+\frac{3}{2}}}{14181975} \right], \]

\[ y_{n+\frac{1}{2}} = y_n + \frac{5 h y_n'}{2} + \frac{25 h^2 y_n''}{8} + \frac{125 h^3 y_n'''}{48} + \frac{100507625 f_n}{715309056} + \frac{2448125 f_{n+1}}{4105728} \\
- \frac{49159375 f_{n+2}}{357654528} - \frac{19990625 h^4 f_{n+3}}{140507136} - \frac{836875 h^4 f_{n+4}}{131383296} + \frac{153150625 h^4 f_{n+\frac{1}{2}}}{168567399}, \]

\[ y_{n+3} = y_{n+2} + \frac{9 h^2 y_n'''}{2} + \frac{9 h^3 y_n''''}{2} + \frac{399411 f_n}{1724800} + \frac{2373 f_{n+1}}{1760} - \frac{12123 f_{n+2}}{172480} \\
+ \frac{31111 f_{n+3}}{-135520} - \frac{381 f_{n+4}}{3248384} + \frac{3649536000}{1403325} - \frac{3649536000}{3284582400} + \frac{75413099 h^4 f_{n+\frac{1}{2}}}{3284582400} + \frac{410572800}{168567399}, \]

\[ y_{n+\frac{3}{2}} = y_n + \frac{7 h y_n'}{2} + \frac{49 h^2 y_n''}{8} + \frac{343 h^3 y_n'''}{48} + \frac{129886897 f_n}{364953600} + \frac{1056840967 f_{n+1}}{410572800} \\
+ \frac{8453921 f_{n+2}}{36495360} - \frac{1529437 h^4 f_{n+3}}{5018112} - \frac{5218507 h^4 f_{n+4}}{3284582400} + \frac{241701467 h^4 f_{n+\frac{1}{2}}}{3284582400} + \frac{86003775}{153964800} + \frac{37065600}{153964800}, \]

\[ y_{n+4} = y_{n+3} + \frac{8 h^2 y_n''}{3} + \frac{32 h^3 y_n'''}{3} + \frac{566528 f_n}{1091475} + \frac{6163456 f_{n+1}}{1403325} + \frac{38656 f_{n+2}}{43659} \\
- \frac{103424 h^4 f_{n+3}}{343035} - \frac{29728 h^4 f_{n+4}}{1403325} + \frac{18312331264 h^4 f_{n+\frac{1}{2}}}{4214184975} + \frac{2916352 h^4 f_{n+\frac{3}{2}}}{4209975} + \frac{14181975}{14181975}, \]

\[ y'_{n+\frac{1}{2}} = y_n + \frac{h y_n'}{4} + \frac{h^2 y_n''}{32} + \frac{h^3}{4} \left[ \frac{94652819 f_n}{55490641920} + \frac{7795423 f_{n+1}}{29727129600} + \frac{11201017 f_{n+2}}{2774532060} \right], \]

\[ y'_{n+1} = y_n + \frac{h y_n'}{2} + \frac{h^2 y_n''}{2} + \frac{h^3}{4} \left[ \frac{1787 f_n}{52920} + \frac{4259 f_{n+1}}{226800} - \frac{703 f_{n+2}}{52920} - \frac{1499 f_{n+3}}{166320} \right], \]

\[ y'_{n+\frac{3}{2}} = y_n + \frac{5 h y_n'}{2} + \frac{25 h^2 y_n''}{8} + \frac{h^3}{4} \left[ \frac{1653625 f_n}{10838016} + \frac{644125 f_{n+1}}{580608} - \frac{124375 f_{n+2}}{33875 f_{n+\frac{1}{2}} + 252625 f_{n+\frac{3}{2}}} \right], \]

\[ y'_{n+2} = y_n + 2 h y''_n + 2 h^2 y'''_n + \frac{h^3}{4} \left[ \frac{151 f_n}{1470} + \frac{22004 f_{n+1}}{42525} - \frac{944 f_{n+2}}{6615} - \frac{452 f_{n+3}}{3465} \right]. \]
\[
\begin{align*}
\frac{493f_{n+4}}{85050} & + \frac{7307264f_{n+\frac{1}{2}}}{9823275} + \frac{5248f_{n+\frac{1}{2}}}{2551} + \frac{1408f_{n+\frac{1}{2}}}{33075}, \\
y_{n+\frac{1}{2}}' & = y_n' + \frac{7hyn}{2} + \frac{49h^2yn}{8} + h^3\left[105301f_n + \frac{2336173f_{n+1}}{777600} + \frac{506611f_{n+2}}{552960}\right]. \\
-16807f_{n+3} & + 20017f_{n+4} + 6876464f_{n+\frac{1}{2}} + \frac{319333f_{n+\frac{1}{2}}}{933120} + \frac{154693f_{n+\frac{1}{2}}}{2246400}. \\
y_{n+3} & = y_n + 3hyn + \frac{9h^2yn}{2} + h^3\left[837f_n + \frac{3457f_{n+1}}{2800} + \frac{81f_{n+2}}{245} + \frac{99f_{n+3}}{560}\right]. \\
-27f_{n+4} & + 2304f_{n+\frac{1}{2}} + \frac{9f_{n+\frac{1}{2}}}{35} + \frac{81f_{n+\frac{1}{2}}}{1225}, \\
y_{n+4} & = y_n' + 4hyn + 8h^2yn + h^3\left[488f_n + \frac{6784f_{n+1}}{1323} + \frac{11488f_{n+2}}{6615} + \frac{128f_{n+3}}{945}\right]. \\
-136f_{n+4} & + 1048576f_{n+\frac{1}{2}} + \frac{4096f_{n+\frac{1}{2}}}{8505} + \frac{4096f_{n+\frac{1}{2}}}{33075}, \\
y_{n+\frac{3}{2}}' & = y_n'' + \frac{hyn''}{4} + \frac{305073173f_n}{17340825600} - \frac{18915997f_{n+1}}{5573836800} + \frac{8977231f_{n+2}}{1734082560} + \frac{168203f_{n+3}}{38928384}. \\
+437982f_{n+4} & + 25479379f_{n+\frac{1}{2}} + \frac{444427f_{n+\frac{1}{2}}}{2511401f_{n+\frac{1}{2}}} - \frac{16345929600}{65318400} + \frac{1761177600}{5639f_{n+3}} - \frac{3923f_{n+4}}{1360800}. \\
y_{n+1} & = y_n + hyn + \frac{1058400}{340200} + \frac{42353f_{n+1}}{21168} - \frac{1993f_{n+2}}{83160} - \frac{3923f_{n+4}}{1360800}. \\
+3441664f_{n+\frac{1}{2}} + \frac{14416f_{n+\frac{1}{2}}}{712f_{n+\frac{1}{2}}} + \frac{9823275}{127575} + \frac{33075}{127575}, \\
y_{n+\frac{1}{2}}' & = y_n'' + \frac{5hyn''}{2} + \frac{600475f_n}{5419008} - \frac{2474875f_{n+1}}{1741824} + \frac{2709504}{2128896} - \frac{104375f_{n+3}}{2128896}. \\
-26875f_{n+4} & + 5807500f_{n+\frac{1}{2}} + \frac{4825f_{n+\frac{1}{2}}}{26125f_{n+\frac{1}{2}}} + \frac{26125f_{n+\frac{1}{2}}}{6967296}. \\
y_{n+2} & = y_n + 2hyn + \frac{5851f_n}{66150} + \frac{40232f_{n+1}}{42525} - \frac{179f_{n+2}}{6615} - \frac{152f_{n+3}}{1485} - \frac{451f_{n+4}}{85050}. \\
+112984064f_{n+\frac{1}{2}} + \frac{15872f_{n+\frac{1}{2}}}{15872f_{n+\frac{1}{2}}} + \frac{127702575}{127575} + \frac{429975}{429975}, \\
y_{n+\frac{3}{2}}' & = y_n'' + \frac{7hyn''}{2} + \frac{429919f_n}{2764800} + \frac{14734937f_{n+1}}{6220800} - \frac{55296}{5108103} + \frac{55296}{326592} + \frac{1520640}{1100736}. \\
-40817f_{n+4} & + 4279268f_{n+\frac{1}{2}} + \frac{10241f_{n+\frac{1}{2}}}{561600} + \frac{10241f_{n+\frac{1}{2}}}{24883200}. \\
y_{n+3} & = y_n + 3hyn + \frac{5217f_n}{39200} + \frac{379f_{n+1}}{200} - \frac{368f_{n+2}}{3920} - \frac{9f_{n+3}}{3080} - \frac{13f_{n+4}}{5600}. \\
+730112f_{n+\frac{1}{2}} + \frac{8f_{n+\frac{1}{2}}}{31f_{n+\frac{1}{2}}} + \frac{8f_{n+\frac{1}{2}}}{127575} + \frac{8f_{n+\frac{1}{2}}}{127575} + \frac{8f_{n+\frac{1}{2}}}{8f_{n+\frac{1}{2}}} + \frac{35}{31f_{n+\frac{1}{2}}} + \frac{15925}{52525}, \\
y_{n+4} & = y_n'' + \frac{hyn''}{2} + \frac{5896f_n}{33075} + \frac{121024f_{n+1}}{42525} + \frac{12304f_{n+2}}{6615} + \frac{1216f_{n+3}}{2079} + \frac{704f_{n+4}}{42525}. \\
+241696768f_{n+\frac{1}{2}} + \frac{47104f_{n+\frac{1}{2}}}{108544f_{n+\frac{1}{2}}} + \frac{47104f_{n+\frac{1}{2}}}{108544f_{n+\frac{1}{2}}} + \frac{47104f_{n+\frac{1}{2}}}{127575} + \frac{429975}{429975}, \\
y_{n+\frac{3}{2}}' & = y_n'' + \frac{hyn''}{2} + \frac{11532343f_n}{115605504} + \frac{4821053f_{n+1}}{185794560} + \frac{11277377f_{n+2}}{289013760} + \frac{7363793f_{n+3}}{227082240}. \\
+156203f_{n+4} & + \frac{44982053f_{n+\frac{1}{2}}}{272432160} - \frac{1781909f_{n+\frac{1}{2}}}{34836480} - \frac{251009f_{n+\frac{1}{2}}}{23482368}. \\
y_{n+\frac{1}{2}}' & = y_n'' + \frac{hyn''}{4} + \frac{11532343f_n}{115605504} + \frac{4821053f_{n+1}}{185794560} + \frac{11277377f_{n+2}}{289013760} + \frac{7363793f_{n+3}}{227082240}. \\
+156203f_{n+4} & + \frac{44982053f_{n+\frac{1}{2}}}{272432160} - \frac{1781909f_{n+\frac{1}{2}}}{34836480} - \frac{251009f_{n+\frac{1}{2}}}{23482368}. 
\end{align*}
\]
3. Properties of the Method

3.1. Order of the Method

The linear difference operator $L$ associated with equation (2.6) is defined as

$$L[y(x); h] = Y_m - A^{-1}BR_1 - h^4 [A^{-1}DR_2 + A^{-1}ER_3]$$  \hspace{1cm} (3.1)$$

Expanding the terms $Y_m$ and $R_3$ using Taylor series about $x_n$ respectively and then collecting their like elements to the powers of $h$ gives

$$L[y(x), h] = C_0 y(x) + C_1 y'(x) + C_2 h^2 y''(x) + \cdots$$  \hspace{1cm} (3.2)$$

Definition 3.1. The hybrid block method (2.6) and its linear operator (3.1) are said to have order $p$ if, in (9), $C_0 = \hat{C}_1 = \hat{C}_2 = \cdots = \hat{C}_{p+3} = 0$ and $C_{p+4} \neq 0$ with error constants vector $C_{p+4}$.

Expanding equation (2.6) about $x = x_n$ using Taylor series and then comparing the coefficients of $h$ lead to $\hat{C}_0 = \hat{C}_1 = \hat{C}_2 = \cdots = \hat{C}_{11} = 0$ and $\hat{C}_{12} \neq 0$. Hence the order of the new method is $[8, 8, 8, 8, 8, 8]^T$. This implies that, the new block method is consistent since its order is greater than one.

3.2. Zero stability

In order to find the zero stability of the method, we have extended the definition in Fatunla [8]. That is, the hybrid block method (7) is said to be zero stable if the first characteristic polynomial $\Pi(t)$ having roots such that $t_r \leq 1$; and if $t_r = 1$; then the multiplicity of $t_r$ must not greater than 4. Applying this
in our main block method 
\[
\begin{bmatrix}
y_{n+1}; y_{n+\frac{1}{4}}; y_{n+1}; y_{n+2}; y_{n+\frac{3}{4}}; y_{n+3}; y_{n+2}; y_{n+4}\end{bmatrix}^T
\]
where \(I\) is \(7 \times 7\) identity matrix and \(B\) is coefficients matrix of \(y_n\),

\[
\Pi(t) = |t I - B| = t^6(t - 1)
\]

which gives \(t = 0, 0, 0, 0, 0, 0, 1\). Hence our method is zero stable.

3.3. Convergence

According to Henrici [9], zero stability and consistency are sufficient conditions of the numerical method to be convergent. Hence, the new method (2.6) is convergent.

4. Numerical Results

4.1. Numerical Examples

In order to show the accuracy of the new method, three fourth order IVPs problems are solved and compared with existing methods as shown in Table 1-3 where

h: Step size.
Step: total number of steps taken to obtain solution.
E(x): magnitude of the maximum error of the computed solution.
Error: absolute error.

**Problem 1:** \(y^{iv} = -y''\), \(y(0) = 0\), \(y'(0) = \frac{-1.1}{72 - 50\pi}\), \(y''(0) = \frac{1}{144 - 100\pi}\).
\(y''(0) = \frac{1.2}{144 - 100\pi}\), \(x \in [0, 101325]\)

*Exact solution*: \(y(x) = 1 - x - \cos x - \frac{1.2 \sin x}{144 - 100\pi}\)

**Problem 2:** \(y^{iv} = y''' + y'' + y' + 2y = 0\), \(y(0) = 1\), \(y'(0) = 0\), \(y''(0) = 30\), \(x \in [0, 2]\).

*Exact solution*: \(y(x) = 2e^{2x} - 5e^{-x} + 3 \cos x - 9 \sin x\)

**Problem 3:** \(y^{iv} = 4y''\), \(y(0) = 1\), \(y'(0) = 3\), \(y''(0) = 0\), \(y'''(0) = 16\)

*Exact solution*: \(y(x) = 1 - x + e^{2x} - e^{-2x}\), \(x \in [0, 1]\)

In general, the obtained numerical results show the efficiency of the developed method in term of accuracy. Its performance is clear better than several existing method through Table 1, Table 2 and Table 3.
Table 1: Comparison of the new method with some existing methods for solving problem 1

<table>
<thead>
<tr>
<th>$h$</th>
<th>Method</th>
<th>Error at $x=1.01325$ Steps.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>New method</td>
<td>$5.28 \times 10^{-18}$</td>
</tr>
<tr>
<td>0.103125</td>
<td>Method in [15]</td>
<td>$2.95 \times 10^{-17}$</td>
</tr>
<tr>
<td></td>
<td>Method in [6]</td>
<td>$5.37 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$h$</th>
<th>Error at $x=1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>New method</td>
</tr>
<tr>
<td>0.01</td>
<td>Method in [4]</td>
</tr>
</tbody>
</table>

Table 2: Comparison of the new method with some existing methods for solving problem 2

<table>
<thead>
<tr>
<th>$h$</th>
<th>Method</th>
<th>Error at $x=2$ Step</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>New method</td>
<td>$1.70 \times 10^{-10}$</td>
</tr>
<tr>
<td>0.1</td>
<td>Method in [15]</td>
<td>$8.07 \times 10^{-10}$</td>
</tr>
<tr>
<td></td>
<td>Method in [10]</td>
<td>$1.74 \times 10^{-8}$</td>
</tr>
<tr>
<td></td>
<td>Adams Method</td>
<td>$2.11 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>Method in [10]</td>
<td>$1.26 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.05</td>
<td>New method</td>
<td>$4.15 \times 10^{-12}$</td>
</tr>
<tr>
<td></td>
<td>Method in [15]</td>
<td>$8.45 \times 10^{-11}$</td>
</tr>
<tr>
<td></td>
<td>Method in [10]</td>
<td>$5.37 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>Adams Method</td>
<td>$1.91 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>Method in [10]</td>
<td>$2.13 \times 10^{-13}$</td>
</tr>
<tr>
<td>0.025</td>
<td>New method</td>
<td>$3.69 \times 10^{-13}$</td>
</tr>
<tr>
<td></td>
<td>Method in [10]</td>
<td>$5.09 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td>Method in [10]</td>
<td>$2.96 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 3: Comparison of the new method with some existing methods for solving problem 3

<table>
<thead>
<tr>
<th>$h$</th>
<th>Method</th>
<th>Error at $x=0.031250$ Step</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>New method</td>
<td>$9.17 \times 10^{-17}$</td>
</tr>
<tr>
<td>0.003125</td>
<td>Method in [15]</td>
<td>$3.29 \times 10^{-15}$</td>
</tr>
<tr>
<td></td>
<td>Method in [4]</td>
<td>$3.60 \times 10^{-13}$</td>
</tr>
</tbody>
</table>
4.2. Application

The new method is applied for solving physical status appear in ship dynamics. In particular, Twizell [13] and Cortell [7] have presented numerical solution of this problem which is fourth order IVPs describes how the sinusoidal wave of frequency $\Omega$ passes along a ship or offshore structure to relate fluids action with time $t$ as below.

$$y'''' = -3y'' - y(2 + \epsilon \cos(\Omega t)) \quad (4.1)$$

which is imposed to the following conditions: $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 0$. where $\epsilon = 0$ for the existence of the theoretical solution, $y(t) = 2 \cos t - \cos(t\sqrt{2})$ . The theoretical solution is undefined when $\epsilon \neq 0$.

In Table 4, the accuracy of the new method is compared with [13] and [7] for solving previous application at the end point $t = 15$, where $h = 0.25$ and $h = 0.1$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>Method</th>
<th>Error at $t=15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>New method</td>
<td>$8.15 \times 10^{-8}$</td>
</tr>
<tr>
<td></td>
<td>Method in [15]</td>
<td>$5.2 \times 10^{-7}$</td>
</tr>
<tr>
<td></td>
<td>Method in [13]</td>
<td>$1.9 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.1</td>
<td>New method</td>
<td>$3.6 \times 10^{-11}$</td>
</tr>
<tr>
<td></td>
<td>Method in [15]</td>
<td>$2.8 \times 10^{-10}$</td>
</tr>
<tr>
<td></td>
<td>Method in [7]</td>
<td>$3.7 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

5. Conclusion

Four steps block method with three hybrid points for solving fourth order initial value problems has been developed. The implementation of the method was then made through direct self-starting block method. Since this method is capable of solving forth order IVPs directly, it is more robust and flexible. Besides having good numerical properties the method is also claim superior to the existing method in terms of error.

Conflicts of Interest: The author declare no conflict of interest.

References