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An effective algorithm to solve option pricing problems

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Abstract

We are aimed to develop a fast and direct algorithm to solve linear complementarity problems (LCP's) arising from option pricing problems. We discretize the free boundary problem of American options in temporal direction and obtain a sequence of linear complementarity problems (LCP's) in the finite dimensional Euclidian space $\mathbb{R}^m$. We develop a fast and direct algorithm based on the active set strategy to solve the LCP's. The active set strategy in general needs $O(2^m m^3)$ operations to solve m dimensional LCP's. Using Thomas algorithm, we develop an algorithm with order of complexity $O(m)$ which can extremely speed up the computations.

Keywords: American options, variational inequalities, linear complementarity problems
2010 MSC: 58E35, 65M70, 91G60.

1. Introduction

Many mathematical problems can be written as variational inequalities introduced by Hartman and Stampacchia [6]. Variational inequality problems are closely related to a system of inequalities called the "linear complementarity problems" (LCP's). It is possible to formulate a wide range of free boundary problems as a variational inequalities and linear complementarity problems (see e.g . [3], [2]). In financial mathematics, option pricing problems can be state as variational inequalities [10, 13]. In the first section of this paper, we review some properties of linear complementarity problems in finite dimensional Euclidian spaces. In section 2, we investigate a simple example of linear complementarity problems called "the obstacle problem" and then we demonstrate a direct algorithm to solve it. In section 3, we focus on the problem of valuation of American options under the well known Black-Scholes model. We develop a fast and direct algorithm for solving the LCP of the Black-Scholes model. Finally, in section 4, we include some illustrative examples to demonstrate the validity and applicability of the proposed algorithm.

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1.1. Linear complementarity problems in finite dimensional Euclidian spaces

Problem 1.1. *(Linear complementarity problem) For a given matrix* $\mathbf{M} \in \mathbb{R}^{m\times m}$ *and a vector* $\mathbf{b} \in \mathbb{R}^m$*,*

$$\text{Find } \ \mathbf{x} \in \mathbb{R}^m \text{ such that } \begin{cases} \mathbf{x} \geq 0, \\ \mathbf{M}\mathbf{x} + \mathbf{b} \geq 0, \\ \mathbf{x}^\top(\mathbf{M}\mathbf{x} + \mathbf{b}) = 0, \end{cases} \tag{1.1}$$

where the inequalities have meaning element-wise. The linear complementarity problems have a rich mathematical theory, variety of algorithms, and a wide range of applications in applied science and technology. For a detailed discussion of LCP's see [1]. The sufficient condition for existence and uniqueness of solutions to the problem (1.1) is provided in the following theorem.

Theorem 1.2. *(See [1, Theorem 3.1.6 page 141]) Let* $\mathbf{M} \in \mathbb{R}^{m\times m}$ *be a positive definite matrix, then the linear complementarity problem* (1.1) *has a unique solution for all vectors* $\mathbf{b} \in \mathbb{R}^m$*.*

Several direct and iterative methods have been proposed to solve problems of type (1.1), for example the well known PSOR method has been applied in [10] to solve LCP's. The sufficient conditions for convergance of the PSOR algorithm to the unique solution of LCP is provided in [5]. In [8], the LU decomposition method is developed for solving complementarity problems arising from the pricing of American options, in [11] a fast algorithm is introduced for the symmetric linear complementarity problems. In the sequel, we describe a direct algorithm called "the active set strategy" to solve LCP's of the form (1.1).

1.2. The active set strategy

Let $\mathbf{y} = \mathbf{M}\mathbf{x} + \mathbf{b}$, the third statement of (1.1) means that for each i we have $\mathbf{x}_i\mathbf{y}_i = 0$ i.e. at least one of $\mathbf{x}_i, \mathbf{y}_i$ is zero. Given the index set $\mathcal{I} = \{1, 2, \ldots, m\}$ we partition $\mathcal{I}$ into two sets

$$\mathcal{A} = \{\ i \mid i \in \mathcal{I} \ \text{ and } \ \mathbf{y_i} > 0\}$$

$$\mathcal{F} = \{\ i \mid i \in \mathcal{I} \ \text{ and } \ \mathbf{y_i} = 0\}$$

We rewrite the system $\mathbf{y} = \mathbf{M}\mathbf{x} + \mathbf{b}$ as

$$\begin{bmatrix} \mathbf{y}_\mathcal{A} \\ \mathbf{y}_\mathcal{F} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\mathcal{A}\mathcal{A}} & \mathbf{M}_{\mathcal{A}\mathcal{F}} \\ \mathbf{M}_{\mathcal{F}\mathcal{A}} & \mathbf{M}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_\mathcal{A} \\ \mathbf{x}_\mathcal{F} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_\mathcal{A} \\ \mathbf{b}_\mathcal{F} \end{bmatrix}$$

but $\mathbf{y}_\mathcal{A} > 0$ implies that $\mathbf{x}_\mathcal{A} = 0$ so

$$\begin{bmatrix} \mathbf{y}_\mathcal{A} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\mathcal{A}\mathcal{A}} & \mathbf{M}_{\mathcal{A}\mathcal{F}} \\ \mathbf{M}_{\mathcal{F}\mathcal{A}} & \mathbf{M}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{x}_\mathcal{F} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_\mathcal{A} \\ \mathbf{b}_\mathcal{F} \end{bmatrix}$$

hence

$$\begin{bmatrix} \mathbf{y}_\mathcal{A} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\mathcal{A}\mathcal{F}}\mathbf{x}_\mathcal{F} + \mathbf{b}_\mathcal{A} \\ \mathbf{M}_{\mathcal{F}\mathcal{F}}\mathbf{x}_\mathcal{F} + \mathbf{b}_\mathcal{F} \end{bmatrix}$$

the elements of solution $\mathbf{x}$ which are correspond to $\mathcal{F}$ are computed by

$$\mathbf{x}_\mathcal{F} = -\mathbf{M}_{\mathcal{F}\mathcal{F}}^{-1}\mathbf{b}_\mathcal{F}. \tag{1.2}$$

Algorithm 1.1. $\big($***The active set strategy***$\big)$

1. *choose a partition* $\{\mathcal{A}, \mathcal{F}\}$ *of the index set* $\mathcal{I}$
2. *compute* $\mathbf{x}_{\mathcal{F}} = -\mathbf{M}_{\mathcal{F}\mathcal{F}}^{-1}\mathbf{b}_{\mathcal{F}}$ *and* $\mathbf{y}_{\mathcal{A}} = \mathbf{M}_{\mathcal{A}\mathcal{F}}\mathbf{x}_{\mathcal{F}} + \mathbf{b}_{\mathcal{A}}$
3. *if* $\mathbf{x}_{\mathcal{F}} \geq 0, \mathbf{y}_{\mathcal{A}} \geq 0$ *then* $\begin{bmatrix} 0 \\ \mathbf{x}_{\mathcal{F}} \end{bmatrix}$ *is a solution for* (1.1) *and stop the algorithm*
   else choose another partition of $\mathcal{I}$ *and go to stage 2.*

There are 2^m possible partitions for the index set $\mathcal{I}$. Computing $\mathbf{x}_{\mathcal{F}}$ from (1.2) needs $O(m^3)$ operation in the worst case (for full matrices), hence the complexity order of the active set strategy in the worst case is $O(2^m m^3)$. In the next sections, we develop a modification of the active set strategy with order of complexity $O(m)$ which can extremely speed up the computations.

2. One dimensional obstacle problem

In this section, we describe a simple example of variational inequalities called "the obstacle problem".

2.1. Obstacle problem

Assume an obstacle $g(x) \in \mathcal{C}^1(0,1)$ with following conditions (compare Figure 1)

- $g(x) > 0$ for $0 < x < \alpha$,
- $g'' < 0$ for $x \in (0,1)$,
- $g(0) = 0$ and $g(1) < 0$.

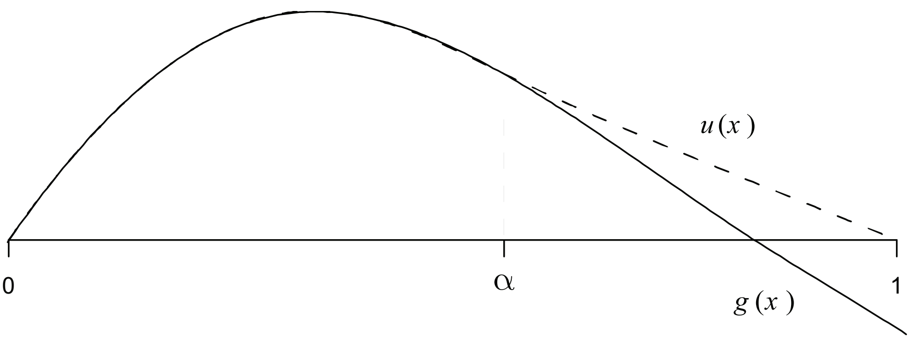

Figure 1: Function $u(x)$ (dashed) across the obstacle $g(x)$ (solid). Before the free boundary $x = \alpha$, the solution $u(x)$ coincides with the obstacle $g(x)$ and after that the solution becomes a straight line. At the free boundary $x = \alpha$ the curve $u(x)$ touches the obstacle tangentially.

Across the obstacle, a function $u(x)$ with minimal length is stretched like a rubber thread. Before $x = \alpha$, the curve $u(x)$ clings to the obstacle $g(x)$. At $x = \alpha$ the curve of $u(x)$ touches the obstacle tangentially. The value of α is unknown initially. The function u shown in Figure 1 is defined by the requirement $u \in \mathcal{C}^1(0,1)$ and by

$$\begin{array}{lll} \text{for} \quad 0 < x < \alpha: & u'' \leq 0 & (\text{because } u = g \text{ and } u'' = g'' < 0), \\ \text{for} \quad \alpha < x < 1: & u'' = 0 & (\text{because } u \text{ becomes a straight line, also } u > g). \end{array}$$

The obstacle problem can be states as following problem [13]

$$
\begin{aligned}
&\text{Find function } u(x) \in \mathcal{C}^1(0,1) \text{ such that} \\
&-u'' \geq 0, && \text{(2.1a)} \\
&u - g \geq 0, && \text{(2.1b)} \\
&(u-g)u'' = 0, && \text{(2.1c)} \\
&u(0) = u(1) = 0. && \text{(2.1d)}
\end{aligned}
$$

Notice that the unknown free boundary α does not appear in (2.1) explicitly. After solving (2.1) one can read off the position of α from the solution. Obstacle problems have been studied as a part of theory of variational inequalities and wide abstract theory has been developed for them [6].

2.2. The finite difference discretization of the obstacle problem

We discretize the interval $[0,1]$ using m equidistant points $\mathcal{T} = \{x_1, x_2, \ldots, x_m\}$ with step size h. We approximate the derivatives using the following finite difference method

$$u''(x_i) = \frac{u(x_{i+2}) - 2u(x_{i+1}) + u(x_i)}{h^2}, \tag{2.2}$$

substituting (2.2) in (2.1) we get

$$\text{find } \mathbf{u} \in \mathbb{R}^m \text{ such that } \begin{cases} \mathbf{u} - \mathbf{g} \geq 0, \\ \mathbf{A}\mathbf{u} \geq 0, \\ (\mathbf{u}-\mathbf{g})^\top \mathbf{A}\mathbf{u} = 0. \end{cases} \tag{2.3}$$

The matrix $\mathbf{A}$ is a sparse matrix with the following tridiagonal structure.

$$\mathbf{A} = \begin{pmatrix} \frac{2}{h^2} & \frac{-1}{h^2} & 0 & \cdots & 0 \\ \frac{-1}{h^2} & \frac{2}{h^2} & \frac{-1}{h^2} & & 0 \\ 0 & & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \frac{-1}{h^2} & \frac{2}{h^2} & \frac{-1}{h^2} \\ 0 & \cdots & 0 & \frac{-1}{h^2} & \frac{2}{h^2} \end{pmatrix} \tag{2.4}$$

By change of variable $\mathbf{x} = \mathbf{u} - \mathbf{g}$ we achieve the standard LCP of the form (1.1).

Theorem 2.1. *The coefficient matrix* $\mathbf{A}$ *in* (2.3) *is positive definite and in view of theorem 1.2 the LCP* (2.3) *has a unique solution.*

Proof . Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)^\top$ be an arbitrary vector in $\mathbb{R}^m$, then

$$
\begin{aligned}
\boldsymbol{\alpha}^\top \mathbf{A}\boldsymbol{\alpha} &= \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i \mathbf{A}_{ij}\alpha_j \\
&= \frac{1}{h^2}\left(2\alpha_1\alpha_1 - \alpha_1\alpha_2 + \sum_{i=2}^{m-1}\alpha_i(-\alpha_{i-1} + 2\alpha_i - \alpha_{i+1}) - \alpha_m\alpha_{m-1} + 2\alpha_m\alpha_m\right) \\
&= \frac{1}{h^2}\Bigg(2\alpha_1\alpha_1 - \alpha_1\alpha_2 + (-\alpha_2\alpha_1 + 2\alpha_2\alpha_2 - \alpha_2\alpha_3) \\
&\qquad\qquad + (-\alpha_3\alpha_2 + 2\alpha_3\alpha_3 - \alpha_3\alpha_4) \\
&\qquad\qquad + \dots \\
&\qquad\qquad - \alpha_m\alpha_{m-1} + 2\alpha_m\alpha_m\Bigg) \\
&= \frac{1}{h^2}\left(\alpha_1^2 + (\alpha_1 - \alpha_2)^2 + (\alpha_2 - \alpha_3)^2 + \dots + (\alpha_{m-1} - \alpha_m)^2 + \alpha_m^2\right) \geq 0,
\end{aligned}
$$

with equality if and only if $\alpha_1 = \alpha_2 = \dots \alpha_m = 0$. □

3. The Black-Scholes model for pricing American options

3.1. Linear complementarity problem of the Black-Scholes model

Under the standard Black-Scholes model, the value of an American option satisfies a free boundary partial differential equation which is equivalent to a variational inequality problem as follows (see [10, 13])

$$
\begin{cases}
P_\tau(S,\tau) + \mathcal{L}P(S,\tau) \geq 0, \quad (S,\tau) \in (0,\infty) \times (0,T), \\
P(S,\tau) \geq h(S), \\
(P - h)\left(P_\tau(S,\tau) + \mathcal{L}P(S,\tau)\right) = 0, \\
P(S,\tau) = E, \quad \lim_{S\to\infty} P(S,\tau) = 0,
\end{cases}
\tag{3.1}
$$

where $h(S) = \max\{E - S, 0\}$ and $\mathcal{L}$ is the Black-Scholes partial differential operator introduced by

$$
\mathcal{L}P(S,\tau) = \frac{1}{2}\sigma^2 S^2 P_{SS}(x,t) + rSP_S(S,\tau) - rP(S,\tau).
$$

In (3.1), $P(S,\tau)$ denotes the price of option at time τ when the spot price of underlying asset is S. The "strike price" E, "volatility" σ, "interest rate" r and "maturity" T are all positive constants. The domain of problem (3.1) has shown in Figure 2.

The theory of free boundary problems is closely related to linear complementarity problems and variational inequality problems. For a detailed discussion of this relation we refer the reader to [2]. Two basic references on the complementarity problems and variational inequalities are [1] and [9].

Now we turn our attention to the Black-Scholes inequality (3.1). By some change of variables we can transform (3.1) into a LCP of heat equation type

$$
\begin{cases}
u_t - u_{xx} \geq 0, \quad -\infty < x < \infty, \quad 0 \leq t \leq t_{max}, \\
u \geq g, \\
(u_t - u_{xx})(u - g) = 0, \\
u(x,0) = g(x,0), \\
\lim_{x\to\pm\infty} u(x,t) = g(x,t).
\end{cases}
\tag{3.2}
$$

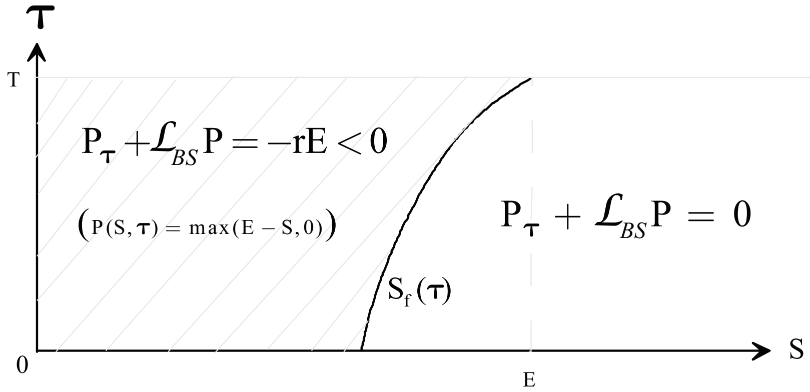

Figure 2: The domain of problem (3.1). Before the free boundary (early exercise curve) $S_f(\tau)$, the solution P coincides with the payoff function $h(S)$ and we have $P_\tau + \mathcal{L}P > 0$. In the right hand of free boundary, the equation $P_\tau + \mathcal{L}P = 0$ holds.

where

$$g(x,t) = e^{\frac{t}{4}(q+1)^2} \max\{e^{\frac{x}{2}(q-1)} - e^{\frac{x}{2}(q+1)}, 0\}.$$

This equivalence can be proved by means of the transformations given in table 1. For more discussion

$x = \ln(S/E), \quad t = \frac{\sigma^2}{2}(T-\tau), \quad t_{max} = \frac{\sigma^2}{2}T, \quad q = \frac{2r}{\sigma^2},$
$p(x,t) = Ee^{-\frac{x}{2}(q-1)-\frac{t}{4}(q+1)^2}u(x,t),$
$P(S,\tau) = p\left(\ln(S/E), \frac{\sigma^2}{2}(T-\tau)\right).$

Table 1: Change of variables for the Black-Scholes equation .

about this equivalence the reader is referred to [13] .

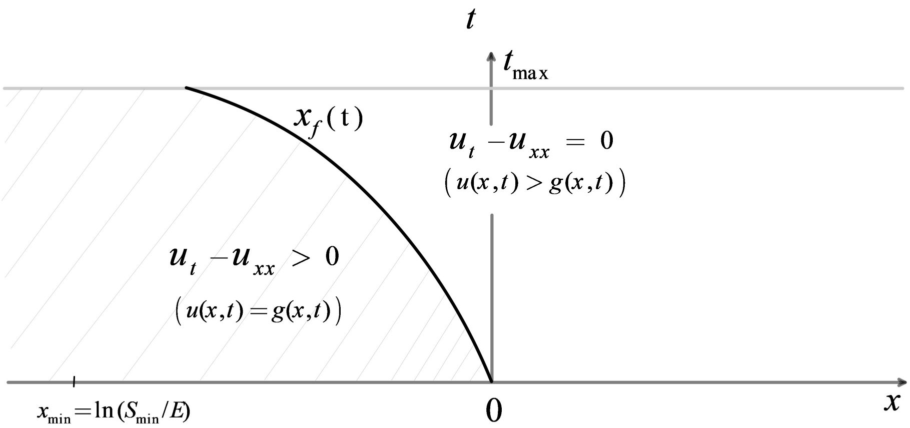

Figure 3: The domain of problem (3.1).

In what follows, we use the implicit Euler method to discretize (3.1) in temporal direction and obtain a sequence of linear complementarity problems (LCPs) in a finite dimensional Euclidean space.

3.2. The finite difference discretization of the Black-Scholes equation

We truncate the infinite domain $\mathbb{R}$ in (3.2) for x to be $[x_{min}, x_{max}]$ with a positive and sufficiently large x_{max} . The truncation error with respect to x has been shown to decrease exponentially [7]. Notice that the last condition in (3.2) states that $g(x,t)$ is the asymptote of $u(x,t)$ when x tends to $\pm\infty$, however we replace it by

$$u(x_{min}, t) = g(x_{min}, t), \quad u(x_{max}, t) = g(x_{max}, t).$$

Now we discretize (3.2) according to the following implicit Euler method

$$u_t(x, t_n) = \frac{u(x, t_n) - u(x, t_{n-1})}{\delta t},$$

using notation $u^n(x) := u(x, t_n)$, the semi-discrete version of (3.2) will be achieved

$$\begin{cases} u^n - \delta t u^{n\prime\prime} - u^{n-1} \geq 0, \\ u^n \geq g^n, \\ (u^n - \delta t u^{n\prime\prime} - u^{n-1})(u^n - g^n) = 0, \\ u^0 = g(x, 0), \\ \lim_{x\to\pm\infty} u^n = g(x, t_n). \end{cases} \tag{3.3}$$

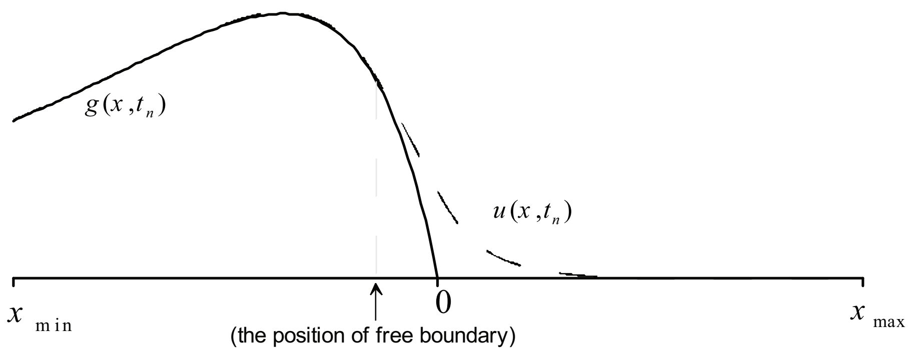

Figure 4: Solution $u(x,t)$ across the obstacle $g(x,t)$ at time t_n. Before the position of free boundary the solution u coincides with obstacle g. At $x = 0$ the obstacle $g(x, t_n)$ has a slope discontinuity.

Time discretization using implicit Euler method deduces accuracy of order $O(\delta t)$. To achieve second order accuracy of time, we can discretize (3.2) using the Crank-Nicholson method. In this case, the first equation of (3.3) becomes

$$u^n - \frac{1}{2}\delta t u^{n\prime\prime} - \frac{1}{2}\delta t u^{n-1\prime\prime} - u^{n-1} \geq 0.$$

The initial value $u(x, 0)$ as shown in figure 4 has a slope discontinuity at $x = 0$. In such cases the popular Crank-Nicholson method can lead to a numerical solution with oscillations due to the lack of L-stability [14]. To reduces these oscillations one can use the Rannacher time stepping scheme proposed in [12]. The Rannacher time stepping starts the computations with two implicit Euler steps and after that it uses the Crank-Nicholson method. Using this scheme the full second order accuracy of the Crank-Nicholson scheme can be achieved. A convergence analysis of Rannacher time-stepping is given in [4].

Now we discretize the spatial domain $[x_{min}, x_{max}]$ similar to section 2 . Since the transformed payoff function $g(x,0)$ as shown in figure 4 has a slope discontinuity at $x = 0$, it is reasonable to assume that $0 \in \mathcal{T}$. Let

$$\begin{aligned} \mathbf{u}^n &= \left(u_1^n, u_2^n, \ldots, u_m^n\right)^\top, \quad u_j^n = u^n(x_j), \\ \mathbf{g}^n &= \left(g_1^n, g_2^n, \ldots, g_m^n\right)^\top, \quad g_j^n = g(x_j, t_n). \end{aligned} \tag{3.4}$$

The discretization of problem (3.2) is

$$\text{Find} \quad \mathbf{u}^n \in \mathbb{R}^m \text{ such that } \begin{cases} \mathbf{u}^n - \mathbf{g}^n \geq 0, \\ \mathbf{M}\mathbf{u}^n - \mathbf{u}^{n-1} \geq 0, \\ \left(\mathbf{u}^n - \mathbf{g}^n\right)^\top \left(\mathbf{M}\mathbf{u}^n - \mathbf{u}^{n-1}\right) = 0. \end{cases} \tag{3.5}$$

In equation (3.5), $\mathbf{M} = \mathbf{I} + \delta t\mathbf{A}$ where $\mathbf{A}$ is introduced by (2.4) and the m-vector $\mathbf{g}^n$ can be easily replaced by $g(x, t_n)$ computed at points of $\mathcal{T}$. By change of variable $\mathbf{x} = \mathbf{u}^n - \mathbf{g}^n$ we achieve a sequence of LCPs of the standard form (1.1).

Theorem 3.1. *The linear complementarity problem* (3.5) *has a unique solution .*

Proof .from theorem 2.1, we know that the matrix $\mathbf{A}$ is positive definite, so the coefficients matrix $\mathbf{M} = \mathbf{I} + \delta t\mathbf{A}$ is also positive definite. $\square$

The linear complementarity problem (3.5) most be solved at every time step t_n. In the sequel we propose a very fast modification of the active set strategy to solve LCPs.

3.3. Modification of the active set strategy for American options

For American options, the unknown boundary of the problem, divides the computational domain into two partitions (see figure 4). Before the position of free boundary, the solution u coincides with the payoff function and the inequality $\mathbf{M}\mathbf{x} + \mathbf{b} > 0$ holds. In the right hand side of free boundary we have the equation $\mathbf{M}\mathbf{x} + \mathbf{b} = 0$. We define two index sets $\mathcal{A} = \{1, 2, \ldots i-1\}$ and $\mathcal{F} = \{i, i+2, \ldots, m\}$. We choose an initial guess for i then we compute $\mathbf{x}_\mathcal{F}$ form formula (1.2) and check the conditions in the third stage of algorithm 1.1. Since $\mathbf{M}$ is a tridiagonal matrix, we use the Thomas algorithm to compute $\mathbf{x}_\mathcal{F}$.

Algorithm 3.1. (***Active set strategy for American options***)

1. *Choose the partitions* $\mathcal{A} = \{1, 2, \ldots i-1\}$ *and* $\mathcal{F} = \{i, i+1, \ldots, m\}$
2. *Solve the system* $\mathbf{M}_{\mathcal{F}\mathcal{F}}\mathbf{x}_\mathcal{F} = \mathbf{b}_\mathcal{F}$ *by Thomas algorithm and compute* $\mathbf{y}_\mathcal{A} = \mathbf{M}_{\mathcal{A}\mathcal{F}}\mathbf{x}_\mathcal{F} + \mathbf{b}_\mathcal{A}$
3. *if* $\mathbf{x}_\mathcal{F} \geq 0, \mathbf{y}_\mathcal{A} \geq 0$ *then* $\begin{bmatrix} 0 \\ \mathbf{x}_\mathcal{F} \end{bmatrix}$ *is a solution for* (1.1) *and stop the algorithm*
   else set $i := i-1$ *and go to stage 2.*

In the third stage of algorithm 3.1 when we decrease i, the dimension of matrix $\mathbf{M}_{\mathcal{F}\mathcal{F}}$ will be increased. If the initial guess of i would be chosen near the free boundary, after a few iterations the solution $\begin{bmatrix} 0 \\ \mathbf{x}_\mathcal{F} \end{bmatrix}$ will be found. Since the Thomas algorithm for tridiagonal matrices has the complexity of order $O(m)$, we can state the following theorem.

Theorem 3.2. *The algorithm 3.1 to solve the LCP* (1.1) *only needs* $O(m)$ *operations.*

4. Numerical experiments

In this section, the proposed numerical algorithm is validated through experimental tests.

Example 4.1. In the first example, we consider the obstacle problem (2.1) with obstacle function $g(x) = x(\frac{\sqrt{2}}{2} - x)$. The exact position of the free boundary is $\alpha = 1 - \frac{1}{2}\sqrt{4 - 2\sqrt{2}}$ and the exact solution of the obstacle problem is

$$u(x) = \begin{cases} x(\frac{\sqrt{2}}{2} - x), & 0 \le x \le \alpha, \\ \frac{g(\alpha)}{\alpha - 1}(x - 1) & \alpha \le x \le 1. \end{cases}$$

We discretize the domain $[0, 1]$ using m equidistance points with different values of m. We apply the finite difference method and reduce the obstacle problem to a linear complementarity problem. We solve the LCP using modified active set strategy proposed in algorithm 3.1. We compare the estimated solutions with exact solutions in the sense of root mean square (RMS) error:

$$\text{RMS error} = \sqrt{\frac{\sum_{i=1}^{m}[u(x_i) - u_h(x_i)]^2}{m}}$$

Figure 5 illustrates the RMS error for various values of m (number of elements). Figure 6 shows computational times for modified active set strategy. The programs have been tested and run on Windows 10 operated by Intel Core i5-7200U processor with 8 GB of RAM. We can see that the computational time grows linearly with respect to m.

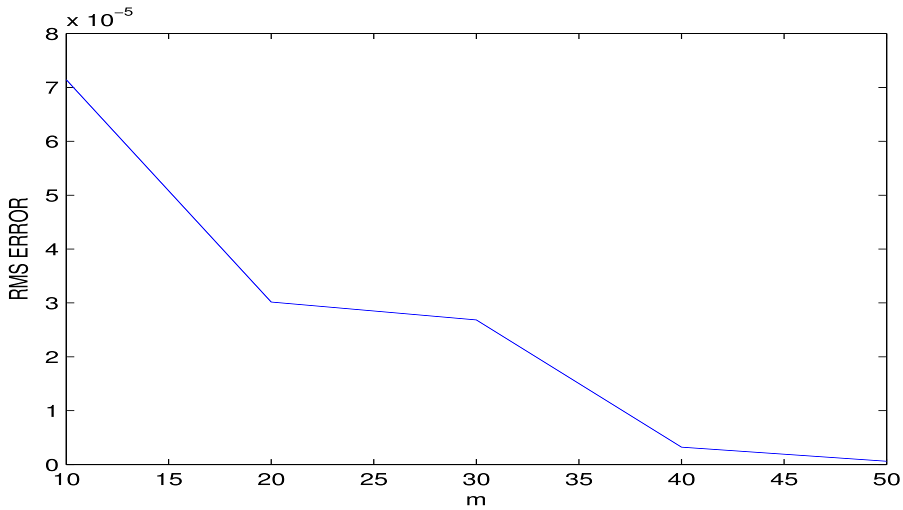

Figure 5: RMS error respect to various values of m for obstacle problem (2.1)

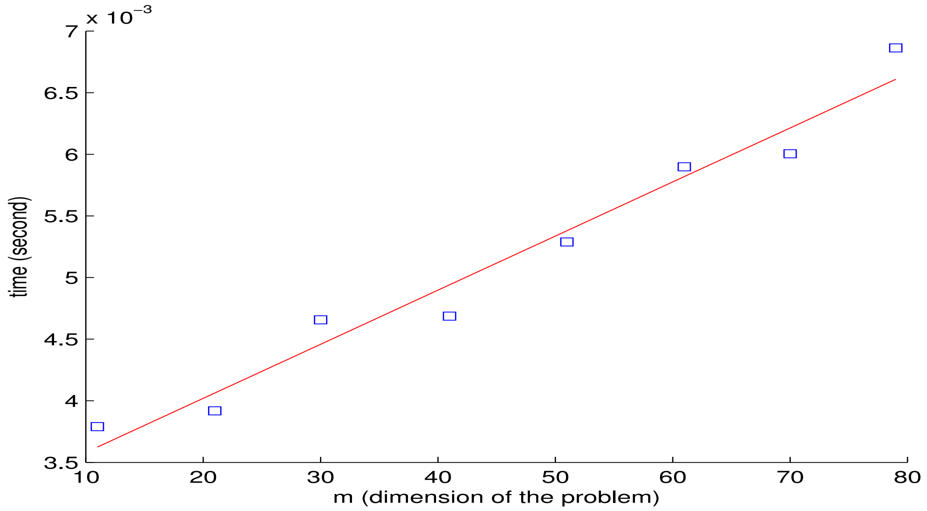

Figure 6: computation time for various values of m for problem (2.1).

Example 4.2. In the second example, we focus on the Black Scholes linear complementarity problem of American put option (3.1) with parameters

$$T = 0.5, \qquad r = 0.06, \qquad \sigma = 0.4, \qquad E = 100.$$

We solve the equivalent problem (3.2) using the spatial computational domain $[x_{min}, x_{max}] = [-1, 1]$ and discretize it with $m = 101$ equidistant points, then we discretize the time interval $[0, t_{max}]$ using $N = 20$ temporal levels. We solve the complementarity problems in each time step using the modified active set strategy explained in section 3.3. Financially, the value of option at time $\tau = 0$ is important because $P(S, 0)$ denotes the option prices at present time. Figure 7 shows the solutions to the problem (3.1) at times $\tau = 0$ across the obstacle function $\max\{E - S, 0\}$.

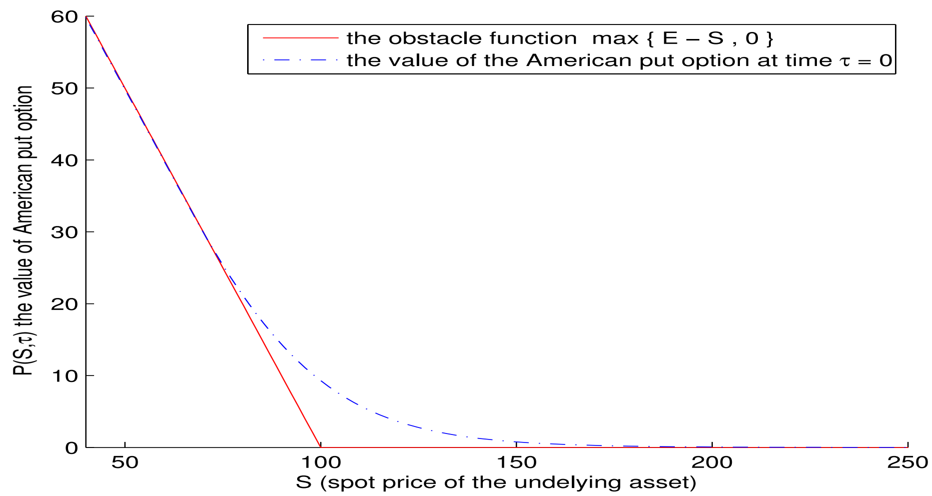

Figure 7: Approximation of the American put option price $P(S, \tau)$ at time $\tau = 0$ across the obstacle function $\max\{E - S, 0\}$.

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