

On certain properties for new subclass of meromorphic starlike functions

Sirous Moradi^{a,*}, Mohammad Taati^b

^aDepartment of Mathematics, Faculty of science, Lorestan University, 68151-4-4316, Khorramabad, Iran.

^bDepartment of Mathematics, Faculty of science, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran.

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper we studying some properties of starlike function of order λ which satisfy in the condition

$$\Re\left(\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}\right) < 1 - \lambda + \alpha$$

for all $z \in \mathbb{U} = \{z : |z| < 1\}$, where $f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$ is analytic in \mathbb{U} , $0 \leq \alpha < 2$ and $0 \leq \lambda < 1$.

Our results extend previos results given by Aghalary et al. (2009) and Wang et al.(2014).

Keywords: Starlike function, Meromorphic function, Hadamard product, Analytic function.

2010 MSC: Primary: 30C45; Secondary: 30C80.

1. Introduction

Let Σ denote the class of functions f of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* = \{z : 0 < |z| < 1\} = \mathbb{U} - \{0\}.$$

*Corresponding author

Email addresses: moradi.s@lu.ac.ir (Sirous Moradi), m_taati@pnu.ac.ir (Mohammad Taati)

A function $f \in \Sigma$ is said to be in the class $\mathcal{MS}^*(\alpha)$ of meromorphically starlike functions of order α if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1).$$

Let \mathcal{P} denote the class of functions p given by

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad (z \in \mathbb{U}) \tag{1.2}$$

which are analytic in \mathbb{U} and satisfy the condition

$$\Re(p(z)) > 0 \quad (z \in \mathbb{U}).$$

Many authors have studied analytic starlike functions. For some recent investigation, see, for example [1, 2, 8, 12, 14, 15, 18, 19, 20, 23] and the references therein.

Wang et al. [23] introduced a new class of starlike analytic functions on \mathbb{U}^* as follows:

$$\mathcal{H}(\beta, \lambda) = \left\{ f \in \Sigma : \Re\left(\frac{zf'(z)}{f(z)} + \beta \frac{z^2 f''(z)}{f(z)}\right) < \beta\lambda\left(\lambda + \frac{1}{2}\right) + \frac{\beta}{2} - \lambda, (z \in U^*) \right\},$$

where $\beta \geq 0$ and $\frac{1}{2} \leq \lambda < 1$.

In [23] Wang et al. had proved that $\mathcal{H}(\beta, \lambda)$ is a subclass of $\mathcal{MS}^*(\lambda)$. Also Wang et al. [22] introduced the following subclass of $\mathcal{H}(\beta, \lambda)$.

Let $\mathcal{H}^+(\beta, \lambda)$ denote the subset of $\mathcal{H}(\beta, \lambda)$ such that all functions $f \in \mathcal{H}(\beta, \lambda)$ having the following form:

$$f(z) = \frac{1}{z} - \sum_{k=1}^{\infty} a_k z^k \quad (a_k \geq 0).$$

The following two lemmas can be derived from ([6], Theorem 1) (see also [7]).

Lemma 1.1. *Let*

$$1 + \beta\lambda\left(\lambda + \frac{1}{2}\right) - \lambda - \frac{3}{2}\beta > 0. \tag{1.3}$$

Suppose also that $f \in \Sigma$ is given by (1.1). If

$$\sum_{k=1}^{\infty} [k + \beta k(k - 1) + \gamma] |a_k| \leq 1 - \gamma - 2\beta.$$

where (and throughout this paper unless otherwise mentioned) the parameter γ is constrained as follows:

$$\gamma = \lambda - \beta\lambda\left(\lambda + \frac{1}{2}\right) - \frac{\beta}{2}, \tag{1.4}$$

then $f \in \mathcal{H}(\beta, \lambda)$.

Lemma 1.2. *Let $f \in \Sigma$ be given by (1.1). Suppose also that γ is defined by (1.4) and the condition (1.3) holds. Then $f \in \mathcal{H}^+(\beta, \lambda)$ if and only if*

$$\sum_{k=1}^{\infty} [k + \beta k(k - 1) + \gamma] a_k \leq 1 - \gamma - 2\beta.$$

Recently Wang et al. [23] proved some coefficient inequalities, neighborhoods, partial sums and inclusion relationships for two classes $\mathcal{H}(\beta, \lambda)$ and $\mathcal{H}^+(\beta, \lambda)$.

In Section 2, we introduced a new class of analytic starlike function. In Section 3 and Section 4, we prove some coefficient inequalities, neighborhoods and partial sums. Our results extend previous results given by Aghalary et al. [1] as well as by Wang et al. [23].

2. Preliminaries

In this section we introduce the notation \mathcal{A} for the class of all functions f of the form

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \quad (2.1)$$

which are analytic in the open unit disk \mathbb{U} . Let $0 \leq \alpha < 2$ and $0 \leq \lambda < 1$ and $\Lambda(\alpha, \lambda)$ denotes the class of functions $f \in \mathcal{A}$ and satisfies the condition

$$\Re \left(\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \right) < 1 - \lambda + \alpha. \quad (2.2)$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{V}(\alpha, \lambda)$ if it satisfies the condition

$$\Re \left(\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \right) < 1 - \lambda + \alpha \left(\frac{1}{2} + \lambda^2 - \frac{3}{2} \lambda \right), \quad (2.3)$$

such that $0 \leq \alpha < 2$ and $0 \leq \lambda < 1$. Obviously $\mathcal{V}(\alpha, \lambda) \subseteq \Lambda(\alpha, \lambda)$. Also suppose that $\Lambda^*(\lambda)$ denotes the class of functions $f \in \mathcal{A}$ such that satisfies the following condition

$$\Re \left(\frac{zf'(z)}{f(z)} \right) < 1 - \lambda. \quad (2.4)$$

Obviously $\Lambda(0, \lambda) = \mathcal{V}(0, \lambda) = \Lambda^*(\lambda)$.

Given two functions $f, g \in \mathcal{A}$ where f is given by (2.1) and g is given by

$$g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k.$$

The Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) = 1 + \sum_{k=1}^{\infty} a_k b_k z^k := (g * f)(z).$$

At first we prove the following lemma.

Lemma 2.1. *Let $0 \leq \alpha < 2$, $0 \leq \lambda < 1$ and $f \in \mathcal{A}$. Then $f \in \mathcal{V}(\alpha, \lambda)$ if and only if $\frac{1}{z}f \in \mathcal{H}\left(\frac{\alpha}{1+2\alpha}, \lambda\right)$.*

Proof . Let $f \in \mathcal{A}$. Then $\frac{1}{z}f \in \mathcal{H}(\frac{\alpha}{1+2\alpha}, \lambda)$ if and only if

$$\Re \left(\frac{z(\frac{1}{z}f(z))'}{(\frac{1}{z}f(z))} + \frac{\alpha}{1+2\alpha} \frac{z^2(\frac{1}{z}f(z))''}{(\frac{1}{z}f(z))} \right) < \frac{\alpha}{1+2\alpha} \lambda(\lambda + \frac{1}{2}) + \frac{\alpha}{2(1+2\alpha)} - \lambda.$$

Which is equivalent to

$$\Re \left((1 - \frac{2\alpha}{1+2\alpha}) \frac{zf'(z)}{f(z)} + \frac{\alpha}{1+2\alpha} \frac{z^2f''(z)}{f(z)} \right) < \frac{\alpha}{1+2\alpha} \lambda(\lambda + \frac{1}{2}) + \frac{\alpha}{2(1+2\alpha)} - \lambda + 1 - \frac{2\alpha}{1+2\alpha}.$$

Hence $\frac{1}{z}f \in \mathcal{H}(\frac{\alpha}{1+2\alpha}, \lambda)$ if and only if

$$\Re \left(\frac{zf'(z)}{f(z)} + \alpha \frac{z^2f''(z)}{f(z)} \right) < 1 - \lambda + \alpha + \alpha(\frac{1}{2} + \lambda^2 - \frac{3}{2}\lambda).$$

This inequality is equivalent to $f \in \mathcal{V}(\alpha, \lambda)$, and completes the proof. \square

Remark 2.2. If $0 \leq \lambda < 1$ and $0 \leq \beta < \frac{2}{5}$ and $\alpha = \frac{\beta}{1-2\beta}$ then by similar method in proof of Lemma 2.1 we can prove that $h \in \mathcal{H}(\beta, \lambda)$ if and only if $zh \in \mathcal{V}(\frac{\beta}{1-2\beta}, \lambda)$.

In order to prove our main results, we need the following useful lemma.

Lemma 2.3. (See[11]) If the function $p \in \mathcal{P}$ is given by (1.2), then

$$|p_k| \leq 2 \quad (k \in \mathbb{N}).$$

A function $f \in \Lambda(\alpha, \lambda)$ of the form

$$f(z) = 1 - \sum_{k=1}^{\infty} a_k z^k \quad (a_k \geq 0) \tag{2.5}$$

is said to be in the class $\Lambda^+(\alpha, \lambda)$.

3. Main Results

We start this section by the following lemmas.

Lemma 3.1. Let $0 \leq \alpha < 2$, $0 \leq \lambda < 1$ and $f \in \mathcal{A}$ is given by (2.1). If

$$\sum_{k=1}^{\infty} [\alpha k^2 - \alpha k + k - \gamma] |a_k| < \gamma \quad (\gamma = 1 - \lambda + \alpha) \tag{3.1}$$

then $f \in \Lambda(\alpha, \lambda)$.

Proof . For $z = re^{i\theta}, 0 \leq r < 1$ and $0 \leq \theta < 2\pi$, from (3.1), we get

$$\Re \left(\frac{\sum_{k=1}^{\infty} [\alpha k^2 - \alpha k + k] a_k z^k}{1 + \sum_{k=1}^{\infty} a_k z^k} \right) \leq \frac{\sum_{k=1}^{\infty} [\alpha k^2 - \alpha k + k] |a_k| r^k}{1 + \sum_{k=1}^{\infty} |a_k| r^k} = \frac{\sum_{k=1}^{\infty} [\alpha k^2 - \alpha k + k] |a_k|}{1 + \sum_{k=1}^{\infty} |a_k|} < \gamma, \quad (r \rightarrow 1).$$

The above inequalities show that $f \in \Lambda(\alpha, \lambda)$. \square

Lemma 3.2. Let $0 \leq \alpha < 2$ and $0 \leq \lambda < 1$ and $f \in \mathcal{A}$ is given by (2.5). Then

$$\sum_{k=1}^{\infty} [\alpha k^2 - \alpha k + k - \gamma] a_k < \gamma \quad \gamma = (1 - \lambda + \alpha)$$

if and only if $f \in \Lambda^+(\alpha, \lambda)$.

Proof . In view of Lemma 3.1, we need only show that $f \in \Lambda^+(\alpha, \lambda)$ satisfies the coefficient condition. We give $f \in \Lambda^+(\alpha, \lambda)$, so

$$\Re \left(\frac{-\sum_{k=1}^{\infty} [\alpha k^2 - \alpha k + k] a_k z^k}{1 - \sum_{k=1}^{\infty} a_k z^k} \right) < \gamma,$$

for $z = re^{i\theta}, 0 \leq r < 1$ and $0 \leq \theta < 2\pi$, we have

$$\frac{-\sum_{k=1}^{\infty} [\alpha k^2 - \alpha k + k] a_k r^k}{1 - \sum_{k=1}^{\infty} a_k r^k} < \gamma.$$

The result follows upon letting $r \rightarrow 1$.

$$\frac{\sum_{k=1}^{\infty} [\alpha k^2 - \alpha k + k] a_k}{1 + \sum_{k=1}^{\infty} a_k} \leq \frac{\sum_{k=1}^{\infty} [\alpha k^2 - \alpha k + k] a_k}{-1 + \sum_{k=1}^{\infty} a_k} < \gamma.$$

\square

Lemma 3.3. Let $0 \leq \alpha < 2$ and $0 \leq \lambda < 1$. Suppose also that the sequence $\{B_k\}_{k=1}^{\infty}$ is defined by

$$B_1 = 2(1 - \lambda + \alpha) \quad \text{and} \quad B_{k+1} = \frac{2(1 - \lambda + \alpha)}{k + 1 + \alpha k(k + 1)} \left(1 + \sum_{i=1}^k B_i \right) \quad (k \in \mathbb{N}). \quad (3.2)$$

Then

$$B_k = 2(1 - \lambda + \alpha) \prod_{j=1}^{k-1} \frac{j + \alpha j(j - 1) + 2(1 - \lambda + \alpha)}{j + 1 + \alpha j(j + 1)} \quad (k \in \mathbb{N}). \quad (3.3)$$

Proof . By virtue of (3.2), we easily get

$$[k + 1 + \alpha k(k + 1)]B_{k+1} = 2(1 - \lambda + \alpha) \left(1 + \sum_{i=1}^k B_i \right)$$

and

$$[k + \alpha k(k - 1)]B_k = 2(1 - \lambda + \alpha) \left(1 + \sum_{i=1}^{k-1} B_i \right).$$

We obtain that

$$\frac{B_{k+1}}{B_k} = \frac{k + \alpha k(k - 1) + 2(1 - \lambda + \alpha)}{k + 1 + \alpha k(k + 1)}$$

Thus, for $k \geq 2$, so we give

$$B_k = \frac{B_k}{B_{k-1}} \cdot \frac{B_{k-1}}{B_{k-2}} \cdot \dots \cdot \frac{B_2}{B_1} \cdot B_1 = 2(1 - \lambda + \alpha) \prod_{j=1}^{k-1} \frac{j + \alpha j(j - 1) + 2(1 - \lambda + \alpha)}{j + 1 + \alpha j(j + 1)}$$

and this evidently completes the proof. \square

By using induction and (3.2) we conclude the following proposition.

Proposition 3.4. *Let $0 \leq \alpha < 2$, $0 \leq \lambda < 1$ and the sequence $\{B_k\}$ is given by (3.2). Then*

$$B_k \leq \frac{k + 1}{3}(1 + B_1) \quad (k \geq 2). \tag{3.4}$$

Theorem 3.5. *Let $0 \leq \alpha < 2$ and $0 \leq \lambda < 1$. If $f \in \Lambda(\alpha, \lambda)$, then*

$$|a_1| \leq 2(1 - \lambda + \alpha) \tag{3.5}$$

and

$$|a_k| \leq 2(1 - \lambda + \alpha) \prod_{j=1}^{k-1} \frac{j + \alpha j(j - 1) + 2(1 - \lambda + \alpha)}{j + 1 + j(j + 1)} \quad (k \geq 2). \tag{3.6}$$

Proof . Suppose that

$$q(z) = -\frac{zf'(z)}{f(z)} - \alpha \frac{z^2 f''(z)}{f(z)} + 1 - \lambda + \alpha$$

Then, by $f \in \Lambda(\alpha, \lambda)$, we know that q is analytic in \mathbb{U} , $q(0) = 1 - \lambda + \alpha > 0$ and $\Re[q(z)] > 0$. Hence

$$h(z) = \frac{q(z)}{q(0)} = \frac{q(z)}{1 - \lambda + \alpha} \in \mathcal{P}.$$

If we put

$$q(z) = c_0 + \sum_{k=1}^{\infty} c_k z^k \quad (c_0 = 1 - \lambda + \alpha),$$

then by Lemma 2.3,

$$|c_k| \leq 2(1 - \lambda + \alpha) \quad (k \in \mathbb{N}).$$

Also

$$q(z)f(z) = -zf'(z) - \alpha z^2 f''(z) + (1 - \lambda + \alpha)f(z)$$

and so

$$\left(c_0 + \sum_{k=1}^{\infty} c_k z^k\right) \left(1 + \sum_{k=1}^{\infty} a_k z^k\right) = -\sum_{k=1}^{\infty} k a_k z^k - \alpha \sum_{k=1}^{\infty} k(k-1) a_k z^k + (1 - \lambda + \alpha) \left(1 + \sum_{k=1}^{\infty} a_k z^k\right).$$

Thus

$$c_0 a_1 + c_1 = -a_1 + (1 - \lambda + \alpha) a_1$$

and

$$c_{k+1} + c_0 a_{k+1} + \sum_{i=1}^k a_i c_{k+1-i} = -(k+1)a_{k+1} - \alpha k(k+1)a_{k+1} + (1 - \lambda + \alpha)a_{k+1} \quad (k \in \mathbb{N}).$$

Therefore

$$|a_1| \leq 2(1 - \lambda + \alpha)$$

and

$$|a_{k+1}| \leq \frac{2(1 - \lambda + \alpha)}{k + 1 + \alpha k(k + 1)} \left(1 + \sum_{i=1}^k a_i\right) \quad (k \in \mathbb{N}).$$

Next, we define the sequence $\{B_k\}$ as follows:

$$B_1 = 2(1 - \lambda + \alpha) \quad \text{and} \quad B_{k+1} = \frac{2(1 - \lambda + \alpha)}{k + 1 + \alpha k(k + 1)} \left(1 + \sum_{i=1}^k B_i\right) \quad (k \in \mathbb{N}).$$

Hence, by the principle of mathematical induction, we easily have

$$|a_k| \leq B_k \quad (k \in \mathbb{N}).$$

By using Lemma 3.3, the conditions (3.5) and (3.6) are hold and this completes the proof. \square

By using above theorem and Proposition 3.4 we can conclude the following corollaries.

Corollary 3.6. *Let $0 \leq \alpha < 2$ and $0 \leq \lambda < 1$. If $f \in \Lambda(\alpha, \lambda)$, then*

$$|a_k| \leq \frac{k+1}{3}(1 + B_1) \quad (k \geq 2)$$

where $B_1 = 2(1 - \lambda + \alpha)$.

Corollary 3.7. *Let $0 \leq \lambda < 1$ and $f \in \Lambda^*(\lambda)$. Then*

$$|a_1| \leq 2(1 - \lambda) \quad \text{and} \quad |a_k| \leq \frac{k+1}{3}(3 - 2\lambda) \quad (k \geq 2).$$

Proof . Let $\alpha = 0$ and apply Corollary 3.6. \square

4. Neighborhoods

We can see the earlier works (based upon the familiar concept of neighborhood of analytic functions) by Goodman [10] and Ruscheweyh [19], and (more recently) by Altıntaş et al. [3], Cataş [4], Cho et al. [5], Liu and Srivastava [15], Frasin [9], Keerthi et al. [13], Srivastava et al. [21] and Wang et al. [24]. Let $0 \leq \alpha < 2$ and $0 \leq \lambda < 1$ and $f \in \mathcal{A}$ of the form (2.1). For $\delta > 0$, we denote the δ -neighborhood of f by the notation $\mathcal{N}(f, \delta)$ and the following definition:

$$\mathcal{N}(f, \delta) = \left\{ g \in \mathcal{A} : g(z) = 1 + \sum_{k=1}^{\infty} d_k z^k \text{ and } \sum_{k=1}^{\infty} \frac{\alpha k^2 - \alpha k + k - 1 + \lambda - \alpha}{\eta} |d_k - a_k| < \delta \right\} \tag{4.1}$$

where

$$\eta = \begin{cases} 1, & \lambda - \alpha < 0; \\ 1 - \lambda + \alpha, & \lambda - \alpha \geq 0. \end{cases} \tag{4.2}$$

By above definition, we now prove the following useful theorem.

Theorem 4.1. *Let $0 \leq \alpha < 2$ and $0 \leq \lambda < 1$. If $f \in \mathcal{A}$ is given by (2.1) and satisfies the following condition*

$$\frac{f + \epsilon}{1 + \epsilon} \in \Lambda(\alpha, \lambda) \quad (\epsilon \in \mathcal{C} : |\epsilon| < \delta; \delta > 0)$$

then

$$\mathcal{N}(f, \delta) \subset \Lambda(\alpha, \lambda).$$

Proof . Suppose that

$$g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k \in \Lambda(\alpha, \lambda).$$

Hence

$$\Re \left(\frac{zg'(z)}{g(z)} + \alpha \frac{z^2 g''(z)}{g(z)} - 1 + \lambda - \alpha \right) < 0. \tag{4.3}$$

The condition (4.3) can be written as

$$\left| \frac{\frac{zg'(z)}{g(z)} + \alpha \frac{z^2 g''(z)}{g(z)} - 1 + \lambda - \alpha + 1}{\frac{zg'(z)}{g(z)} + \alpha \frac{z^2 g''(z)}{g(z)} - 1 + \lambda - \alpha - 1} \right| < 1 \quad (z \in \mathbb{U}),$$

which is equivalent to

$$\left| \frac{zg'(z) + \alpha z^2 g''(z) + (\lambda - \alpha)g(z)}{zg'(z) + \alpha z^2 g''(z) + (\lambda - \alpha - 2)g(z)} \right| < 1 \quad (z \in \mathbb{U}). \tag{4.4}$$

We easily find from (4.4) that $g \in \Lambda(\alpha, \lambda)$ if and only if

$$\frac{zg'(z) + \alpha z^2 g''(z) + (\lambda - \alpha)g(z)}{zg'(z) + \alpha z^2 g''(z) + (\lambda - \alpha - 2)g(z)} \neq \sigma, \quad (z \in \mathbb{U}, \sigma \in \mathcal{C}, |\sigma| = 1)$$

or equivalently

$$1 + \sum_{k=1}^{\infty} \frac{k + \alpha k(k - 1) + \lambda - \alpha - \sigma[k + \alpha k(k - 1) + \lambda - \alpha - 2]}{\lambda - \alpha - \sigma(\lambda - \alpha - 2)} b_k z^k \neq 0$$

which is equivalent to $(g * h)(z) \neq 0$ where

$$h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad c_k = \frac{k + \alpha k(k-1) + \lambda - \alpha - \sigma[k + \alpha k(k-1) + \lambda - \alpha - 2]}{\lambda - \alpha - \sigma(\lambda - \alpha - 2)}. \quad (4.5)$$

It follows from (4.5) that

$$\begin{aligned} |c_k| &= \left| \frac{k + \alpha k(k-1) + \lambda - \alpha - \sigma[k + \alpha k(k-1) + \lambda - \alpha - 2]}{\lambda - \alpha - \sigma(\lambda - \alpha - 2)} \right| \\ &\leq \frac{k + \alpha k(k-1) + \lambda - \alpha + |\sigma|[k + \alpha k(k-1) + \lambda - \alpha - 2]}{|\sigma|(\lambda - \alpha - 2) - |\lambda - \alpha|} \\ &= \frac{k + \alpha k(k-1) - 1 + \lambda - \alpha}{\eta} \quad (|\sigma| = 1). \end{aligned}$$

If $f \in \mathcal{A}$ and $g(z) = \frac{f + \epsilon}{1 + \epsilon} \in \Lambda(\alpha, \lambda)$, we deduce from $(g * h)(z) \neq 0$ that

$$(f * h)(z) \neq -\epsilon \quad (\epsilon \in \mathcal{C} : |\epsilon| < \delta; \delta > 0)$$

or equivalently,

$$|(f * h)(z)| \geq \delta \quad (z \in \mathbb{U}; \delta > 0). \quad (4.6)$$

Now suppose that

$$q(z) = 1 + \sum_{k=1}^{\infty} d_k z^k \in \mathcal{N}(f, \delta).$$

It follows from (4.1) that

$$|(q - f) * h(z)| = \left| \sum_{k=1}^{\infty} (d_k - a_k) c_k z^k \right| \leq |z| \sum_{k=1}^{\infty} \frac{k + \alpha k(k-1) - 1 + \lambda - \alpha}{\eta} |d_k - a_k| < \delta. \quad (4.7)$$

By combining (4.6) and (4.7), we can find that

$$|(q * h)(z)| = |([f + (q - f)] * h)(z)| \geq |(f * h)(z)| - |([q - f] * h)(z)| > 0,$$

which implies that

$$(q * h)(z) \neq 0 \quad (z \in \mathbb{U}).$$

Hence

$$q(z) \in \Lambda(\alpha, \lambda).$$

Therefore $\mathcal{N}(f, \delta) \subset \Lambda(\alpha, \lambda)$ and this completes the proof. \square

By taking $\alpha = 0$ we can conclude the following corollary.

Corollary 4.2. *Let $0 \leq \lambda < 1$. If $f \in \mathcal{A}$ is given by (2.1) satisfies the following condition*

$$\frac{f + \epsilon}{1 + \epsilon} \in \Lambda^*(\lambda) \quad (\epsilon \in \mathcal{C} : |\epsilon| < \delta; \delta > 0)$$

then

$$\mathcal{N}(f, \delta) \subset \Lambda^*(\lambda).$$

Theorem 4.3. Let $f \in \mathcal{A}$ be given by (2.1) and define the partial sums $f_n(z)$ of f by

$$f_n(z) = 1 + \sum_{k=1}^n a_k z^k \quad (n \in \mathbb{N}). \tag{4.8}$$

If

$$\sum_{k=1}^{\infty} \frac{k + \alpha k(k - 1) - \gamma}{\eta} |a_k| \leq 1, \tag{4.9}$$

where $\gamma = 1 - \lambda + \alpha$ and η is given by (4.2), then

1. $f \in \Lambda(\alpha, \lambda)$;
- 2.

$$\Re \left(\frac{f(z)}{f_n(z)} \right) \leq \frac{n + 1 + \alpha n(n + 1) - \gamma - \eta}{n + 1 + \alpha n(n + 1) - \gamma} \tag{4.10}$$

and

$$\Re \left(\frac{f_n(z)}{f(z)} \right) \leq \frac{n + 1 + \alpha n(n + 1) - \gamma}{n + 1 + \alpha n(n + 1) - \gamma + \eta} \tag{4.11}$$

Also the bounds in (4.10) and (4.11) are sharp.

Proof . First of all, we suppose that $f_0(z) = 1$. We know that

$$\frac{f_0 + \epsilon}{1 + \epsilon} = 1 \in \Lambda(\alpha, \lambda).$$

From (4.9), we easily find that

$$\sum_{k=1}^{\infty} \frac{k + \alpha k(k - 1) - \gamma}{\eta} |a_k - 0| \leq 1,$$

which implies that $f \in \mathcal{N}(f_0, 1) \subset \Lambda(\alpha, \lambda)$ (by virtue of Theorem 4.1).

Next, it is easy to see that

$$\frac{n + 1 + \alpha n(n + 1) - \gamma}{\eta} > \frac{n + \alpha n(n - 1) - \gamma}{\eta} > 1 \quad (n \in \mathbb{N}).$$

Therefore, we have

$$\sum_{k=1}^n |a_k| + \frac{n + 1 + \alpha n(n + 1) - \gamma}{\eta} \sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=1}^{\infty} \frac{k + \alpha k(k - 1) - \gamma}{\eta} |a_k| \leq 1 \tag{4.12}$$

We now suppose that

$$\begin{aligned} h(z) &= \frac{n + 1 + \alpha n(n + 1) - \gamma}{\eta} \left(\frac{f(z)}{f_n(z)} - \frac{n + 1 + \alpha n(n + 1) - \gamma - \eta}{n + 1 + \alpha n(n + 1) - \gamma} \right) \\ &= 1 + \frac{\frac{n + 1 + \alpha n(n + 1) - \gamma}{\eta} \sum_{k=n+1}^{\infty} a_k z^k}{1 + \sum_{k=1}^n a_k z^k} \end{aligned} \tag{4.13}$$

It follows from (4.12) and (4.13) that

$$\left| \frac{h(z) - 1}{h(z) + 1} \right| \leq \frac{\frac{n+1 + \alpha n(n+1) - \gamma}{\eta} \sum_{k=n+1}^{\infty} |a_k|}{2 - 2 \sum_{k=1}^n |a_k| - \frac{n+1 + \alpha n(n+1) - \gamma}{\eta} \sum_{k=n+1}^{\infty} |a_k|} \leq 1,$$

which implies that $\Re(h(z)) > 0$.

Therefore, we deduce that the assertion (4.10) holds true. Furthermore, if we put

$$f(z) = 1 - \frac{\eta}{n+1 + \alpha n(n+1) - \gamma} z^{n+1}, \quad (4.14)$$

then

$$\frac{f(z)}{f_n(z)} = 1 - \frac{\eta}{n+1 + \alpha n(n+1) - \gamma} z^{n+1} \rightarrow \frac{n+1 + \alpha n(n+1) - \gamma - \eta}{n+1 + \alpha n(n+1) - \gamma} (|z| \rightarrow 1^-),$$

which implies that the bound in (4.10) is the best possible for each $n \in \mathbb{N}$.

Similarly, we suppose that

$$h(z) = \frac{n+1 + \alpha n(n+1) - \gamma + \eta}{\eta} \left(\frac{f_n(z)}{f(z)} - \frac{n+1 + \alpha n(n+1) - \gamma}{n+1 + \alpha n(n+1) - \gamma + \eta} \right),$$

we readily get the assertion (4.10) of Theorem 4.3. The bound in (4.10) is sharp with the extremal function f given by (4.14). We thus complete the proof of Theorem. \square

The proof of the following theorem is similar to that of Theorem 4.3, we here choose to omit the analogous details.

Theorem 4.4. *Let $f \in \mathcal{A}$ be given by (2.1) and define the partial sums $f_n(z)$ of f by (4.8). If the conditions (4.9) hold, where $\gamma = 1 - \lambda + \alpha$ and η is given by (4.2), then*

$$\Re \left(\frac{f'(z)}{f'_n(z)} \right) \leq \frac{n+1 + \alpha n(n+1) - \gamma - (n+1)\eta}{n+1 + \alpha n(n+1) - \gamma} \quad (4.15)$$

and

$$\Re \left(\frac{f'_n(z)}{f'(z)} \right) \leq \frac{n+1 + \alpha n(n+1) - \gamma}{n+1 + \alpha n(n+1) - \gamma + (n+1)\eta}. \quad (4.16)$$

The bounds in (4.15) and (4.16) are sharp with the extremal function given by (4.14). Finally, we prove the following inclusion relationship for the function class $\Lambda(\alpha, \lambda)$.

Theorem 4.5. *If $0 \leq \alpha_2 < \alpha_1 < 2$ and $0 \leq \lambda_2 < \lambda_1 < 1$, then*

$$\Lambda(\alpha_1, \lambda_1) \subset \Lambda(\alpha_2, \lambda_2).$$

Proof . Let $f \in \Lambda(\alpha_1, \lambda_1)$. Then

$$\Re \left(\frac{zf'(z)}{f(z)} + \alpha_1 \frac{z^2 f''(z)}{f(z)} \right) < 1 - \lambda_1 + \alpha_1 < 1 - \lambda_2 + \alpha_1,$$

which shows that $f \in \Lambda(\alpha_1, \lambda_2)$, and subsequently, we see that $f \in \Lambda^*(\lambda_2)$, that is

$$\Re \left(\frac{zf'(z)}{f(z)} \right) < 1 - \lambda_2.$$

Now, by setting $\mu = \frac{\alpha_2}{\alpha_1}$, so that $0 < \mu < 1$. Therefore, we have

$$\begin{aligned} & \Re \left(\frac{zf'(z)}{f(z)} + \alpha_2 \frac{z^2 f''(z)}{f(z)} \right) - 1 + \lambda_2 - \alpha_2 = \\ & \mu \left[\Re \left(\frac{zf'(z)}{f(z)} + \alpha_1 \frac{z^2 f''(z)}{f(z)} \right) - 1 + \lambda_2 - \alpha_1 \right] + (1 - \mu) \left[\Re \left(\frac{zf'(z)}{f(z)} \right) + \lambda_2 - 1 \right] < 0, \end{aligned}$$

that is, $f \in \Lambda(\alpha_2, \lambda_2)$. \square

From Theorem 4.5 and the definition of the function class $\Lambda^+(\alpha_1, \lambda_1)$, we easily get the following inclusion relationship.

Corollary 4.6. *If $0 \leq \alpha_2 < \alpha_1 < 2$ and $0 \leq \lambda_2 < \lambda_1 < 1$, then*

$$\Lambda^+(\alpha_1, \lambda_1) \subset \Lambda^+(\alpha_2, \lambda_2) \subset \Lambda^*(\lambda_2).$$

By virtue of Lemma 3.2, we obtain the following result.

Corollary 4.7. *If $f \in \Lambda^+(\alpha, \lambda)$, then*

$$a_k \leq \frac{\gamma}{k + \alpha k(k-1) - \gamma} \quad (\gamma = 1 - \lambda + \alpha).$$

Each of these inequalities is sharp, with the extremal function given by

$$f_k(z) = 1 + \frac{\gamma}{k + \alpha k(k-1) - \gamma} z^k.$$

Acknowledgments

The authors would like to thank the referees for useful comments and suggestions.

References

- [1] R. Aghalary, A. Ebadian and M. Eshaghi Gordji, Subclasses of Meromorphic Starlike Functions Connected to Multiplier Family, *Australian Journal of Basic and Applied Sciences*, **3(4)**(2009) 4416-4421.
- [2] O. P. Ahuja, J. M. Jahangiri and H. Silverman, Subclasses of starlike functions related to a multiplier family, *Journal of natural geometry*, **15** (1999) 65-72.
- [3] O. Altinta, Neighborhoods of certain p-valently analytic functions with negative coefficients, *Appl. Math. Comput.* **187**(2007) 47-53.
- [4] A. C. Ata, Neighborhoods of a certain class of analytic functions with negative coefficients, *Banach J. Math. Anal.* **3** (2009) 111-121.
- [5] N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion relationships for certain subclasses of meromorphic functions associated with a family of multiplier transformations, *Integral Transforms Spec. Funct.* **16** (2005) 647-658.
- [6] J. Dziok, Classes of meromorphic functions associated with conic regions, *Acta Math. Sci.* **32** (2012) 765-774.

- [7] J. Dziok, Classes of multivalent analytic and meromorphic functions with two fixed points, *Fixed Point Theory Appl.* **2013**,**86** (2013).
- [8] R. Fournier and S. T. Ruscheweyh, Remarks on a multiplier conjecture for univalent functions, *Proc. Amer. Math. Soc.* **116** (1992) 35-43.
- [9] B. A. Frasin, Neighborhoods of certain multivalent functions with negative coefficients, *Appl. Math. Comput.***193**(2007) 1-6.
- [10] A. W. Goodman, Univalent functions and nonanalytic curves, *Proc. Am. Math. Soc.***8** (1957) 598-601.
- [11] A. W. Goodman, *Univalent Functions*, vol. 1. Polygonal Publishing House, Washington, New Jersey, 1983.
- [12] N. K. Jain and V. Ravichandran, Product and convolution of certain univalent functions, *Honam Math. J.* **38** , no. 4 (2016) 701–724.
- [13] B. S. Keerthi, A. Gangadharan and H. M. Srivastava, Neighborhoods of certain subclasses of analytic functions of complex order with negative coefficients, *Math. Comput. Model.***47** (2008) 271-277.
- [14] V. Kumar, N. E. Cho, V. Ravichandran and H. M. Srivastava, Sharp coefficient bounds for starlike functions associated with the Bell numbers, *Math Slovaca*,**2019** , 69(5) (2019) 1053–1064.
- [15] J. L. Liu and H. M. Srivastava, A linear operator and associated families of meromorphically multivalent functions *Functions*, *J. Math. Anal. Appl.***259** (2001) 566-581.
- [16] M. S. Liu, Y. C. Zhu and H. M. Srivastava, Properties and characteristics of certain subclasses of starlike functions of order β , *Math.Comput.Model.***48** (2008) 402-419.
- [17] M. Nunokawa and OP. Ahuja, On meromorphic starlike and convex functions, *Indian J. Pure Appl. Math.***32** (2001) 1027-1032.
- [18] M. S. Robertson, Quasi-subordinate function, *Mathematical essays dedicated to A. J. Macintyre*,(Ohio Univ. Press Athens, Ohio 1967, pp. 311-330
- [19] S. Ruscheweyh, Neighborhoods of univalent functions, *Proc. Am. Math. Soc.* **81** (1981) 521-527.
- [20] K. Sharma, N. E. Cho and V. Ravichandran, Sufficient conditions for strong starlikeness, *Bull. Iranian Math. Soc.* (2020), DOI:10.1007/s41980-020-00452-z.
- [21] H. M. Srivastava, S. S. Eker and B. Seker, Inclusion and neighborhood properties for certain classes of multivalently analytic functions of complex order associated with the convolution structure, *Appl. Math. Comput.***212** (2009) 66-71.
- [22] Z. G. Wang, Z. H. Liu and R. G. Xiang, Some criteria for meromorphic multivalent starlike functions, *Appl. Math. Comput.* **218** (2011) 1107-1111.
- [23] Z. G. Wang, H. M. Srivastava and S. M. Yuan, Some basic properties of certain subclasses of meromorphically starlike functions, *J. Inequal. Appl.***2014**. 2014:29 (2014).
- [24] Z. G. Wang, X. S. Yuan and L. Shi, Neighborhoods and partial sums of certain subclass of starlike functions, *J. Inequal. Appl.* **2013** (2013) 163.