On the efficient of adaptive methods to solve nonlinear equations

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Abstract

The main goal of this work, obtaining a family of Steffensen-type iterative methods adaptive with memory for solving nonlinear equations, which uses three self-accelerating parameters. For this aim, we present a new scheme to construct the self-accelerating parameters and obtain a family of Steffensen-type iterative methods with memory. The self-accelerating parameters have the properties of simple structure and easy calculation, which do not increase the computational cost of the iterative methods. The convergence order of the new iterative methods has increased from 4 to 8. Also, these methods possess very high computational efficiency. Another advantage of the new method is that they remove the severe condition \( f'(x) \) in a neighborhood of the required root imposed on Newton’s method. Numerical comparisons have made to show the performance of the proposed methods, as shown in the illustrative examples.

Keywords: Nonlinear equations, Newton’s interpolatory polynomial, Adaptive method with memory, The order of convergence, Self accelerating parameter.


1. Introduction

Solving nonlinear equations is a classical problem that has interesting applications in various branches of science and engineering. To solve nonlinear equations, iterative methods such as Newton’s method are usually used. Throughout this paper, we consider iterative methods to find a simple root \( \xi \), i.e., \( f(\xi) = 0 \) and \( f'(\xi) \neq 0 \), of a nonlinear equation \( f(x) = 0 \), where \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) for an open

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interval $I$. Newton’s method (NM) for the calculation of $\xi$ is probably the most widely used iterative scheme defined by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots.$$  \hspace{1cm} (1.1)

This well-known method is quadratically convergent to compute simple roots \[20\]. This method is not applicable when the derivative of any function has been defined. Therefore, Steffensen modified Newton’s method. He replaced the first derivative $f'(x_k)$ with the forward difference approximation.

$$f'(x_k) = \frac{f(x_k + \beta f(x_k)) - f(x_k)}{\beta f(x_k)}$$

and can obtain the famous Steffensen’s method \[41\]:

$$x_{k+1} = x_k - \frac{\beta f(x_k)^2}{f(x_k + \beta f(x_k)) - f(x_k)}, \quad k = 0, 1, 2, \ldots.$$  \hspace{1cm} (1.2)

where the parameter $\beta$ to be freely chosen in $\mathbb{R} - \{0\}$ and used to generate a class of Steffensen’s methods provided that the denominator is not equal to zero. Newton’s and Steffensen’s methods are of second-order convergence require two function evaluations per step, but in contrast to Newton’s method, Steffensen’s method is free from the derivative of function because sometimes the applications of the iterative methods which depend upon derivatives are restricted in engineering. These are two sample schemes of a one-point iteration, i.e., in each iteration step of the evaluations have taken at one point. Multiple-point methods evaluate at several points in each iteration step, and principle allows for a higher convergence order with a lower number of function evaluations. Kung and Traub \[19\] conjectured that multi-point optimal method without memory with $k$ evaluations could have a convergence order larger than $2^k - 1$. For well-known two-point without-memory methods, one can consult e.g., Jarrat \[16\], King \[17\], Ostrowski \[27\], and Maheshwari \[24\]. Soleymani et al. \[36\], \[39\] developed an optimal three-point iterative method with convergence order 8. Sharma-Arora \[20\] used weight functions to construct optimal three-point methods and optimal convergence order eight. Geum and Kim \[14\] and Sharifi et al. \[34\] utilizing parametric weight functions. The efficiency index sees \[27\] gives a measure of the balance between those quantities, according to the formula $p^{1/d}$, where $p$ is the order of convergence of the method and $d$ the number of functional evaluations per step. Some of the people who have worked on increasing the efficiency index of numerical methods for solving nonlinear equations after the Traub (which is leading in with memory methods) and high-efficiency indexing methods are Cordero et al. \[7\], \[8\], Dzunic et al. \[11\], \[12\], Lotfi et al. \[21\], \[22\], \[23\], Petkovic et al. \[28\], \[29\], Soleymani et al. \[38\], \[39\], Wang et al. \[46\], \[47\], \[48\]. This paper aims to state a two-point family adaptive with the memory of very high computational efficiency. We start from a family of two-point methods without memory with order 4, derived in \[22\], and increase the convergence order to 6, 7, 7.22, 7.53, 7.77 and 8 (depending on the accelerating technique) without additional calculations. In this manner, we have obtained new methods for finding simple roots of nonlinear equations. Computational efficiency is higher than the efficiency of existing methods known in literature in the class of two-point methods and even higher than the efficiency of optimal three-, four-, and five-point methods of optimal order. The main idea has based on the use of suitable two-valued functions and the variation of three free parameters in each iterative step. These parameters have calculated using information from the current and previous iteration so that the developed methods can regard as methods with memory following Traub’s classification \[45\]. An additional motivation for studying adaptive methods with memory arises from a surprising fact that such classes of the methods have been considered in literature very seldom despite their high computational efficiency. If one can increase the order of convergence in a without memory method by reusing the old information, he/she
develops it as a with-memory method. The adaptive with-memory methods reuse the information from all previous steps. Our motivated focus on this problem. Therefore, in this work, we have developed with-memory methods; i.e., that uses the information not only from the last two-steps but also from all previous iterations. The adaptive technique enables theoretically and practically us to achieve the highest efficiency. Indeed, we will develop the adaptive-method with-memory has efficiency index 2, hence, competes with all the existing methods without and with memory in the literature.

This paper has organized as follows: Section 2 deals with modifying the optimal two-points methods with memory introduced by Lotfi et al. [22]. In section 3, the aim of this work has been presented by contributing iterative-adaptive with memory method for solving nonlinear equations, improved order of convergence from 4 to 8 without adding more evaluations, has presented, and has achieved in the maximum performance index. It means that, without any new function calculations, we can improve convergence order by 100%. The numerical study presented in section 4 confirms the theoretical results and the excellent convergence properties of the presented methods in comparison with some optimal iterative methods, without memory and with memory. To show applicability, and competitive of the developed-methods some have tested nonlinear equations have solved.

2. Without memory methods

In this section, we will discuss the convergence analysis of without-memory methods that can build with memory methods in section 3. In 2015, Lotfi et al. proposed a two-step method as following: (LSGAM4) [22]

\[
\begin{aligned}
\begin{cases}
w_k = x_k + \gamma f(x_k), & B_k = f(x_k, w_k) / f(y_k, w_k), k \in \mathbb{N} \\
y_k = x_k - f(x_k) f(w_k) f(y_k) / f(y_k - x_k) = B_k (B_k - 1)^4, & x_k+1 = y_k - \lambda f(y_k - x_k) y_k (y_k - x_k) (B_k + (B_k - 1)^4),
\end{cases}
\end{aligned}
\]

(2.1)

where \(\gamma, \lambda\) and \(q\) are arbitrary nonzero real parameters, and \(f[x, y] = \frac{f(x) - f(y)}{x - y}\) stands for the divided difference of the first order. This method is an optimal-order without-memory method. In other words, it uses three function evaluations per iteration that it has optimal convergence order 4. The error equation of the method (2.1) is:

\[
e_{k+1} = \frac{(1 + \gamma f'(\alpha)(q - c_2)(-\lambda + f'(\alpha)(q - 2c_2)c_2 + c_3)}{f'(\alpha)} e^4_k + O(e^5_k).
\]

(2.2)

The next theorem states of the error equation of the method (2.1).

**Theorem 2.1.** Let \(I \subseteq \mathbb{R}\) be an open interval, \(f : I \rightarrow \mathbb{R}\) be a differentiable function, and has a simple zero, say \(\alpha\). If \(x_0\) is an initial guess to \(\alpha\), then the error equation of the method (2.1) is given by

\[
e_{k+1} = \frac{(1 + \gamma f'(\alpha)(q - c_2)(-\lambda + f'(\alpha)(q - 2c_2)c_2 + c_3)}{f'(\alpha)} e^4_k + O(e^5_k),
\]

(2.3)

**Proof.** Let \(e_k = x_k - \alpha, \tilde{e}_k = w_k - \alpha, \hat{e}_k = y_k - \alpha, \) and \(e_{k+1} = x_{k+1} - \alpha\). Denote \(c_k = \frac{f''(\alpha)}{k!f'(\alpha)}\) for \(k = 2, 3, \ldots\). Using Taylor expansion and taking into account \(f(\alpha) = 0\), we have:

\[
f(x_k) = f'(\alpha)(e_k + c_2 e^2_k + c_3 e^3_k + c_4 e^4_k + O(e^5_k)).
\]

(2.4)

Then, computing \(w_k = x_k + \gamma f(x_k)\), we attain

\[
\tilde{e}_k = e_k + e_k \gamma f'(\alpha)(1 + e_k(c_2 + e_k(c_3 + e_k c_4))) + O(e^5_k).
\]

(2.5)
Considering $f[x, y] = \frac{f(x) - f(y)}{x - y}$ is Newton’s first order divided difference. we get

\[
f[x_k, w_k] = -1/(e_k f'(\alpha)(1 + e_k(c_2 + e_k(c_3 + e_k c_4))))(1 + e_k(c_2 + e_k(c_3 + e_k c_4))))^{-1}(e_k f'(\alpha)(1 + e_k(c_2 + e_k(c_3 + e_k c_4))))
\]

By a simple calculation, we get:

\[
\frac{f(w_k)}{f[x_k, w_k]} = -((e_k f'(\alpha)^2)\gamma(1 + e_k(c_2 + e_k(c_3 + e_k c_4))))(e_k + e_k f'(\alpha)(1 + e_k(c_2 + e_k(c_3 + e_k c_4))))
\]

and

\[
\frac{f(x_k)}{f[x_k, w_k]} = -((e_k f'(\alpha)^2)\gamma(1 + e_k(c_2 + e_k(c_3 + e_k c_4))))(e_k + e_k f'(\alpha)(1 + e_k(c_2 + e_k(c_3 + e_k c_4))))
\]

By substituting (2.7) and (2.8) into (2.4), we obtain

\[
y_k = \alpha + (1 + \gamma f'(\alpha))(q - c_2) e_1^2 + ((2 + \gamma f'(\alpha)(2 + \gamma f'(\alpha))(q - c_2)(c_2 + (1 + \gamma f'(\alpha)))
\]

Using (2.4) and (2.9) we conclude that

\[
f[y_k, x_k] = f'(\alpha) + f'(\alpha)c_2e_k + f'(\alpha)(-1 + \gamma f'(\alpha)(q - c_2)(2 + c_3)e_1^2 + f'(\alpha)((2
\]

(2.10)
Using (2.5) and (2.9) we can get
\[
f[y_k, w_k] = f'(\alpha) + f'(\alpha)(1 + \gamma f'(\alpha)e_k(c_2 + f'(\alpha)(q - q\gamma f'(\alpha)c_2 + (1 + 2\gamma f'(\alpha)c_2^2 + (1 + \gamma f'(\alpha)^2c_2^2 + f'(\alpha)(q(2 + \gamma f'(\alpha)(2 + \gamma f'(\alpha)))c_2^2 - (2 + \gamma f'(\alpha)(2 + \gamma f'(\alpha))c_2^3 + y_1)q(5 + \gamma f'(\alpha)(7 + \gamma f'(\alpha)(4 + \gamma f'(\alpha))))c_2^4 + (4 + \gamma f'(\alpha)(5 + \gamma f'(\alpha)(3 + \gamma f'(\alpha))))c_2^4 + (q + \gamma f'(\alpha))^2c_3 - (8 + \gamma f'(\alpha)(11 + \gamma f'(\alpha)(7 + 3\gamma f'(\alpha)))c_3^2c_3 + (1 + \gamma f'(\alpha)(2 + \gamma f'(\alpha)(5 + \gamma f'(\alpha)))c_3^2 + (q + \gamma f'(\alpha)(13 + 5\gamma f'(\alpha)))c_4)\epsilon_k^4 + O(\epsilon_k^5).
\]
(2.11)

By dividing the relation (2.6) to (2.11), it follows
\[
B = \frac{f[x_k, w_k]}{f[y_k, w_k]} = 1 + c_2\epsilon_k + ((1 + \gamma f'(\alpha)(q - 2c_2)c_2 + (2 + \gamma f'(\alpha)c_3)c_2^2 + (-q(2 + \gamma f'(\alpha)(3 + \gamma f'(\alpha)))c_2^3 + (3 + \gamma f'(\alpha)(4 + 3 + \gamma f'(\alpha)c_2^3 + q(1 + \gamma f'(\alpha))^2c_3 - (6 + \gamma f'(\alpha)(9 + 4\gamma f'(\alpha)c_2^3c_3 + 3 + \gamma f'(\alpha)(3 + \gamma f'(\alpha))c_2^4 + (q + \gamma f'(\alpha)(2 + \gamma f'(\alpha)(1 + 3\gamma f'(\alpha))c_2^4 + (1 + \gamma f'(\alpha))(3 + \gamma f'(\alpha))(1 + \gamma f'(\alpha))c_2^4 - (q + \gamma f'(\alpha))^2c_3 - (4 + \gamma f'(\alpha)(9 + \gamma f'(\alpha)(4 + \gamma f'(\alpha)c_2^3)^2 + c_2^2((q + \gamma f'(\alpha))^2c_3 - (11 + \gamma f'(\alpha)(19 + 3\gamma f'(\alpha)(7 + 3\gamma f'(\alpha)))c_3 + (q + \gamma f'(\alpha))^3c_4 - c_2(3 + \gamma f'(\alpha)(7 + 3\gamma f'(\alpha)(9 + 4\gamma f'(\alpha)))c_3 + (8 + \gamma f'(\alpha)(15 + 4\gamma f'(\alpha)(3 + \gamma f'(\alpha))))c_4) + \epsilon_k^4 + O(\epsilon_k^5).
\]
(2.12)

By substituting (2.9), (2.5) and (2.10) into (2.1), we find
\[
\frac{f(y_k)}{f[x_k, y_k] + \lambda(y_k - x_k)(y_k - w_k)} = -(1 + \gamma f'(\alpha)(q - c_2)c_2 + (3 + \gamma f'(\alpha)(3 + \gamma f'(\alpha))(q - c_2)c_2 + (1 + \gamma f'(\alpha)(2 + \gamma f'(\alpha)c_3)c_2^2 + (f'(\alpha))^{-1}(q f'(\alpha)(2 + \gamma f'(\alpha)) + (4 + \gamma f'(\alpha)(3 + \gamma f'(\alpha))c_2^3 + f'(\alpha)(7 + \gamma f'(\alpha)(8 + \gamma f'(\alpha)(4 + \gamma f'(\alpha))))c_2^4 + q f'(\alpha)(5 + \gamma f'(\alpha)(8 + \gamma f'(\alpha)(5 + \gamma f'(\alpha)))c_3c_2(-\lambda(1 + \gamma f'(\alpha))^2 - 2 f'(\alpha)(5 + \gamma f'(\alpha))(7 + \gamma f'(\alpha)(4 + \gamma f'(\alpha)))c_3 + (1 + \gamma f'(\alpha))(q\lambda + (1 + \gamma f'(\alpha)) + f'(\alpha)(3 + \gamma f'(\alpha))(3 + \gamma f'(\alpha)))c_4) + \epsilon_k^4 + O(\epsilon_k^5).
\]
(2.13)

Substituting equations (2.4)-(2.13) into equation (2.14), we obtain
\[
e_{k+1} = \frac{(1 + \gamma f'(\alpha))^2(q - c_2)(-\lambda + f'(\alpha)(q - 2c_2)c_2 + c_3)\epsilon_k^4 + O(\epsilon_k^5)}{f'(\alpha)}.
\]
(2.14)

This reveals that the proposed scheme (2.1) reaches fourth-order convergence. □
3. Family of two-point methods with memory

By considering (2.3) it is clear that there are some possibilities to vanish the coefficient of $e_k^2$. For example, if $(1 + \gamma f'(\alpha)) = 0$, $(q - c_2) = 0$, or $(-\lambda + f'(\alpha)((q - 2c_2)c_2 + c_3)) = 0$, then the coefficient of $e_k^3$ vanishes at once. We propose Steffensen-type methods with memory as follows:

1. The new with-memory methods order 6.(TEM6)

If $(1 + \gamma f'(\alpha)) = 0$, it can be seen that this relation leads to $\gamma = -\frac{1}{f'(\alpha)}$, since $\alpha$ is unknown, it is impossible to compute $f'(\alpha)$. If we assume that $\alpha$ is known, computing $f'(\alpha)$ has been not suggested since it increases these function evaluations. Fortunately, during the iterative process (2.1), finder approximations to $\alpha$ are generated by the sequence $x_k$, and therefore we try to obtain a good approximate for $f'(\alpha)$. Each iteration, $x_k$, $w_k$, $y_k$, and $x_{k+1}$, are accessible, except at the initial step. Hence, we can interpolate $f'(\alpha)$ using these nodes. Now, we consider Newton interpolating polynomial as follows:

$$
\begin{align*}
N_3^0(x_k) &= \left[\frac{d}{dt}N_3(t; x_{k-1}, w_{k-1}, y_{k-1}, x_k)\right]_{t=x_k} = \left[\frac{d}{dt}(f(x_k) + f[x_k, y_{k-1}]
\right.
\left.\cdot(t - x_k) + f[x_k, y_{k-1}, w_{k-1}](t - x_k)(t - y_{k-1}) + f[x_k, y_{k-1}, w_{k-1}, x_k]
\right.
\left.\cdot(t - x_k)(t - y_{k-1})(t - w_{k-1})\right)_{t=x_k} = f[x_k, y_{k-1}] + f[x_k, y_{k-1}, w_{k-1}]
\right.
\left.\cdot(x_k - y_{k-1}) + f[x_k, y_{k-1}, w_{k-1}, x_k - y_{k-1}](x_k - y_{k-1})(x_k - w_{k-1}).
\end{align*}
$$

If $\gamma = -\frac{1}{f'(\alpha)} = \frac{-1}{N_3^0(x_k)}$, we obtain the following with-memory method order 6:

$$
\begin{align*}
\gamma_k &= -\frac{1}{N_3^0(x_k)}, \quad k \in \mathbb{N}, \\
w_k &= x_k + \gamma_k f(x_k), \quad k \in \mathbb{N}, \\
y_k &= x_k - \frac{f(x_k)}{f[y_k, w_k]}(1 + q \frac{f(w_k)}{f[y_k, w_k]}), \quad B_k = \frac{f[x_k, w_k]}{f[y_k, w_k]}, \\
x_{k+1} &= y_k - \frac{f(y_k)}{f[y_k, w_k] + \lambda(y_k - x_k)(y_k - w_k)}(B_k + (B_k - 1)^4).
\end{align*}
$$

2. The new with-memory methods order 7.(TEM7)

If $(1 + \gamma f'(\alpha)) = 0$ and $(q - c_2) = 0$ it can be seen that these relations lead to $\gamma = -\frac{1}{f'(\alpha)} = \frac{-1}{N_3^0(x_k)}$ and $q = c_2 = \frac{f''(\alpha)}{2f'(\alpha)}$. Each iteration, $x_k$, $w_k$, $y_k$, $x_{k+1}$ and $w_{k+1}$, are accessible, except at the initial step. Hence, we can interpolate $f'(\alpha)$ using these nodes, and as a result, we consider Newton interpolating polynomial as follows:

$$
\begin{align*}
N_4^0(w_k) &= \left[\frac{d}{dt}N_4(t; x_{k-1}, w_{k-1}, y_{k-1}, x_k, w_k)\right]_{t=w_k} = \left[\frac{d}{dt}(f(w_k) + f[w_k, x_k]
\right.
\left.\cdot(t - w_k) + f[w_k, x_k, y_{k-1}](t - w_k)(t - x_k) + f[w_k, x_k, y_{k-1}, w_{k-1}](t - w_k)
\right.
\left.\cdot(t - x_k)(t - y_{k-1}) + f[w_k, x_k, y_{k-1}, w_{k-1}, x_k - y_{k-1}](t - w_k)(t - x_k)(t - y_{k-1})
\right.
\left.\cdot(t - w_{k-1})(t - w_k - x_k)(w_k - x_k)(w_k - y_{k-1}) + f[w_k, x_k, y_{k-1}, w_{k-1}, x_k - y_{k-1}]
\right.
\left.\cdot(w_k - x_k)(w_k - y_{k-1})(w_k - w_{k-1}).
\end{align*}
$$

Now if $\gamma = -\frac{1}{f'(\alpha)} = \frac{-1}{N_3^0(x_k)}$ and $q = c_2 = \frac{f''(\alpha)}{2f'(\alpha)} = \frac{N_4^0(w_k)}{2N_3^0(w_k)}$ then, we have a new method with memory as follows order 7:

$$
\begin{align*}
\gamma_k &= -\frac{1}{N_3^0(x_k)}, \quad q_k = \frac{N_4^0(w_k)}{2N_3^0(w_k)}, \quad k \in \mathbb{N}, \\
w_k &= x_k + \gamma_k f(x_k), \quad k \in \mathbb{N}, \\
y_k &= x_k - \frac{f(x_k)}{f[y_k, w_k]}(1 + q_k \frac{f(w_k)}{f[y_k, w_k]}), \quad B_k = \frac{f[x_k, w_k]}{f[y_k, w_k]}, \\
x_{k+1} &= y_k - \frac{f(y_k)}{f[y_k, w_k] + \lambda(y_k - x_k)(y_k - w_k)}(B_k + (B_k - 1)^4).
\end{align*}
$$
3. The new with-memory methods order 7.23 (LSGAM7.2)
If \((1 + \gamma f'(\alpha)) = 0, (q - c_2) = 0\) and \((-\lambda + f'(\alpha)((q - 2c_2)c_2 + c_3)) = 0\) it can be seen that these relations lead to \(\gamma = -\frac{1}{f'2(x_0)}, q = c_2 = \frac{f'(\alpha)}{f'2(x_0)} = \frac{N^2_y(w_k)}{2N^4_y(w_k)}\) and \(\lambda = f'(\alpha)((q - 2c_2)c_2 + c_3) = \frac{f'^2(\alpha)}{6} + \frac{f''(\alpha)}{6} = \frac{N^2_y(w_k)}{4N^4_y(w_k)} + \frac{N^4_y(w_k)}{6}\), then, we earn the new-family with-memory method following order 7.23:

\[
\begin{aligned}
\gamma_k &= -\frac{1}{N'_y(x_k)}, q_k = \frac{N^2_y(w_k)}{2N^4_y(w_k)}, \lambda_k = \frac{N^2_y(w_k)}{4N^4_y(w_k)} + \frac{N^4_y(w_k)}{6}, k \in \mathbb{N}, \\
w_k &= x_k + \gamma_k f(x_k), y_k = x_k - \frac{f(x_k)}{f'(x_k)}(1 + q_k f(w_k)), k \in \mathbb{W}, \\
B_k &= \frac{f(x_k)}{f'(x_k)}, x_{k+1} = y_k - \frac{f(x_k) + \lambda_k(y_k - x_k)(y_k - w_k)}{f'(x_k)}(B_k + (B_k - 1)^4). 
\end{aligned}
\]

4. The new with-memory methods order 7.53 (LSGAM7.5)
If \((1 + \gamma f'(\alpha)) = 0, (q - c_2) = 0\) and \((-\lambda + f'(\alpha)((q - 2c_2)c_2 + c_3)) = 0\) it can be seen that these relations lead to \(\gamma = -\frac{1}{f'2(x_0)} = \frac{1}{N'_y(x_k)}\) and \(q = c_2 = \frac{f''(\alpha)}{f'2(\alpha)} = \frac{N^2_y(w_k)}{2N^4_y(w_k)}\). Each iteration, \(x_k, w_k, y_k, x_{k+1}, w_{k+1}\) and \(y_{k+1}\) are accessible, except at the initial step. Hence, we can interpolate \(f'(\alpha)\) using these nodes, and as a result, we consider Newton interpolating polynomial as follows:

\[
\begin{aligned}
N'_y(y_k) &= \frac{d}{dx} N_5(t; x_{k-1}, w_{k-1}, y_{k-1}, x_k, w_k, y_k) \bigg|_{t=y_k} = \frac{d}{dx}(f(y_k) + f(y_k, w_k) \\
(t - y_k) + f(y_k, w_k, x_k) (t - y_k)(t - w_k) + f(y_k, w_k, x_k, y_{k-1})(t - y_k) \\
(t - y_k)(t - w_k)(t - x_k) + f(y_k, w_k, x_k, y_{k-1}, w_{k-1})(t - y_k)(t - w_k) \\
(t - y_k)(t - x_k) + f(y_k, w_k, x_k, y_{k-1}, w_{k-1})(t - y_k)(t - w_k) \\
(t - y_k)(t - w_k)(t - x_k) + f(y_k, w_k, x_k, y_{k-1}, w_{k-1})(t - y_k)(t - w_k). 
\end{aligned}
\]

Now if \(\gamma = -\frac{1}{f'(\alpha)} = \frac{1}{N'_y(x_k)}\), \(q = c_2 = \frac{f''(\alpha)}{f'2(\alpha)} = \frac{N^2_y(w_k)}{2N^4_y(w_k)}\) and \(\lambda = f'(\alpha)((q - 2c_2)c_2 + c_3) = \frac{f'^2(\alpha)}{6} + \frac{f''(\alpha)}{6} = \frac{N^2_y(w_k)}{4N^4_y(w_k)} + \frac{N^4_y(w_k)}{6}\), then, we earn the new-family with-memory method following order 7.53:

\[
\begin{aligned}
\gamma_k &= -\frac{1}{N'_y(x_k)}, q_k = \frac{N^2_y(w_k)}{2N^4_y(w_k)}, \lambda_k = \frac{N^2_y(w_k)}{4N^4_y(w_k)} + \frac{N^4_y(w_k)}{6}, k \in \mathbb{N}, \\
w_k &= x_k + \gamma_k f(x_k), y_k = x_k - \frac{f(x_k)}{f'(x_k)}(1 + q_k f(w_k)), k \in \mathbb{W}, \\
B_k &= \frac{f(x_k)}{f'(x_k)}, x_{k+1} = y_k - \frac{f(x_k) + \lambda_k(y_k - x_k)(y_k - w_k)}{f'(x_k)}(B_k + (B_k - 1)^4). 
\end{aligned}
\]

5. The new with-memory methods order 7.77 (LSGAM7.7)
If \(\gamma = -\frac{1}{N'_y(x_k)}\), \(q = \frac{N^2_y(w_k)}{2N^4_y(w_k)}\) and \(\lambda = \frac{N^2_y(w_k)}{4N^4_y(w_k)} + \frac{N^4_y(w_k)}{6}\), then we have a new method with memory following with order 7.77:

\[
\begin{aligned}
\gamma_k &= -\frac{1}{N'_y(x_k)}, q_k = \frac{N^2_y(w_k)}{2N^4_y(w_k)}, \lambda_k = \frac{N^2_y(w_k)}{4N^4_y(w_k)} + \frac{N^4_y(w_k)}{6}, k \in \mathbb{N}, \\
w_k &= x_k + \gamma_k f(x_k), y_k = x_k - \frac{f(x_k)}{f'(x_k)}(1 + q_k f(w_k)), k \in \mathbb{W}, \\
B_k &= \frac{f(x_k)}{f'(x_k)}, x_{k+1} = y_k - \frac{f(x_k) + \lambda_k(y_k - x_k)(y_k - w_k)}{f'(x_k)}(B_k + (B_k - 1)^4). 
\end{aligned}
\]

The proof of the order of convergence of the methods mentioned in relations 3.2, 3.4, 3.5, 3.7 and 3.8 is similar. The order of convergence has been obtained with memory methods 3.5, 3.7, and 3.8 in [22].
4. Family of adaptive methods with memory

To get the best result, we suggest that all these relations hold simultaneously. The following equations hold:

\[
\begin{align*}
\gamma_k &= \frac{-1}{f'(\alpha)}, \\
q_k &= \frac{f''(\alpha)}{2f'(\alpha)}, \\
\lambda_k &= \frac{f'''(\alpha)}{6} - \frac{f''(\alpha)}{4f'(\alpha)}. \\
\end{align*}
\]

(4.1)

Since \(\alpha\) is unknown, it is impossible to compute \(f'(\alpha), f''(\alpha),\) and \(f'''(\alpha)\). Even worse, if we assume that \(\alpha\) is known, computing \(f'(\alpha), f''(\alpha),\) and \(f'''(\alpha)\) are not suggested since it increases function evaluations and it spoils that optimality of the method (2.1). Following the same idea in the with memory methods, this issue can be resaved. However, we are going to do it more efficiently, say recursively adaptively. Let us describe it a little more. The accelerators are updated using the information from all previous iterations in such a way that the highest efficiency indices obtain. Hence

\[
\begin{align*}
\gamma_k &= \frac{-1}{f'(\alpha)} \approx -\frac{1}{N'_3(x_k)}, \\
q_k &= \frac{f''(\alpha)}{2f'(\alpha)} \approx \frac{N''_3(w_k)}{2N'_3(w_k)}, \\
\lambda_k &= \frac{f'''(\alpha)}{6} - \frac{f''(\alpha)}{4f'(\alpha)} \approx \frac{N''_3(y_k)}{4N'_3(y_k)} + \frac{N''_3(y_k)}{6}, \\
\end{align*}
\]

(4.2)

where \(N'_3(x_k), N''_3(w_k)\) and \(N''_3(y_k)\) are Newton’s interpolation polynomials for the nodes \(\{x_k, x_{k-1}, w_{k-1}, y_{k-1}\}, \{w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}\}\) and \(\{y_k, w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}\}\), respectively. To construct a recursive adaptive method with memory, we use the information not only in the current and its previous iterations but also in all the previous iterations, i.e., from the beginning to the current iteration. Thus, as iterations proceed, the degree of interpolation polynomials increase, and the best-updated approximations for computing the self-accelerator \(\gamma_k, q_k,\) and \(\lambda_k\) are obtained. Indeed, we have developed the following recursive adaptive method with memory. Let \(x_0, \gamma_0, q_0,\) and \(\lambda_0\) are given suitably. Then:

\[
\begin{align*}
\gamma_k &= \frac{-1}{N_3'(x_k)}, \quad q_k = \frac{N''_3(w_k)}{2N''_3(w_k)}, \quad \lambda_k = \frac{N''_3(y_k)}{4N'_3(y_k)} + \frac{N''_3(y_k)}{6}, \quad k \in \mathbb{N}, \\
w_k &= x_k + \gamma_k f(x_k), \quad y_k = x_k - \frac{f(x_k)}{f(x_k, w_k)} (1 + q_k f(w_k)), \quad k \in \mathbb{N}, \\
B_k &= \frac{f(x_k, w_k)}{f(y_k, w_k)}, \quad x_{k+1} = y_k - \frac{f(y_k)}{f(y_k, x_k) + \lambda_k (y_k - x_k)(y_k - w_k)} (B_k + (B_k - 1)^4). \\
\end{align*}
\]

(4.3)

In what follows, we discuss the general convergence analysis of the recursive adaptive method with memory (4.15). It should be noted that the convergence order varies as the iteration go ahead. First, we need the following lemma:

**Lemma 4.1.** If \(\gamma_k = \frac{-1}{N_3'(x_k)}, q_k = \frac{N''_3(w_k)}{2N''_3(w_k)},\) and \(\lambda_k = \frac{N''_3(y_k)}{4N'_3(y_k)} + \frac{N''_3(y_k)}{6},\) then

\[
(1 + \gamma_k f'(\alpha)) \sim \prod_{s=0}^{k-1} e_s e_s w e_s y, \\
(q_k - c_2) \sim \prod_{s=0}^{k-1} e_s e_s w e_s y, \\
(-\lambda_k + f'(\alpha)((q_k - 2c_2)c_2 + c_3)) \sim \prod_{s=0}^{k-1} e_s e_s w e_s y,
\]

where \(e_s = x_s - \alpha, e_s w = w_s - \alpha, e_s y = y_s - \alpha.\)
Theorem 4.2. Let $x_0$ be a suitable initial guess to the simple root $\alpha$ of $f(x) = 0$. Also, suppose the initial values $\gamma_0, q_0$, and $\lambda_0$ are chosen appropriately. Then the $R$-order of the recursive adaptive method with memory (4.3) can be obtained from the following system of nonlinear equations:

$$
\begin{align*}
\begin{cases}
    r^k r_1 - (1 + r_1 + r_2)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - r^k = 0, \\
    r^k r_2 - 2(1 + r_1 + r_2)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 2r^k = 0, \\
    r^{k+1} - 4(1 + r_1 + r_2)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 4r^k = 0,
\end{cases}
\end{align*}
$$

(4.7)

where $r, r_1$ and $r_2$ are the order of convergence of the sequences $\{x_k\}, \{w_k\}$, and $\{y_k\}$, respectively. Also, $k$ indicates the number of iterations.

Proof. Let $\{x_k\}, \{w_k\}$, and $\{y_k\}$, are convergent with orders $r$, $r_1$, and $r_2$, respectively. Then:

$$
\begin{align*}
\begin{cases}
    e_{k+1} \sim e_k' \sim e_{k-1}^r \sim \ldots \sim e_0^r, \\
    e_{k,w} \sim e_k' \sim e_{k-1}^r \sim \ldots \sim e_0^r, \\
    e_{k,y} \sim e_k' \sim e_{k-1}^r \sim \ldots \sim e_0^r,
\end{cases}
\end{align*}
$$

(4.8)

where $e_k = x_k - \alpha, e_{k,w} = w_k - \alpha$ and $e_{k,y} = y_k - \alpha$. Now, by using Lemma (4.1) and Equation (4.8), we obtain

$$
(1 + \gamma_k f'(\alpha)) \sim \prod_{s=0}^{k-1} e_s e_s w e_s = (e_0 e_{0,w} e_{0,y}) \ldots (e_{k-1} e_{k-1,w} e_{k-1,y})
\begin{align*}
&= (e_0 e_0^r e_0^r) (e_0 e_0^r e_0^r) \ldots (e_0^{k-1} e_0^{k-1} e_0^{k-1} e_0^{k-1}) \\
&= e_0 (1 + r_1 + r_2 + (1 + r_1 + r_2) r + \ldots + (1 + r_1 + r_2)^{r-1}) \\
&= e_0 (1 + r_1 + r_2)(1 + r + \ldots + r^{k-1}).
\end{align*}
$$

(4.9)

Similarly, we get:

$$
(q_k - c_2) \sim e_0 (1 + r_1 + r_2)(1 + r + \ldots + r^{k-1}),
$$

(4.10)

and

$$
(-\lambda_k + f'(\alpha)((q_k - 2c_2)c_2 + c_3)) \sim e_0 (1 + r_1 + r_2)(1 + r + \ldots + r^{k-1}).
$$

(4.11)

By considering the errors of $w, y, and x_{k+1}$ in Equation (4.8), and Equations (2.12)-(4.4), we conclude that

$$
e_{k,w} \sim (1 + \gamma_k f'(\alpha)) e_k \sim e_0 (1 + r_1 + r_2)(1 + r + \ldots + r^{k-1}) e_0^r,
$$

(4.12)

$$
e_{k,y} \sim - (1 + \gamma_k f'(\alpha))(q_k + c_2) e_k^2 \sim e_0 (1 + r_1 + r_2)(1 + r + \ldots + r^{k-1})^2 e_0^2 e_0^r,
$$

(4.13)

$$
e_{k+1} \sim (1 + \gamma_k f'(\alpha))^2(q_k - c_2)(-\lambda_k + f'(\alpha)((q_k - 2c_2)c_2 + c_3)) e_k^4
\begin{align*}
& \sim e_0 (1 + r_1 + r_2)(1 + r + \ldots + r^{k-1})^4 e_0^4 e_0^r.
\end{align*}
$$

(4.14)
Equating the powers of error exponents of $e_{k-1}$ in pairs of relations in Equations (4.8)-(4.12), (4.8)-(4.13), and (4.8)-(4.14), we have:

\[
\begin{align*}
& r^k r_1 - (1 + r_1 + r_2)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - r^k = 0, \quad k = 1, 2, \ldots, \\
& r^k r_2 - 2(1 + r_1 + r_2)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 2r^k = 0, \\
& r^{k+1} - 4(1 + r_1 + r_2)(1 + r + r^2 + r^3 + \ldots + r^{k-1}) - 4r^k = 0.
\end{align*}
\]

Remark 4.3. For, $k = 1$, we use the information from the current and the one previous steps. In this case, the order of convergence of the with-memory method can compute from the following system

\[
\begin{align*}
& rr_1 - (1 + r_1 + r_2) - r = 0, \\
& rr_2 - 2(1 + r_1 + r_2) - 2r = 0, \\
& r^2 - 4(1 + r_1 + r_2) - 4r = 0.
\end{align*}
\]

This case special gives the solutions result of Lotfi et al. [22]. This system of equations has the following solution:

\[
r_1 = \frac{1}{8}(7 + \sqrt{65}) \simeq 1.88, \quad r_2 = \frac{1}{4}(7 + \sqrt{65}) \simeq 3.76 \quad \text{and} \quad r = \frac{1}{2}(7 + \sqrt{65}) \simeq 7.53. \quad \text{The same order as in 3.7.}
\]

Remark 4.4. For, $k = 2$, we obtain the order of convergence as follows:

\[
r_1 \simeq 1.98612, \quad r_2 \simeq 3.97225, \quad \text{and} \quad r \simeq 7.94449. \quad \text{Also, for} \quad k = 3, \quad \text{the system of equations (4.3) has the solution:} \quad r_1 \simeq 1.99829, \quad r_2 \simeq 3.99657, \quad \text{and} \quad r \simeq 7.99315. \quad \text{(regarding TEM)}
\]

Remark 4.5. Likewise, for $k = 4$, we obtain the order of convergence: $r_1 \simeq 1.99979, r_2 \simeq 3.99957$ and $r \simeq 7.99915$ (regarding TEM8). In this case the efficiency index is $7.99915^{\frac{3}{2}} = 1.99993 \cong 2$, which shows that our developed method competes with all the existing with memory methods.

Remark 4.6. Can easily see that the improvement of the order of convergence from 4 to 8 (100% improvement) has attained without any additional functional evaluations. Therefore, the efficiency index of the proposed method (4.3) is $EI = 8^{\frac{1}{3}} = 2$.

5. Numerical results and comparisons

The errors $|x_k - \alpha|$ of approximations to the sought zeros, produced by the different methods at the first three iterations have given in Table 2 where $m(-n)$ stands for $m \times 10^{-n}$. Tables 1 – 3 also include, for each test function, the initial estimation values and the last value of the computational order of convergence $COC$ [15] and order convergence $p$ [50] computed by the expressions (if it is stable)

\[
COC = \frac{\log |f(x_n)/f(x_{n-1})|}{\log |f(x_{n-1})/f(x_{n-2})|} \approx p.
\]

The package Mathematica 10, with 2000 arbitrary precision arithmetic, has been used in our computations. The following test functions have used:

\[
\begin{align*}
& f_1(x) = e^{\frac{5x^2}{2}} + x^4 + x^3 + \frac{1}{1+x^2}, \quad \alpha = 1, \quad x_0 = 1.4, \\
& f_2(x) = x \log(1 + x \sin(x)) + e^{-1+x^2+x \cos(x)} \sin(\pi x), \quad \alpha = 0, \quad x_0 = 0.6, \\
& f_3(x) = e^{x^2-3x} \sin(x) + \log(x^2 + 1), \quad \alpha = 0, \quad x_0 = 0.35,
\end{align*}
\]
\[ f_4(x) = e^{-x^2+2} + e^{-1+x^2+x \cos(x) \sin(\pi x)} + 1, \alpha = 1.55031, x_0 = 1.3. \]

Here, we compare the performance of the proposed methods \((2.1), (3.1), (3.3), (3.4), (3.6), (3.7), (4.15)\) and Abbasbandy’s method order 3 (AM) \[1\], Artidiello et al.’s method order 8 (ACTVM) \[2\], Babajee et al.’s method order 8 (BCSTM) \[3\], Chun’s method order 4 (CM) \[4\], Chun-Neta’s method order 8 (CNM) \[5\], two-step with memory derivative-free Cordero et al. order 7 (CLTAM) \[6\], two-step with memory Cordero et al. order 6 (CLKTM) \[7\], two-step without memory Cordero et al. order 8 (CFGTM) \[8\], Choubey-Jaiswal’s method order 8 (CJM) \[9\], Dzunic’s method order 7 (DM) \[10\], Dzunic-Petkovic’s method order 3 (DPM) \[12\], two-step with derivative Kou’s et al. order 4 (KILW) \[13\], Kung-Traub’s order 4 and 8 (KTM) \[14\], two-step without memory Lee-Kim order 4 (LKM) \[20\], three-step with memory Lotfi-Assari order 15.5 (LAM15) \[21\], three-step with memory Lotfi et al. order 12 (LS-GAM) \[22\], three-step without memory Fardi et al. order 7 (FDGM) \[23\], Maheshwari’s method order 4(MM) \[24\], Neta-Scott’s method order 8(NSM) \[25\], Newton’s method (NM) \[26\], Salimi et al.’s method order 8 (SLSSM) \[27\], Sharma-Arora’s method order 8 (SAM) \[28\], Sharifi et al.’s methods by order 16 \[29\], order 8 \[30\] and order 12 \[31\], Soleymani’s method order 10 (SM) \[34\], three-step with memory Soleymani et al. order 12 (SLTKM) \[35\], three-step without memory Soleymani et al. order 8 (SSMM) \[36\], two-step without memory Soleymani et al. order 4 (SJKKM) \[39\], Steffensen’s method with order 2 (SM) \[41\], Thukral-Petkovic’s method order 8 (TPM) \[42\], one-step adaptive method Torkashvand et al. \[43\], Wang’s method order 4.23 (WM) \[46\], two-step with memory Wang et al. order 7.53 (WZQM) \[48\], three-step with memory Wang et al. order 10, 11, 11.66 and 12 (WDZM) \[49\], two-step without memory Zheng et al. order 4 (ZLHM4) \[50\] and 8 (ZLHM8) \[51\], two-step with memory Zheng et al. order 4.2361 (ZZLM) and 4.74483 (ZZLM) \[52\]. The results of comparison of the test functions are summarized in Tables 1-3. From Tables 1 and 3, we observe that the new scheme is superior than some existing methods.

### 6. Conclusion

In this work, the general general Steffensen-like with-memory adaptive-family method proposed for solving nonlinear equations. To this end, Newton’s interpolatory polynomial with different degrees has applied. The numerical results show that the proposed method is more useful to find an acceptable approximation of the exact solution of nonlinear equations, especially when the function is non-differentiable. In Tables 1 – 3, we have examined some methods with different kinds of convergence order. Table 1 compares iterative methods with and without memory and the proposed method on functions \(f_1(t), f_2(t), f_3(t), f_4(t)\). It has observed that these methods support their theoretical aspects. The fourth column of Table 1 shows the computational order of convergence by \(COC\) and the last column of the tables show the efficiency index defined by \(EI = COC^{1/n}\), which is asymptotically 2. In other words, the proposed adaptive methods with memory \((4.3)\) show behavior as optimal \(n\)-point methods without memory. Therefore, we have developed an adaptive-family with-memory method that has efficiency index 2. Moreover, the developed methods \((4.3)\) do not need any derivatives and can be used even for non-smooth functions. Table 3 shows the convergence rate of adaptive-methods in comparison with the corresponding with-memory methods. This table results from the fact that we have reached the maximum improvement of the order of 100%. Figure 1 shows a comparison of without-memory methods, with memory and adaptive \((\%25, \%50, \%75, and \%100\) of improvements) in terms of the highest possible convergence order. Figure 2 shows a comparison of methods without memory, with memory and adaptive \((\%25, \%50, \%75, and \%100\) of improvements) in terms of the highest possible efficiency index. Studying basins of attraction of the methods can be considered for future researches. These methods are under development for the general case. In other words, the efficiency index of the proposed adaptive family with memory is...
$8^{\frac{1}{4}} = 2$, which is much better than the optimal one-,..., five-point optimal methods without memory having efficiency indexes $2^{1/2} \approx 1.41421$, $4^{1/3} \approx 1.58740$, $8^{1/4} \approx 1.68179$, $16^{1/5} \approx 1.74110$, $32^{1/6} \approx 1.78180$, respectively, also, $7.77^{1/3} \approx 1.98065$. Adaptive methods with memory have minimum evaluation function, not evaluation derivative, and most efficiency index, hence competes with existing methods with- and without memory.

Figure 1: Comparison of methods without memory, with memory and adaptive (%25, %50, and %75 of improvements) in terms of highest possible convergence order.

Figure 2: Comparison of methods without memory, with memory and adaptive (%25, %50, %75, and %100 of improvements) in terms of highest possible efficiency index.
Table 1: Comparison evaluation function and efficiency index of proposed method by with and without memory methods

<table>
<thead>
<tr>
<th>Method</th>
<th>without memory</th>
<th>with memory</th>
<th>EF</th>
<th>p</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSAGM</td>
<td>0.11558(0)</td>
<td>0.11181(1)</td>
<td>0.11558(0)</td>
<td>0.11181(1)</td>
<td>1.8587</td>
</tr>
<tr>
<td>TEMO</td>
<td>0.17222(0)</td>
<td>0.1180(1)</td>
<td>0.17222(0)</td>
<td>0.1180(1)</td>
<td>1.8572</td>
</tr>
<tr>
<td>TEMF</td>
<td>0.1722(0)</td>
<td>0.2435(1)</td>
<td>0.1722(0)</td>
<td>0.2435(1)</td>
<td>1.8572</td>
</tr>
<tr>
<td>LSGAM</td>
<td>0.1722(0)</td>
<td>0.6273(1)</td>
<td>0.1722(0)</td>
<td>0.6273(1)</td>
<td>1.8572</td>
</tr>
<tr>
<td>LSAGM</td>
<td>0.1722(0)</td>
<td>0.5929(1)</td>
<td>0.2853(2)</td>
<td>0.5929(1)</td>
<td>2.8629</td>
</tr>
<tr>
<td>TEMS</td>
<td>0.1722(0)</td>
<td>0.5929(1)</td>
<td>0.2853(2)</td>
<td>0.5929(1)</td>
<td>2.8629</td>
</tr>
<tr>
<td>LAM</td>
<td>0.1722(0)</td>
<td>0.5929(1)</td>
<td>0.2853(2)</td>
<td>0.5929(1)</td>
<td>2.8629</td>
</tr>
<tr>
<td>TEMS</td>
<td>0.1722(0)</td>
<td>0.2853(2)</td>
<td>1.8572</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Comparison of the absolute errors and COC of proposed methods

<table>
<thead>
<tr>
<th>Methods</th>
<th>$f(x) = e^x + x^2 + x$</th>
<th>$x = 0.5, 0.6$</th>
<th>$\alpha = 0.1, 0.5$</th>
<th>$p = 0.1, 0.5$</th>
<th>$q = 1, 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSAGM</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>1.8587</td>
</tr>
<tr>
<td>TEMO</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>1.8587</td>
</tr>
<tr>
<td>TEMF</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>1.8587</td>
</tr>
<tr>
<td>LSGAM</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>1.8587</td>
</tr>
<tr>
<td>LSAGM</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>1.8587</td>
</tr>
<tr>
<td>TEMS</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>1.8587</td>
</tr>
<tr>
<td>LAM</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>1.8587</td>
</tr>
<tr>
<td>TEMS</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>0.01693(1)</td>
<td>1.8587</td>
</tr>
</tbody>
</table>

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### Table 3: Comparison of the percentage improvement of the convergence rate

<table>
<thead>
<tr>
<th>with memory methods</th>
<th>number of steps</th>
<th>optimal order</th>
<th>$\eta$</th>
<th>percentage increase</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLKTM[7]</td>
<td>2</td>
<td>4.0000</td>
<td>6.0000</td>
<td>50</td>
</tr>
<tr>
<td>CLTAM[8]</td>
<td>2</td>
<td>4.0000</td>
<td>7.0000</td>
<td>75</td>
</tr>
<tr>
<td>DM[9]</td>
<td>2</td>
<td>4.0000</td>
<td>7.0000</td>
<td>75</td>
</tr>
<tr>
<td>DFM[10]</td>
<td>1</td>
<td>2.0000</td>
<td>4.0000</td>
<td>50</td>
</tr>
<tr>
<td>LA[11]</td>
<td>2</td>
<td>8.0000</td>
<td>15.0000</td>
<td>50.75</td>
</tr>
<tr>
<td>LSGAM[22]</td>
<td>2</td>
<td>4.0000</td>
<td>7.7700</td>
<td>94.25</td>
</tr>
<tr>
<td>LSNNK[23]</td>
<td>4</td>
<td>8.0000</td>
<td>12.0000</td>
<td>50</td>
</tr>
<tr>
<td>PDPM[28]</td>
<td>2</td>
<td>4.0000</td>
<td>6.0000</td>
<td>50</td>
</tr>
<tr>
<td>LSGAM[22]</td>
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<td>4.0000</td>
<td>7.2300</td>
<td>80</td>
</tr>
<tr>
<td>TEM6[2.14]</td>
<td>2</td>
<td>4.0000</td>
<td>6.0000</td>
<td>50</td>
</tr>
<tr>
<td>TEM7[3.2]</td>
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### References


