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Strong convergence of modified iterative algorithm for family of asymptotically nonexpansive mappings

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Abstract

In this paper we introduce new modified implicit and explicit algorithms and prove strong convergence of the two algorithms to a common fixed point of a family of uniformly asymptotically regular asymptotically nonexpansive mappings in a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Our result is applicable in $L_p(\ell_p)$ spaces, $1 < p < \infty$ and consequently in sobolev spaces.

Keywords: Fixed point; Banach space; Asymptotically nonexpansive mapping.
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1. Introduction

Let E be a real Banach space and E^* be the dual space of E. The normalised duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x^*\| = \|x\| \quad \forall\, x \in E\}, \tag{1.1}$$

where $\langle ., . \rangle$ denotes the pairing between the elements of E and those of E^*.

Let $S(E) := \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then space E is said to have *Gâteaux differentiable norm* if for any $x \in S(E)$ the limit

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} \tag{1.2}$$

exists $\forall y \in S(E)$. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S(E)$, the limit (1.2) is attained uniformly for $x \in S(E)$.

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A mapping $T : E \to E$ is said to be L-*Lipschitz* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \text{ for all } x, y \in E. \tag{1.3}$$

If (1.3) is satisfied with $L \in [0, 1)$, respectively $L = 1$, then the mapping T is called a *contraction,* respectively *nonexpansive.* A mapping $T : K \to K$ is called *asymptotically nonexpansive* if there exists a sequence $\rho_n \in [1, \infty)$, $\lim_{n\to\infty} \rho_n = 1$ such that for all $x, y \in K$

$$||T^n x - T^n y|| \leq \rho_n ||x - y|| \quad \text{for all} \quad n \in N. \tag{1.4}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [14] as an important generalization of the class of nonexpansive mappings. Goebel and Kirk [14] proved that if K is a nonempty, bounded, closed and convex subset of a real uniformly convex Banach space and T is a self asymptotically nonexpansive mapping of K, then T has a fixed point. T is said to be uniformly $L-Lipschitzian$ if there exists $L \geq 0$ such that

$$||T^n x - T^n y|| \leq L||x - y||, \ \forall x, y \in E. \tag{1.5}$$

A point $x \in K$ is called a *fixed point of* T provided $Tx = x$. We denote by $F(T)$ the set of all fixed point of T (i.e., $F(T) = \{x \in E : Tx = x\}$). T is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in K which converges weakly to $x^* \in K$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$. It is well known that if $T : K \to K$ is asymptotically nonexpansive, then T is uniformly $L-Lipschitzian$; $(I-T)$ is demiclosed at 0, and $F(T)$ is closed and convex (see for example [15, 22]). The mapping T is said to be asymptotically regular if

$$\lim_{n\to\infty} ||T^{n+1}x - T^n x|| = 0$$

for all $x \in K$. It is said to be uniformly asymptotically regular if for any bounded subset C of K,

$$\lim_{n\to\infty} \sup_{x\in C} ||T^{n+1}x - T^n x|| = 0.$$

Let C be a closed subset of a Hilbert space H and T be a self-nonexpansive mapping. The classical $Mann$ *iteration method* [20] is given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \ n \geq 1, \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence of real numbers in $[0, 1]$, has extensively been investigated in literature (see, e.g., [6, 28, 37] and references therein). If the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by (1.6) *converges weakly* to a fixed point of T (this is indeed true in a uniformly convex Banach space with Frechét differentiable norm [28]). Related works can also be found in [1, 2, 3, 4, 9, 11, 17, 23, 19, 24, 30, 33]. However, this convergence is in general not strong (see the counter example in [12], see also [13]). Attempts to modify the Mann iteration method (1.6) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [29] proposed the following modification of the Mann iteration method

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily} \\ y_n = \alpha_n x_n + (1 - \alpha_n)T x_n, \\ C_n = \{z \in C : ||y_n - z|| \leq ||x_n - z||\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), n \geq 0. \end{cases} \tag{1.7}$$

They proved that the sequence $\{x_n\}$ defined by (1.7) *converges strongly* to the fixed point of nonexpansive T.

It is worth mentioning that Scheme (1.7) involves computation of closed convex subsets C_n and Q_n of C for each $n \geq 1$ and hence is not easy to compute.

In [31], Schu introduced a Mann type process given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1, \tag{1.8}$$

to approximate fixed point of asymptotically nonexpansive self-mapping. He proved that, if C is a nonempty, closed and bounded and T is completely continuous asymptotically nonexpansive self-mapping with sequence $\{k_n\} \subset [1,\infty)$, for all $n \geq 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ then the sequence $\{x_n\}$ given by (1.9) converges strongly to some fixed point of T.

Rhoades [25] and Chidume et al. [8] extended the results of Schu [31] to uniformly convex Banach spaces which are more general than Hilbert spaces using a modified Ishikawa iteration method [18] under different settings. In [21], Osilike and Aniagbosor proved that the theorems of Schu and Rhoades remain true without the boundedness condition imposed on C, provided that $F(T) \neq \emptyset$.

Recently, Chidume et al. [10] proved that, if T is completely continuous and asymptotically nonexpansive mapping in the intermediate sense with a sequence $\{\nu_n\}$ such that $\sum \nu_n < \infty$ with $F(T) \neq \emptyset$, then, for arbitrary $x_0 \in C$, the sequence defined by:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1, \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in $[\epsilon, 1-\epsilon]$, for some $\epsilon > 0$, converges strongly to some fixed point of T. They also proved weak convergence of the scheme without the assumption that T is completely continuous.

But it is worth mentioning that in all the above results, either compactness assumption or complete continuity, is imposed on the map T or the convergence is weak. A natural question arises:
Question. Besides the concepts mentioned before, could one construct a new Mann iterative algorithm in order to get strong convergence?

In 2009, Yao et al. [35] introduced a new modified Mann iterative algorithm which is different from those in the literature for a nonexpansive mapping in a real Hilbert space. To be more precise, they proved the following theorem.

Theorem 1.1. Let C be a nonempty, closed and convex subset of a real Hilbert space. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0, 1)$. For $x_0 \in C$ given arbitrarily, let the sequence $\{x_n\}, n \geq 0$ be generated iteratively by

$$\begin{cases} v_n = P_C[(1-\alpha_n)x_n], \\ x_{n+1} = (1-\beta_n)v_n + \beta_n T v_n. \end{cases} \tag{1.10}$$

Suppose that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

then the sequence $\{x_n\}$ generated by (1.10) converges strongly to a fixed point of T.

Recently, Shehu and Ugwunnadi [36], extended the result of Yao et al. [35] to uniformly convex Banach space which is also uniformly smooth. Under some assumption on $\{\alpha_n\}, \{\beta_n\}$, they proved that the sequence $\{x_n\}$ generated by (1.10), under their assumption converges strongly to the unique some fixed point T.

It is our purpose in this paper to modified the algorithm (1.10) and prove strong convergence of both implicit and explicit of the modified algorithm to a common fixed point of a family of uniformly asymptotically regular asymptotically nonexpansive mappings in a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Our result is applicable in $L_p(\ell_p)$ spaces, $1 < p < \infty$ and consequently in sobolev spaces.

2. Preliminaries

Let K be a nonempty, closed, convex and bounded subset of a Banach space E and let the diameter of K be defined by $d(K) := \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := \sup\{\|x - y\| : y \in K\}$ and let $r(K) := \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The *normal structure coefficient* $N(E)$ of E (introduced in 1980 by Bynum [5], see also Lim [26] and the references contained therein) is defined by $N(E){:=}\inf\{\frac{d(K)}{r(K)}$: K is a closed convex and bounded subset of E with $d(K) > 0\}$. A space E such that $N(E) > 1$ is said to have *uniform normal structure* . It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see e.g.,[7, 27]).

The following lemmas are used for our main result.

Lemma 2.1. Let E be a real normed space. Then

$$||x + y||^2 \leq ||x||^2 + 2\langle y, j(x + y)\rangle,$$

for all $x, y \in E$ and for all $j(x + y) \in J(x + y)$.

Lemma 2.2. (Suzuki [32]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose that $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ for all integer $n \geq 1$ and $\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$. Then, $\lim_{n\to\infty} ||y_n - x_n|| = 0$.

Lemma 2.3. (Xu [34]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \; n \geq 0$$

where, (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$;; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

3. The main results

In the sequel we assume for the sequences $\{\beta_n\}, \{\sigma_{in}\} \subset (0, 1)$, that $\sum_{i\geq 1}\sigma_{in} := 1 - \beta_n$ for each $n \in \mathbb{N}$.

Theorem 3.1. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let $\{T_i\}_{i=1}^{\infty}$ be a family of uniformly asymptotically regular asymptotically nonexpansive self mappings of E with sequences $\{v_{in}\}$ such that $v_{in} \to 0$ as $n \to \infty$ for each $i \geq 1$ and $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\lim\limits_{n\to\infty} \frac{v_n}{\alpha_n} = 0$*, where* $v_n := \sup\limits_{i\geq 1}\{v_{in}\}$

(C2) $\Sigma_{n=0}^{\infty} \alpha_n = \infty$

(c4) $\beta_n \in [a,b]\ \forall n \geq 1$ *and for some* $a, b \in (0,1)$.

For some fixed $\delta \in (0,1)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by $x_0 \in C$ chosen arbitrarily,

$$\begin{cases} y_n = (1-\alpha_n)x_n \\ x_n = [1-\delta(1-\beta_n)]y_n + \delta \sum_{i\geq 1} \sigma_{in} T_i^n y_n, \quad n \geq 0. \end{cases} \tag{3.1}$$

Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F$.

Proof . First, we show that $\{x_n\}$ defined by (3.1) is well defined. For all $n \in \mathbb{N}$, let define the mapping

$$T_n^{\delta} x := [1-\delta(1-\beta_n)](1-\alpha_n)x + \delta \sum_{i\geq 1} \sigma_{in} T_i^n (1-\alpha_n)x.$$

Indeed, for all $x, y \in E$, we have

$$\begin{aligned} ||T_n^{\delta}x - T_n^{\delta}y|| &\leq [1-\delta(1-\beta_n)](1-\alpha_n)||x-y|| \\ &\quad +\delta \sum_{i\geq 1} \sigma_{in} ||T_i^n(1-\alpha_n)x - T_i^n(1-\alpha_n)y|| \\ &\leq [1-\delta(1-\beta_n)](1-\alpha_n)||x-y|| \\ &\quad +\delta(1-\beta_n)(1+v_n)(1-\alpha_n)||x-y|| \\ &\leq [1-\alpha_n + \delta(1-\beta_n)v_n]||x-y|| \\ &\leq \Big(1-\alpha_n[1-\delta(1-\beta_n)v_n/\alpha_n]\Big)||x-y||. \end{aligned}$$

Since, $\lim\limits_{n\to\infty} \delta(1-\beta_n)v_n/\alpha_n = 0$, then there exist $n_0 \in \mathbb{N}$ such that $\delta(1-\beta_n)v_n/\alpha_n < 1/2$ for all $n \geq n_0$. Therefore, for $n \geq n_0$. we have

$$1-\alpha_n[1-\delta(1-\beta_n)v_n/\alpha_n] < 1.$$

Hence,

$$||T_n^{\delta}x - T_n^{\delta}y|| \leq ||x-y||, \quad n \geq n_0.$$

Thus, $\{x_n\}$ defined by (3.1) is well defined. Therefore, by contraction mapping principle, there exists a unique fixed point $x_n \in E$ of T_n^f for each $n \geq 0$ such that (3.1) holds.

Let $p \in F$, then from (3.1), we obtain

$$\begin{aligned} ||x_n - p|| &\leq [1-\delta(1-\beta_n)]||y_n - p|| + \delta\sum_{i\geq 1}\sigma_{in}||T_i^n y_n - p|| \\ &\leq [1-\delta(1-\beta_n)]||y_n - p|| + \delta(1-\beta_n)(1+v_n)||y_n - p|| \\ &= [1+\delta(1-\beta_n)v_n]||y_n - p|| \\ &\leq [1+\delta(1-\beta_n)v_n]\Big((1-\alpha_n)||x_n - p|| + \alpha_n||p||\Big) \\ &= [1+\delta(1-\beta_n)v_n](1-\alpha_n)||x_n - p|| \\ &\quad +\alpha_n[1+\delta(1-\beta_n)v_n]||p|| \\ &= [1-\alpha_n+\delta(1-\alpha_n)(1-\beta_n)v_n]||x_n - p|| \\ &\quad +\alpha_n[1+\delta(1-\beta_n)v_n]||p|| \\ &\leq [1-\alpha_n+\delta(1-\beta_n)v_n]||x_n - p|| \\ &\quad +\alpha_n[1+\delta(1-\beta_n)v_n]||p|| \\ &= \Big(1-\alpha_n[1-\delta(1-\beta_n)v_n/\alpha_n]\Big)||x_n - p|| \\ &\quad +\alpha_n[1+\delta(1-\beta_n)v_n]||p||. \end{aligned}$$

Therefore

$$||x_n - p|| \leq \frac{[1+\delta(1-\beta_n)v_n]||p||}{1-\delta(1-\beta_n)v_n/\alpha_n}.$$

Since $\delta(1-\beta_n)v_n \to 0$ and $\delta(1-\beta_n)v_n/\alpha_n \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\delta(1-\beta_n)v_n < 1/2$ and $\delta(1-\beta_n)(v_n/\alpha_n) < 1/2$ respectively for all $n \geq n_0$.
Hence $||x_n - p|| \leq 3||p||$, for all $n \geq n_0$. Thus $\{x_n\}$ is bounded, which imply that $\{y_n\}$ is also bounded. From (3.1), we also obtain that

$$||y_n - x_n|| = \alpha_n||x_n|| \to 0 \quad \text{as} \quad n \to \infty \tag{3.2}$$

which implies

$$\sum_{i\geq 1}\sigma_{in}||T_i^n y_n - y_n|| = ||x_n - y_n|| \to 0 \quad \text{as} \quad n \to \infty$$

hence

$$||T_i^n y_n - y_n|| = ||x_n - y_n|| \to 0 \quad \text{as} \quad n \to \infty \tag{3.3}$$

for each $i \geq 1$. Therefore

$$\begin{aligned} ||T_i^n x_n - x_n|| &= ||T_i^n x_n - T_i^n y_n|| + ||T_i^n y_n - y_n|| + ||y_n - x_n|| \\ &\leq (2+v_n)||x_n - y_n|| + ||T_i^n y_n - y_n|| \end{aligned}$$

From (3.2) and (3.3), we obtain

$$||T_i^n x_n - x_n|| \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad i \geq 1. \tag{3.4}$$

For each $i \geq 1$, using the asymptotic regularity of T_i, we obtain

$$\begin{aligned} \lim_{n\to\infty}||T_i x_n - x_n|| &= \lim_{n\to\infty}||x_n - T_i^n x_n|| + \lim_{n\to\infty}||T_i^n x_n - T_i^{n+1}x_n|| \\ &\quad + \lim_{n\to\infty}||T_i^{n+1}x_n - T_i x_n|| \\ &\leq (1+L)\lim_{n\to\infty}||x_n - T_i^n x_n|| \\ &\quad + \lim_{n\to\infty}||T_i^{n+1}x_n - T_i^n x_n|| = 0 \end{aligned} \tag{3.5}$$

where $L = \sup_{i\geq 1} L_i$, hence

$$\begin{aligned}\lim_{n\to\infty} ||y_n - T_i y_n|| &\leq \lim_{n\to\infty} ||y_n - x_n|| + \lim_{n\to\infty} ||x_n - T_i x_n|| \\ &\quad + \lim_{n\to\infty} ||T_i x_n - T_i y_n|| \\ &\leq (1+L) \lim_{n\to\infty} ||x_n - y_n|| \\ &\quad + \lim_{n\to\infty} ||x_n - T_i x_n|| = 0 \end{aligned} \tag{3.6}$$

We next show that $x_n \to p$ (as $n \to \infty$). Indeed, define a map $\phi : E \to \mathbb{R}$ by

$$\phi(y) := \mu_n ||y_n - y||^2, \quad \forall y \in E.$$

Then, $\phi(y) \to \infty$ as $||y|| \to \infty$, ϕ is continuous and convex, so as E is reflexive, there exists $q \in E$ such that $\phi(q) = \min_{u\in E} \phi(u)$. Hence, the set

$$K^* := \{y \in E : \phi(y) = \min_{u\in E} \phi(u)\} \neq \emptyset.$$

Since $\lim_{n\to\infty} ||y_n - T_i y_n|| = 0$, $\lim_{n\to\infty} ||y_n - T_i^m y_n|| = 0$, for any $m \geq 1$ and each $i \geq 1$, by induction. Now let $v \in K^*$, we have

$$\begin{aligned}\lim_{n\to\infty} \phi(T_i v) &= \lim_{n\to\infty} \mu_n ||y_n - T_i v||^2 \\ &= \lim_{n\to\infty} \mu_n ||y_n - T_i y_n + T_i y_n - T_i v||^2 \\ &\leq \lim_{n\to\infty} \mu_n [(1+v_n)||y_n - v||]^2 = \lim_{n\to\infty} \phi(v),\end{aligned}$$

and hence $T_i v \in K^*$.
Now let $z \in F$, then $z = T_i z$. Since K^* is a closed convex set, there exists a unique $v^* \in K^*$ such that

$$||z - v^*|| = \min_{u\in K^*} ||z - u||.$$

But

$$\lim_{n\to\infty} ||z - T_i v^*|| = \lim_{n\to\infty} ||T_i z - T_i v^*|| \leq \lim_{n\to\infty} (1+v_n)||z - v^*||,$$

which implies $v^* = T_i v^*$ and so $K^* \cap F \neq \emptyset$.

Let $p \in K^* \cap F$ and $t \in (0,1)$, then it follows that $\phi(p) \leq \phi(p - tp)$ and using Lemma 2.1, we obtain that

$$||y_n - p + tp||^2 \leq ||y_n - p||^2 + 2t\langle p, j(y_n - p + tp)\rangle$$

which implies that

$$\mu_n \langle -p, j(y_n - p + tp)\rangle \leq 0.$$

Moreover

$$\begin{aligned}\mu_n \langle -p, j(y_n - p)\rangle &= \mu_n \langle -p, j(y_n - p) - j(y_n - p + tp)\rangle \\ &\quad + \mu_n \langle -p, j(y_n - p + tp)\rangle \\ &\leq \mu_n \langle -p, j(y_n - p) - j(y_n - p + tp)\rangle.\end{aligned}$$

Since j is norm-to-weak* uniformly continuous on bounded subsets of E, we have that

$$\mu_n\langle -p, j(y_n - p)\rangle \leq 0. \tag{3.7}$$

Since $\delta(1-\beta_n)v_n \to 0$ and $\delta(1-\beta_n)v_n/\alpha_n \to 0$ as $n \to \infty$, if we denote by w_n the value of $2v_n + v_n^2$, it implies that $\delta(1-\beta_n)w_n \to 0$ and $\delta(1-\beta_n)w_n/\alpha_n \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\delta(1-\beta_n)w_n < 1/2$ and $\delta(1-\beta_n)(w_n/\alpha_n) < 1/2$, for all $n \geq n_0$. From recursion formula (3.1), we obtain

$$\begin{aligned}
||x_n - p||^2 &= ||[1-\delta(1-\beta_n)](y_n - p) + \delta\sum_{i\geq 1}\sigma_{in}(T_i^n y_n - p)||^2\\
&\leq [1-\delta(1-\beta_n)]||y_n - p||^2 + \delta\sum_{i\geq 1}\sigma_{in}||T_i^n y_n - p||^2\\
&\leq [1-\delta(1-\beta_n)]||y_n - p||^2 + \delta(1-\beta_n)(1+v_n)^2||y_n - p||^2\\
&= [1-\delta(1-\beta_n) + \delta(1-\beta_n)(1+w_n)]||y_n - p||^2\\
&= [1+\delta(1-\beta_n)w_n]||(1-\alpha_n)(x_n - p) - \alpha_n p||^2\\
&\leq [1+\delta(1-\beta_n)w_n]\Big((1-\alpha_n)||x_n - p||^2 + 2\alpha_n\langle -p, j(y_n - p)\rangle\Big)\\
&= [1+\delta(1-\beta_n)w_n](1-\alpha_n)||x_n - p||^2\\
&\quad +2\alpha_n[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n - p)\rangle\\
&= [1-\alpha_n + \delta(1-\alpha_n)(1-\beta_n)w_n]\\
&\quad +2\alpha_n[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n - p)\rangle\\
&\leq \Big[1-\alpha_n\Big(1-\delta(1-\beta_n)w_n/\alpha_n\Big)\Big]\\
&\quad +2\alpha_n[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n - p)\rangle.
\end{aligned}$$

Therefore

$$||x_n - p||^2 \leq \frac{2[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n - p)\rangle}{(1-\delta(1-\beta_n)w_n/\alpha_n)}$$

hence

$$\mu_n||x_n - p||^2 \leq 3\mu_n\langle -p, j(y_n - p)\rangle. \tag{3.8}$$

Therefore, from (3.7) we obtain $\mu_n||x_n - p|| \leq 0$. Hence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ as $k \to \infty$. To complete the proof, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \to z$ as $j \to \infty$, from (3.8) we obtain

$$\mu_n||z - p||^2 \leq 0.$$

which implies that $z = p$ and hence $\{x_n\}$ converges strongly to $p \in F$ as $n \to \infty$. This complete the proof. $\square$

Theorem 3.2. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm, K a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{\infty}$ be a family of uniformly asymptotically regular asymptotically nonexpansive self mappings of E with sequences $\{v_{in}\}$ such that $v_{in} \to 0$ as $n \to \infty$ for each $i \geq 1$ and $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\lim_{n\to\infty}\frac{v_n}{\alpha_n} = 0$, *where* $v_n := \sup_{i\geq 1}\{v_{in}\}$ *and* $\sum_{n=1}^{\infty} v_n < \infty$

(C2) $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\sum_{n\geq 1}|\sigma_{i,n+1}-\sigma_{in}| = 0$

(C3) $\sum_{n=1}^{\infty}\beta_n < \infty$

For some fixed $\delta \in (0,1)$, *let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence defined iteratively by* $x_0 \in C$ *chosen arbitrarily,*

$$\begin{cases} y_n = (1-\alpha_n)x_n \\ x_{n+1} = [1-\delta(1-\beta_n)]y_n + \delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n, \quad n \geq 0. \end{cases} \tag{3.9}$$

Then, $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in F$.

Proof . Let $p \in F$ be arbitrary, we obtain from (3.9)

$$\begin{aligned} ||x_{n+1}-p|| &= ||[1-\delta(1-\beta_n)](y_n-p) + \delta\sum_{i\geq 1}\sigma_{in}(T_i^n y_n - p)|| \\ &\leq [1-\delta(1-\beta_n)]||y_n-p|| + \delta(1-\beta_n)(1+v_n)||y_n-p|| \\ &= [1-\delta(1-\beta_n)+\delta(1-\beta_n)(1+v_n)]||y_n-p|| \\ &= [1+\delta(1-\beta_n)v_n]||(1-\alpha_n)(x_n-p)-\alpha_n p|| \\ &\leq [1+\delta(1-\beta_n)v_n]\Big((1-\alpha_n)||x_n-p|| + \alpha_n||p||\Big) \\ &\leq [1+\delta(1-\beta_n)v_n]\max\{||x_n-p||, ||p||\} \\ &\ \ \vdots \\ &\leq \prod_{j=1}^{n}[1+\delta(1-\beta_j)v_j]\max\{||x_1-p||, ||p||\}. \end{aligned} \tag{3.10}$$

Since $\sum_{n=1}^{\infty}v_n < \infty$. it follows from (3.10) that $\{x_n\}$ is bounded. Hence $\{y_n\}$ is also bounded. Furthermore, it follows from (3.9) that

$$||y_n - x_n|| = \alpha_n||x_n|| \to 0 \quad \text{as} \quad n \to \infty \tag{3.11}$$

Define two sequences by $\gamma_n := (1-\delta)\beta_n + \delta$ and $z_n := \frac{x_{n+1}-x_n+\gamma_n x_n}{\gamma_n}$. From the recursion formula (3.9), we observe that

$$z_n = \frac{[1-\delta(1-\beta_n)](y_n-x_n) + \delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n + \beta_n x_n}{\gamma_n}$$

which implies

$$\begin{aligned} z_{n+1}-z_n &= \frac{[1-\delta(1-\beta_{n+1})](y_{n+1}-x_{n+1}) + \delta\sum_{i\geq 1}\sigma_{i,n+1}T_i^{n+1}y_{n+1} + \beta_{n+1}x_{n+1}}{\gamma_{n+1}} \\ &\quad -\frac{[1-\delta(1-\beta_n)](y_n-x_n) + \delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n + \beta_n x_n}{\gamma_n} \\ &= \frac{[1-\delta(1-\beta_{n+1})](y_{n+1}-x_{n+1})}{\gamma_{n+1}} - \frac{[1-\delta(1-\beta_n)](y_n-x_n)}{\gamma_n} \\ &\quad +\frac{\delta\sum_{i\geq 1}\sigma_{i,n+1}\Big(T_i^{n+1}y_{n+1} - T_i^{n+1}y_n\Big)}{\gamma_{n+1}} + \frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big(T_i^{n+1}y_n - T_i^n y_n\Big)}{\gamma_{n+1}} \\ &\quad +\Big(\frac{\delta\sum_{i\geq 1}\sigma_{i,n+1}T_i^n y_n}{\gamma_{n+1}} - \frac{\delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n}{\gamma_n}\Big) + \frac{\beta_{n+1}}{\gamma_{n+1}}x_{n+1} - \frac{\beta_n}{\gamma_n}x_n \end{aligned}$$

therefore

$$\begin{aligned}
||z_{n+1} - z_n|| &= \frac{[1-\delta(1-\beta_{n+1})]\Big\|y_{n+1} - x_{n+1}\Big\|}{\gamma_{n+1}} + \frac{[1-\delta(1-\beta_n)]\Big\|y_n - x_n\Big\|}{\gamma_n} \\
&+ \frac{\delta\sum_{i\geq 1}\sigma_{i,n+1}\Big\|T_i^{n+1}y_{n+1} - T_i^{n+1}y_n\Big\|}{\gamma_{n+1}} + \frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n - T_i^n y_n\Big\|}{\gamma_{n+1}} \\
&+ \Big\|\frac{\delta\sum_{i\geq 1}\sigma_{i,n+1}T_i^n y_n}{\gamma_{n+1}} - \frac{\delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n}{\gamma_n}\Big\| + \frac{\beta_{n+1}}{\gamma_{n+1}}||x_{n+1}|| + \frac{\beta_n}{\gamma_n}||x_n|| \\
&\leq \frac{[1-\delta(1-\beta_{n+1})]\Big\|y_{n+1} - x_{n+1}\Big\|}{\gamma_{n+1}} + \frac{[1-\delta(1-\beta_n)]\Big\|y_n - x_n\Big\|}{\gamma_n} \\
&+ \frac{\delta(1-\beta_n)(1+v_{n+1})\Big\|y_{n+1} - y_n\Big\|}{\gamma_{n+1}} + \frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n - T_i^n y_n\Big\|}{\gamma_{n+1}} \\
&+ \frac{\delta}{\gamma_{n+1}\gamma_n}\Big\|\sum_{i\geq 1}\Big(\gamma_n\sigma_{i,n+1} - \gamma_{n+1}\sigma_{in}\Big)T_i^n y_n\Big\| + \frac{\beta_{n+1}}{\gamma_{n+1}}||x_{n+1}|| + \frac{\beta_n}{\gamma_n}||x_n||.
\end{aligned} \tag{3.12}$$

But

$$\begin{aligned}
y_{n+1} - y_n &= (1-\alpha_{n+1})x_{n+1} - (1-\alpha_n)x_n \\
&= (1-\alpha_{n+1})(x_{n+1} - x_n) + (\alpha_{n+1} - \alpha_n)x_n
\end{aligned}$$

so that

$$||y_{n+1} - y_n|| = (1-\alpha_{n+1})||x_{n+1} - x_n|| + |\alpha_{n+1} - \alpha_n|||x_n||. \tag{3.13}$$

From (3.13) and (3.12), we obtain

$$\begin{aligned}
&||z_{n+1} - z_n|| - ||x_{n+1} - x_n|| \\
&\quad\leq \frac{[1-\delta(1-\beta_{n+1})]\Big\|y_{n+1} - x_{n+1}\Big\|}{\gamma_{n+1}} + \frac{[1-\delta(1-\beta_n)]\Big\|y_n - x_n\Big\|}{\gamma_n} \\
&\quad+ \Big(\frac{\delta(1-\beta_n)}{\gamma_{n+1}}(1+v_{n+1})(1-\alpha_{n+1}) - 1\Big)||x_{n+1} - x_n|| + \frac{\delta(1-\beta)}{\gamma_{n+1}}|\alpha_{n+1} - \alpha_n|||x_n|| \\
&\quad+ \frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n - T_i^n y_n\Big\|}{\gamma_{n+1}} + \frac{\delta}{\gamma_{n+1}\gamma_n}\Big\|\sum_{i\geq 1}\Big(\gamma_n\sigma_{i,n+1} - \gamma_{n+1}\sigma_{in}\Big)T_i^n y_n\Big\| \\
&\quad+ \frac{\beta_{n+1}}{\gamma_{n+1}}||x_{n+1}|| + \frac{\beta_n}{\gamma_n}||x_n|| \\
&\quad\leq \frac{[1-\delta(1-\beta_{n+1})]\Big\|y_{n+1} - x_{n+1}\Big\|}{\gamma_{n+1}} + \frac{[1-\delta(1-\beta_n)]\Big\|y_n - x_n\Big\|}{\gamma_n} \\
&\quad+ \Big(\frac{\delta(1-\beta_n)}{\gamma_{n+1}}(1+v_{n+1})(1-\alpha_{n+1}) - 1\Big)||x_{n+1} - x_n|| + \frac{\delta(1-\beta)}{\gamma_{n+1}}|\alpha_{n+1} - \alpha_n|||x_n|| \\
&\quad+ \frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n - T_i^n y_n\Big\|}{\gamma_{n+1}} + \frac{\delta M^*}{\gamma_{n+1}\gamma_n}\Big[\gamma_n\sum_{i\geq 1}|\sigma_{i,n+1} - \sigma_{in}| + |\gamma_n - \gamma_{n+1}|(1-\alpha_n)\Big] \\
&\quad+ \frac{\beta_{n+1}}{\gamma_{n+1}}||x_{n+1}|| + \frac{\beta_n}{\gamma_n}||x_n||
\end{aligned}$$

for some $M^* > 0$, thus

$$\limsup_{n\to\infty}(||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \le 0,$$

and by Lemma 2.2, we have

$$\lim_{n\to\infty} ||z_n - x_n|| = 0.$$

Hence

$$||x_{n+1} - x_n|| = (1 - \gamma_n)||z_n - x_n|| \to 0 \text{ as } n \to \infty. \tag{3.14}$$

From the recursion formula (3.9), we obtain

$$\delta \sum_{i\ge1} \sigma_{in}||T_i^n y_n - y_n|| = ||x_{n+1} - x_n|| \to 0 \quad \text{as} \quad n \to \infty.$$

For each $i \ge 1$, we get

$$\lim_{n\to\infty} ||T_i^n y_n - y_n|| = 0. \tag{3.15}$$

Therefore

$$\begin{aligned} ||T_i^n x_n - x_n|| &\le ||T_i^n x_n - T_i^n y_n|| + ||T_i^n y_n - y_n|| + ||y_n - x_n|| \\ &\le (2 + v_n)||x_n - y_n|| + ||T_i^n y_n - y_n|| \end{aligned}$$

From (3.11) and (3.15), we obtain

$$||T_i^n x_n - x_n|| \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad i \ge 1. \tag{3.16}$$

For each $i \ge 1$, using the asymptotic regularity of T_i, we obtain

$$\begin{aligned} \lim_{n\to\infty} ||T_i x_n - x_n|| &\le \lim_{n\to\infty} ||x_n - T_i^n x_n|| + \lim_{n\to\infty} ||T_i^n x_n - T_i^{n+1} x_n|| \\ &\quad + \lim_{n\to\infty} ||T_i^{n+1} x_n - T_i x_n|| \\ &\le (1 + L) \lim_{n\to\infty} ||x_n - T_i^n x_n|| + \lim_{n\to\infty} ||T_i^{n+1} x_n - T_i^n x_n|| = 0 \end{aligned} \tag{3.17}$$

hence

$$\begin{aligned} \lim_{n\to\infty} ||y_n - T_i y_n|| &\le \lim_{n\to\infty} ||y_n - x_n|| + \lim_{n\to\infty} ||x_n - T_i x_n|| \\ &\quad + \lim_{n\to\infty} ||T_i x_n - T_i y_n|| \\ &\le (1 + L) \lim_{n\to\infty} ||x_n - y_n|| + \lim_{n\to\infty} ||x_n - T_i x_n|| = 0 \end{aligned} \tag{3.18}$$

For each $m \ge 0$, let $z_m \in E$ be the unique fixed point of the contraction mapping

$$z_m := [1 - \delta(1 - \alpha_m)](1 - \beta_m)z_m + \delta \sum_{i\ge1} \sigma_{im} T_i^m (1 - \beta_n) z_m$$

on E, for $i \geq 1$ (see Theorem 3.1). Then we obtain by letting $y_m = (1-\alpha_m)z_m$ and w_m denote by $2v_m + v_m^2$

$$\begin{aligned}
||z_m - y_n||^2 &= ||[1-\delta(1-\beta_m)](y_m - y_n) + \delta \sum_{i\geq 1} \sigma_{im}(T_i^m y_m - y_n)||^2 \\
&\leq [1-\delta(1-\beta_m)]||y_m - y_n||^2 + \delta \sum_{i\geq 1} \sigma_{im}||T_i^m y_m - y_n||^2 \\
&\leq [1-\delta(1-\beta_m)]||y_m - y_n||^2 \\
&\quad +\delta \sum_{i\geq 1} \sigma_{im}\Big[||T_i^m y_m - T_i^m y_n|| + ||T_i^m y_n - y_n||\Big]^2 \\
&\leq [1-\delta(1-\beta_m)]||y_m - y_n||^2 \\
&\quad +\delta \sum_{i\geq 1} \sigma_{im}\Big[(1+v_m)||y_m - y_n|| + ||T_i^m y_n - y_n||\Big]^2 \\
&\leq [1-\delta(1-\beta_m)]||y_m - y_n||^2 \\
&\quad +\delta(1-\beta_m)(1+v_m)^2||y_m - y_n||^2 \\
&\quad +2\delta \sum_{i\geq 1} \sigma_{im}(1+v_m)||y_m - y_n||||T_i^m y_n - y_n|| \\
&\quad +\delta \sum_{i\geq 1} \sigma_{im}||T_i^m y_n - y_n||^2 \\
&= [1+\delta(1-\beta_m)w_m]||y_m - y_n||^2 \\
&\quad +2\delta \sum_{i\geq 1} \sigma_{im}(1+v_m)||y_m - y_n||||T_i^m y_n - y_n|| \\
&\quad +\delta \sum_{i\geq 1} \sigma_{im}||T_i^m y_n - y_n||^2 \\
&= [1+\delta(1-\beta_m)w_m]||(1-\alpha_m)(z_m - y_n) - \alpha_m y_n||^2 \\
&\quad +2\delta \sum_{i\geq 1} \sigma_{im}(1+v_m)||y_m - y_n||||T_i^m y_n - y_n|| \\
&\quad +\delta \sum_{i\geq 1} \sigma_{im}||T_i^m y_n - y_n||^2 \\
&\leq [1+\delta(1-\beta_m)w_m]\Big[(1-\alpha_m)^2||z_m - y_n||^2 + 2\alpha_m\langle -y_n, j(z_m - y_n)\rangle\Big] \\
&\quad +2\delta \sum_{i\geq 1} \sigma_{im}(1+v_m)||y_m - y_n||||T_i^m y_n - y_n|| \\
&\quad +\delta \sum_{i\geq 1} \sigma_{im}||T_i^m y_n - y_n||^2 \\
&\leq [1+\delta(1-\beta_m)w_m](1-\alpha_m)^2||z_m - y_n||^2 \\
&\quad +2[1+\delta(1-\beta_m)w_m]\alpha_m\langle -z_m, j(z_m - y_n)\rangle \\
&\quad +2\alpha_m[1+\delta(1-\beta_m)w_m]||z_m - y_n||^2 \\
&\quad +2\delta \sum_{i\geq 1} \sigma_{im}(1+v_m)||y_m - y_n||||T_i^m y_n - y_n|| \\
&\quad +\delta \sum_{i\geq 1} \sigma_{im}||T_i^m y_n - y_n||^2
\end{aligned}$$

$$
\begin{aligned}
&\leq [1+\delta(1-\beta_m)w_m](1+\alpha_m^2)||z_m-y_n||^2 \\
&\quad +2\alpha_m[1+\delta(1-\beta_m)w_m]\langle -z_m, j(z_m-y_n)\rangle \\
&\quad +2\delta\sum_{i\geq 1}\sigma_{im}(1+v_m)||y_m-y_n||||T_i^m y_n-y_n|| \\
&\quad +\delta\sum_{i\geq 1}\sigma_{im}||T_i^m y_n-y_n||^2 \\
&\leq \Big[1+\alpha_m\Big\{\delta(1-\beta_m)(w_m/\alpha_m)+\alpha_m[\delta(1-\beta_m)w_m]\Big\}\Big]||z_m-y_n||^2 \\
&\quad +2\alpha_m[1+\delta(1-\beta_m)w_m]\langle -z_m, j(z_m-y_n)\rangle \\
&\quad +2\delta\sum_{i\geq 1}\sigma_{im}(1+v_m)||y_m-x_n||||T_i^m y_n-y_n|| \\
&\quad +\delta\sum_{i\geq 1}\sigma_{im}||T_i^m y_n-y_n||^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\langle -z_m, j(y_n-z_m)\rangle \;\leq\; & \frac{\Big\{\delta(1-\beta_m)(w_m/\alpha_m)+\alpha_m[\delta(1-\beta_m)w_m]\Big\}||z_m-y_n||^2}{2[1+\delta(1-\beta_m)w_m]} \\
&+\frac{2\delta\sum_{i\geq}\sigma_{im}(1+v_m)||y_m-y_n||||T_i^m y_n-y_n||}{2\alpha_m[1+\delta(1-\beta_m)w_m]} \\
&+\frac{\delta\sum_{i\geq 1}\sigma_{im}||T_i^m y_n-y_n||^2}{2\alpha_m[1+\delta(1-\beta_m)w_m]}
\end{aligned}
$$

Now, taking limit superior as $n\to\infty$ firstly, and then as $m\to\infty$, we have

$$
\limsup_{m\to\infty}\limsup_{n\to\infty}\langle -z_m, j(y_n-z_m)\rangle \leq 0 \tag{3.19}
$$

But by Theorem 3.1, $z_m\to p$ as $m\to\infty$ and the fact that E has a uniformly Gâteaux differentiable norm implies that j is norm-to-weak* uniformly continuous on bounded sets. Thus, since

$$
\begin{aligned}
\langle -p, j(y_n-z_m)\rangle &= \langle -p, j(y_n-p)-j(y_n-z_m)\rangle+\langle z_m-p, j(y_n-z_m)\rangle \\
&\quad +\langle -z_m, j(y_n-z_m)\rangle \\
&\leq \langle -p, j(y_n-p)-j(y_n-z_m)\rangle+||z_m-p||||y_n-z_m|| \\
&\quad +\langle -z_m, j(y_n-z_m)\rangle
\end{aligned}
$$

we get that

$$
\begin{aligned}
\limsup_{n\to\infty}\langle -p, j(y_n-p)\rangle &\leq \limsup_{m\to\infty}\limsup_{n\to\infty}\langle -z_m, j(y_n-z_m)\rangle \\
&\leq 0
\end{aligned}
$$

Finally, we prove that $x_n\to p$ as $n\to\infty$. Since $\delta(1-\beta_n)v_n\to 0$ and $\delta(1-\beta_n)v_n/\alpha_n\to 0$ as $n\to\infty$, if we denote by w_n the value of $2v_n+v_n^2$, it implies that $\delta(1-\beta_n)w_n\to 0$ and $\delta(1-\beta_n)w_n/\alpha_n\to 0$ as $n\to\infty$, then there exists $n_0\in\mathbb{N}$ such that $\delta(1-\beta_n)w_n<1/2$ and $\delta(1-\beta_n)(w_n/\alpha_n)<1/2$, for all $n \geq n_0$. From recursion formula (3.1), we obtain

$$\begin{aligned}
||x_{n+1}-p||^2 &= ||[1-\delta(1-\beta_n)](y_n-p)+\delta\sum_{i\geq 1}\sigma_{in}(T_i^n y_n-p)||^2 \\
&\leq [1-\delta(1-\beta_n)]||y_n-p||^2+\delta\sum_{i\geq 1}\sigma_{in}||T_i^n y_n-p||^2 \\
&\leq [1-\delta(1-\beta_n)]||y_n-p||^2+\delta(1-\beta_n)(1+v_n)^2||y_n-p||^2 \\
&= [1-\delta(1-\beta_n)+\delta(1-\beta_n)(1+w_n)]||y_n-p||^2 \\
&= [1+\delta(1-\beta_n)w_n]||(1-\alpha_n)(x_n-p)-\alpha_n p||^2 \\
&\leq [1+\delta(1-\beta_n)w_n]\Big((1-\alpha_n)||x_n-p||^2 \\
&\quad +2\alpha_n\langle -p, j(y_n-p)\rangle\Big) \\
&= [1+\delta(1-\beta_n)w_n](1-\alpha_n)||x_n-p||^2 \\
&\quad +2\alpha_n[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n-p)\rangle \\
&= [1-\alpha_n+\delta(1-\alpha_n)(1-\beta_n)w_n] \\
&\quad +2\alpha_n[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n-p)\rangle \\
&\leq \Big[1-\alpha_n\Big(1-\delta(1-\beta_n)w_n/\alpha_n\Big)\Big] \\
&\quad +\alpha_n\Big(1-\delta(1-\beta_n)w_n/\alpha_n\Big) \\
&\quad \times\frac{2[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n-p)\rangle}{1-\delta(1-\beta_n)w_n/\alpha_n}.
\end{aligned}$$

Observe that $\sum\alpha_n(1-\delta(1-\beta_n)w_n/\alpha_n)=\infty$ and

$$\limsup_{n\to\infty}\Big(\frac{2[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n-p)\rangle}{1-\delta(1-\beta_n)w_n/\alpha_n}.\Big)\leq 0.$$

Applying Lemma 2.3, we obtain $||x_n-p||\to 0$ as $n\to\infty$. This completes the proof. □

Remark 3.3. By Gossez and Lami [16], we know that if E satisfies Opial's condition, then E has a weakly continuous duality mapping. Thus, Theorem 3.2 hold in uniformly convex and uniformly smooth Banach spaces which satisfies Opial's condition and also hold in real Hilbert spaces.

4. Numerical example

In this section, we discuss the direct application of Theorem 3.2 with a typical example on real line. Letting $T : C \subseteq E \to C$, then we consider the following:

$$E=\mathbb{R}, C=[0,1], Tx=x, \alpha_n=\frac{1}{n+1}, \beta_n=\frac{1}{2n^2+1}, \delta=\frac{1}{2},\ \forall n\geq 1$$

T here is nonexpansive which is particular case of our Theorem. Thus the scheme can be simplified as

$$x_{n+1}=\Big(\frac{n(n^2+1)}{(n+1)(2n^2+1)}+\frac{n^{n+2}}{(2n^2+1)(n+1)^n}\Big)x_n,\quad n\geq 1. \tag{4.1}$$

Take the initial point $x_1=0.5$, the numerical experiment result using MATLAB is given in Figure 1, which shows the iteration process of the sequence $\{x_n\}$ converges to 0.

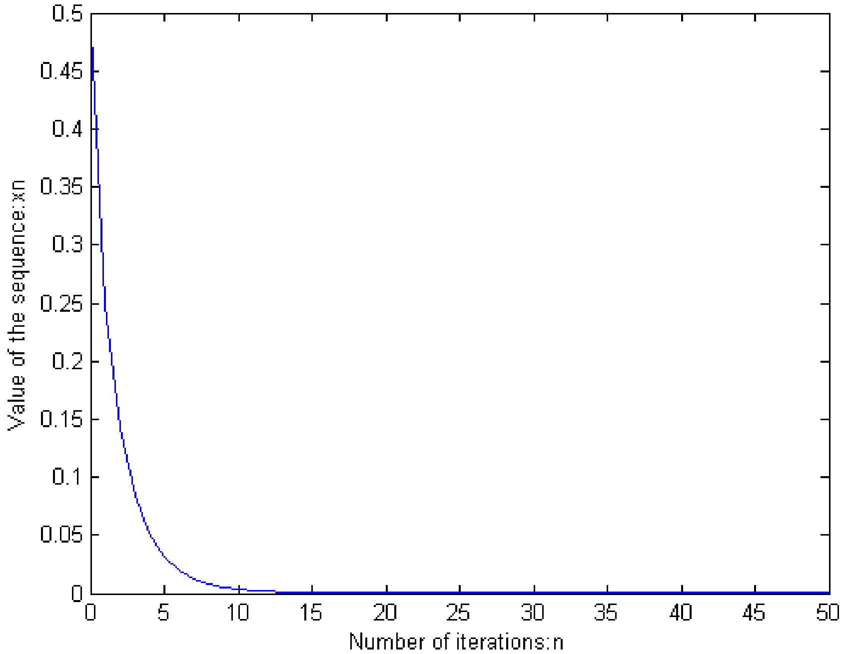

Figure 1: $x_1 = 0.5$, the convergence process of the sequnece $\{x_n\}$ generated by (4.1).

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