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Multi-point boundary value problems of higher-order nonlinear fractional differential equations

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Abstract

We investigate the existence and uniqueness of solutions for multi-point nonlocal boundary value problems of higher-order nonlinear fractional differential equations by using some well known fixed point theorems.

Keywords: boundary value problems; fractional derivative; fixed point theorems.
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1. Introduction

Fractional differential equations have been of great interest recently. This is due to the intensive development of the theory of fractional calculus itself as well as its applications. They arise in various fields of science and engineering such as the mathematical modeling of systems and processes in the fields of physics, chemistry, biology, aerodynamics, porous structures and polymer rheology [6, 15, 20]. Therefore, in recent years, the study of the boundary value problems for fractional differential equations has received considerable attention (see, for instance, [2, 4, 5, 8, 9, 10, 16, 19, 23, 25, 26, 27, 28, 30, 31, 32, 34] and references therein). However, few papers have considered the multi-point boundary value problems for higher-order fractional differential equations (see [7, 13, 17, 21, 29, 33]).

Ahmad and Nieto [1] studied some existence results in a Banach space for a nonlocal fractional boundary value problem given by

$$\begin{cases} {}^{C}D^{q}(x(t)) = f(t, x(t)), \;\; t \in (0,1), \;\; q \in (m-1, m], \;\; m \geq 2, \\ x(0) = x'(0) = \ldots = x^{(m-2)}(0) = 0, \quad x(1) = \alpha x(\eta), \quad 0 < \eta < 1, \end{cases}$$

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where ${}^{C}D^{q}$ is the Caputo fractional derivative.

Salem [22] investigated the existence of Pseudo solutions for the nonlinear m-point fractional boundary value problem

$$\begin{cases} D^{\alpha}_{0+}(x(t)) + q(t)f(t,x(t)) = 0, \ \ t \in (0,1), \ \ \alpha \in (n-1,n], \ \ n \geq 2, \\ x(0) = x'(0) = ... = x^{(n-2)}(0) = 0, \quad x(1) = \sum\limits_{i=1}^{m-2} \zeta_i x(\eta_i), \end{cases}$$

where $0 < \eta_1 < \ldots < \eta_{m-2} < 1$, $\zeta_i > 0$ with $\sum\limits_{i=1}^{m-2} \zeta_i \eta_i^{\alpha-1} < 1$.

Jia and Zhang [12] consider the multi-point boundary value problem of nonlinear fractional differential equation

$$\begin{cases} D^{\alpha}_{0+}u(t) + \lambda f(t,u(t)) = 0, \ \ 0 < t < 1, \ \ \alpha \geq 2, \ \ n-1 < \alpha \leq n, \\ u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, \ \ u^{(i)}(1) = \sum\limits_{j=0}^{m-2} \eta_j u'(\xi_j) ds, \ \ 0 \leq i \leq n-2. \end{cases}$$

Ahmad and Ntouyas [3] are concerned with the existence of solutions for a fractional boundary value problem

$$\begin{cases} {}^{C}D^{q}(x(t)) = f(t,x(t)), \ \ t \in (0,T), \ \ 1 \leq q \leq 2, \\ \alpha_1 x(0) + \beta_1({}^{C}D^{q}(x(0))) = \gamma_1, \\ \alpha_2 x(1) + \beta_2({}^{C}D^{q}(x(1))) = \gamma_2, \ \ 0 < p < 1. \end{cases}$$

Liu and Jia [18] investigated the nonlinear boundary value problem of fractional differential equation

$$\begin{cases} {}^{C}D^{q}(x(t)) = f(t,x(t),x'(t)), \ \ t \in (0,1), \ \ q \in (n-1,n], \ \ n > 2, \\ g_0(x(0),x'(0)) = 0, \\ g_1(x(1),x'(1)) = 0, \\ x''(0) = x'''(0) = ... = x^{(n)}(0) = 0. \end{cases}$$

In [24], authors developed sufficient conditions for multiplicity of positive solutions to the boundary value problem

$$\begin{cases} D^{q}_{0+}(u(t)) + f(t,u(t)) = 0, \ \ t \in (0,1), \\ u^{(i)}(0) = 0, \ \ 0 \leq i \leq n-2, \\ u(1) = \sum\limits_{i=1}^{m-2} \delta_i u(\eta_i), \end{cases}$$

where $n-1 < q \leq n$ and D^{q}_{0+} is the standard Riemann-Liouville fractional derivative of order q, $n \geq 3$, $\delta_i, \eta_i \in (0,1)$ with $\sum\limits_{i=1}^{m-2} \delta_i \eta_i^{q-1} < 1$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

Jiang and Wang [14] studied the existence and uniqueness of solutions to the following boundary value problem for fractional differential equation

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t,u(t),I^{\beta}_{0+}u(t)) = 0, \ \ 0 \leq t \leq 1, \\ u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, \ \ D^{p}_{0+}u(1) = \sum\limits_{i=1}^{m} a_i D^{q}_{0+}u(\xi_i), \end{cases}$$

where D^{α}_{0+} is the Riemann-Liouville fractional derivative of order $n-1 < \alpha \leq n$, $n \geq 2$, $0 < \beta < 1$, $p \in [1, n-2]$, $q \in [0,p]$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $a_i > 0 \ \ (i = 1, 2, \cdots, m)$.

Motivated by the aforementioned works, this paper is concerned with the existence of solutions to multi-point boundary value problem for higher order fractional differential equations:

$$\begin{cases} -{}^{R}D^{\alpha}_{0+}(u(t)) + f(t,u(t)) = 0, \ \ t \in [0,1], \\ u''(0) = u'''(0) = \cdots = u^{(n-2)}(0) = 0, \ u''(1) = \sum\limits_{i=1}^{m-2} \gamma_i u''(\xi_i), \\ au(0) - bu'(0) = \sum\limits_{i=1}^{m-2} a_i u'(\xi_i), \\ cu(1) + du'(1) = \sum\limits_{i=1}^{m-2} b_i u'(\xi_i), \end{cases} \tag{1.1}$$

where ${}^{R}D^{\alpha}_{0+}$ is the Riemann-Liouville fractional derivative of order α. Throughout the paper we suppose that $m \geq 3$, $n \geq 4$, $n-1 < \alpha \leq n$ where $n, m \in \mathbb{N}$, a, b, c, $d > 0$, a_i, $b_i \geq 0$, $0 < \xi_1 < \ldots < \xi_{m-2} < 1$ and $0 \leq \sum\limits_{i=1}^{m-2} \gamma_i \xi_i^{n-1} < 1$. We assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Also, we consider the analogous problem using the Caputo fractional derivative:

$$\begin{cases} -{}^{C}D^{\alpha}_{0+}(u(t)) + f(t,u(t)) = 0, \ \ t \in [0,1], \\ u''(0) = u'''(0) = \cdots = u^{(n-2)}(0) = 0, \ u''(1) = \sum\limits_{i=1}^{m-2} \gamma_i u''(\xi_i), \\ au(0) - bu'(0) = \sum\limits_{i=1}^{m-2} a_i u'(\xi_i), \\ cu(1) + du'(1) = \sum\limits_{i=1}^{m-2} b_i u'(\xi_i). \end{cases} \tag{1.2}$$

We mean a function $u \in C[0,1]$ of class $C^n[0,1]$ which satisfies the nonlocal fractional boundary value problem (1.1) (or (1.2)) by a solution of (1.1) (or (1.2)).

We have organized the paper as follows. First, we provide some definitions and preliminary lemmas which are key tools for our main results. Second, we obtain some existence and uniqueness results of the Riemann-Liouville multi-point boundary value problem (RLMBVP) (1.1) and the Caputo multi-point boundary value problem (CMBVP) (1.2).

We assume that the following conditions are satisfied:

(H1) If $m \geq 3$, then $c \sum\limits_{i=1}^{m-2} a_i \geq a \sum\limits_{i=1}^{m-2} b_i$ and if $m > 3$, then $ad > c \sum\limits_{i=1}^{j-1} a_i \geq a \sum\limits_{i=1}^{j-1} b_i > bc$ where $2 \leq j \leq m-2$.

(H2) $ad > a \sum\limits_{i=1}^{m-2} b_i + c \sum\limits_{i=1}^{m-2} a_i$.

2. Preliminaries

To state the main results of this paper, we will need the following lemmas and we present some notation.

Definition 2.1. *([15]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $u : (0,\infty) \to \mathbb{R}$ is defined by*

$${}^{R}D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right side is pointwise defined on $(0,\infty)$.

Definition 2.2. *([15]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,\infty) \to \mathbb{R}$ is given by*

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$

provided the integral exists.

Lemma 2.3. *([15]) Let $\alpha > 0$. Then the following equality holds for $u \in L(0,1)$, ${}^R D_{0^+}^{\alpha} u \in L(0,1)$;*

$$I_{0^+}^{\alpha}\, {}^R D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

$c_i \in \mathbb{R}$, $i = 1, ..., n$, where $n-1 < \alpha \leq n$.

Definition 2.4. *([15]) The fractional derivative of a function u in the Caputo sense is defined as*

$${}^C D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma 2.5. *([15]) Let $n-1 < \alpha \leq n$, $u \in C^n[0,1]$. Then*

$$I_{0^+}^{\alpha}\, {}^C D_{0^+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for $c_i \in \mathbb{R}$, $i = 1, ..., n$.

In the following, the RLMBVP (1.1) will be reduced to an equivalent integral equation. We know that ${}^R D_{0^+}^{\alpha}(u(t)) = {}^R D_{0^+}^{\alpha-2}\left({}^R D_{0^+}^{2} u(t)\right) = {}^R D_{0^+}^{\alpha-2}(u''(t))$. If $-u''(t) = y(t)$ and $\alpha - 2 = q$, then the problem

$$\begin{cases} -{}^R D_{0^+}^{\alpha-2}(u''(t)) + f(t,u(t)) = 0, \ \ t \in [0,1] \\ u''(0) = u'''(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u''(1) = \sum\limits_{i=1}^{m-2} \gamma_i u''(\xi_i) \end{cases}$$

is turned into problem

$$\begin{cases} {}^R D_{0^+}^{q} y(t) + f(t,u(t)) = 0, \ \ t \in [0,1] \\ y(0) = y'(0) = \cdots = y^{(n-4)}(0) = 0, \ \ y(1) = \sum\limits_{i=1}^{m-2} \gamma_i y(\xi_i). \end{cases} \tag{2.1}$$

Lemma 2.6. *The boundary value problem (2.1) has a unique solution*

$$y(t) = \int\limits_0^1 H_1(t,s) f(s,u(s)) ds$$

where

$$H_1(t,s) = \frac{1}{K_1 \Gamma(q)} \begin{cases} (1-s)^{q-1} t^{q-1} - \sum\limits_{i=k}^{m-2} \gamma_i t^{q-1} (\xi_i - s)^{q-1}, \\ t \leq s, \ s \in J_k, \ \ k = 1, 2, ..., m-1; \\ (1-s)^{q-1} t^{q-1} - \sum\limits_{i=k}^{m-2} \gamma_i t^{q-1} (\xi_i - s)^{q-1} - K_1 (t-s)^{q-1}, \\ t \geq s, \ \ s \in J_k, \ \ k = 1, 2, ..., m-1, \end{cases}$$

and $K_1 = 1 - \sum\limits_{i=1}^{m-2} \gamma_i \xi_i^{q-1}$.

Proof . According to Lemma 2.3, we can obtain that

$$y(t)=-\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,u(s))ds+c_1t^{q-1}+c_2t^{q-2}+\cdots+c_{n-2}t^{q-n+2}.$$

By boundary conditions of (2.1) we get $c_2=c_3=\cdots=c_{n-2}=0$ and

$$c_1=\frac{1}{K_1\Gamma(q)}\left[\int_0^1(1-s)^{q-1}f(s,u(s))ds-\sum_{i=1}^{m-2}\gamma_i\int_0^{\xi_i}(\xi_i-s)^{q-1}f(s,u(s))ds\right].$$

Thus, the unique solution of problem (2.1) is

$$\begin{aligned}
y(t) &= -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,u(s))ds+\frac{1}{K_1\Gamma(q)}\int_0^1(1-s)^{q-1}t^{q-1}f(s,u(s))ds\\
&- \frac{1}{K_1\Gamma(q)}\sum_{i=1}^{m-2}\gamma_i\int_0^{\xi_i}(\xi_i-s)^{q-1}t^{q-1}f(s,u(s))ds\\
&= \int_0^1 H_1(t,s)f(s,u(s))ds.
\end{aligned}$$

The proof is complete.

□

Lemma 2.7. *Let* $D:=ac+ad+bc+\sum_{i=1}^{m-2}(ca_i-ab_i)$ *and* $J_1=[0,\xi_1]$, $J_2=[\xi_1,\xi_2]$,..., $J_{m-2}=[\xi_{m-3},\xi_{m-2}]$, $J_{m-1}=[\xi_{m-2},1]$. *For* $y\in C[0,1]$, *the boundary value problem*

$$\begin{cases} -u''(t)=y(t),\ \ t\in[0,1],\\ au(0)-bu'(0)=\sum_{i=1}^{m-2}a_iu'(\xi_i)\\ cu(1)+du'(1)=\sum_{i=1}^{m-2}b_iu'(\xi_i)\end{cases}\tag{2.2}$$

has a unique solution

$$u(t)=\int_0^1 G(t,s)y(s)ds\tag{2.3}$$

where

$$G(t,s)=\frac{1}{D}\begin{cases}(at+b+\sum_{i=1}^{k-1}a_i)(c(1-s)+d-\sum_{i=k}^{m-2}b_i)+\sum_{i=k}^{m-2}a_i[c(t-s)+\sum_{i=1}^{k-1}b_i],\\ t\le s,\ s\in J_k,\ k=1,2,...,m-1;\\ (as+b+\sum_{i=1}^{k-1}a_i)(c(1-t)+d-\sum_{i=k}^{m-2}b_i)+\sum_{i=1}^{k-1}b_i[a(t-s)+\sum_{i=k}^{m-2}a_i],\\ t\ge s,\ s\in J_k,\ k=1,2,...,m-1.\end{cases}\tag{2.4}$$

Proof . A direct calculation gives that if $y \in C[0,1]$, then the boundary value problem (2.2) has the unique solution

$$\begin{aligned} u(t) &= -\int_0^t (t-s)y(s)ds + \frac{t}{D}\Bigg\{a\int_0^1 (c(1-s)+d)y(s)ds + \sum_{i=1}^{m-2}(ca_i - ab_i)\int_0^{\xi_i} y(s)ds\Bigg\} \\ &+ \frac{1}{D}\Bigg\{(b+\sum_{i=1}^{m-2}a_i)\int_0^1 (c(1-s)+d)y(s)ds - (b+\sum_{i=1}^{m-2}a_i)\sum_{i=1}^{m-2}b_i\int_0^{\xi_i} y(s)ds \\ &+ \frac{1}{a}\left(c(b+\sum_{i=1}^{m-2}a_i) - D\right)\sum_{i=1}^{m-2}a_i\int_0^{\xi_i} y(s)ds\Bigg\}. \end{aligned}$$

Hence, we obtain (2.3). □

Lemma 2.8. *The Green's function $G(t,s)$ in (2.4) satisfies*

$$0 < G(t,s) \le G(s,s)$$

for $(t,s) \in [0,1]\times[0,1]$.

Proof . From $(H1)$ and $(H2)$, we have $G(t,s) > 0$.
Now, we will show that $G(t,s) \le G(s,s)$.

(i) Let $s \in J_k$, $1 \le k \le m-2$ and $t \le s$. Since $G(t,s)$ is increasing in t, we get $G(t,s) \le G(s,s)$.
(ii) Take $s \in J_k$, $1 \le k \le m-2$ and $t \ge s$. From (H1), $G(t,s)$ is decreasing in t. So we obtain $G(t,s) \le G(s,s)$.

□

From Lemma 2.6 and Lemma 2.7, we know that $u(t)$ is a solution of the problem (1.1) if and only if

$$u(t) = \int_0^1 G(t,s)\int_0^1 H_1(s,\tau)f(\tau,u(\tau))d\tau ds. \tag{2.5}$$

Now, let $E = C[0,1]$, with supremum norm $\|y\| = \sup_{t\in[0,1]} |y(t)|$ for any $y \in E$. We can define the operator $A : E \to E$ by

$$Au(t) = \int_0^1 G(t,s)\int_0^1 H_1(s,\tau)f(\tau,u(\tau))d\tau ds, \tag{2.6}$$

where $u \in E$. Therefore solving (2.5) in E is equivalent to finding fixed points of the operator A.

Now, the CMBVP (1.2) will be reduced to an equivalent integral equation. We know that ${}^C D_{0^+}^{\alpha}(u(t)) = {}^C D_{0^+}^{\alpha-2}\left({}^C D_{0^+}^{2} u(t)\right) = {}^C D_{0^+}^{\alpha-2}(u''(t))$. If $-u''(t) = y(t)$ and $\alpha - 2 = q$, then the problem

$$\begin{cases} -{}^C D_{0^+}^{\alpha-2}(u''(t)) + f(t,u(t)) = 0, \;\; t \in [0,1] \\ u''(0) = u'''(0) = \cdots = u^{(n-2)}(0) = 0, \;\; u''(1) = \sum\limits_{i=1}^{m-2}\gamma_i u''(\xi_i) \end{cases}$$

is turned into problem

$$\begin{cases} {}^{C}D_{0^+}^{q}y(t) + f(t,u(t)) = 0, \ \ t \in [0,1] \\ y(0) = y'(0) = \cdots = y^{(n-4)}(0) = 0, \ \ y(1) = \sum\limits_{i=1}^{m-2} \gamma_i y(\xi_i). \end{cases} \tag{2.7}$$

Lemma 2.9. *The boundary value problem (2.7) has a unique solution*

$$y(t) = \int_0^1 H_2(t,s) f(s,u(s))ds$$

where

$$H_2(t,s) = \frac{1}{K_2\Gamma(q)} \begin{cases} (1-s)^{q-1}t^{n-1} - \sum\limits_{i=k}^{m-2} \gamma_i t^{n-1}(\xi_i - s)^{q-1}, \\ t \le s, \ s \in J_k, \ \ k = 1,2,...,m-1; \\ (1-s)^{q-1}t^{n-1} - \sum\limits_{i=k}^{m-2} \gamma_i t^{n-1}(\xi_i - s)^{q-1} - K_2(t-s)^{q-1}, \\ t \ge s, \ \ s \in J_k, \ \ k = 1,2,...,m-1, \end{cases}$$

and $K_2 = 1 - \sum\limits_{i=1}^{m-2} \gamma_i \xi_i^{n-1}$.

Proof . According to Lemma 2.5, we can obtain that

$$y(t) = -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))ds + c_0 + c_1 t + ... + c_{n-1}t^{n-1}.$$

By boundary conditions of (2.7) we get $c_0 = c_1 = \cdots = c_{n-2} = 0$ and

$$c_{n-1} = \frac{1}{K_2\Gamma(q)} \left[\int_0^1 (1-s)^{q-1} f(s,u(s))ds - \sum_{i=1}^{m-2} \gamma_i \int_0^{\xi_i} (\xi_i - s)^{q-1} f(s,u(s))ds \right].$$

Thus, the unique solution of problem (2.7) is

$$\begin{aligned} y(t) &= -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))ds + \frac{1}{K_2\Gamma(q)} \int_0^1 (1-s)^{q-1}t^{n-1} f(s,u(s))ds \\ &- \frac{1}{K_2\Gamma(q)} \sum_{i=1}^{m-2} \gamma_i \int_0^{\xi_i} (\xi_i - s)^{q-1} t^{n-1} f(s,u(s))ds \\ &= \int_0^1 H_2(t,s) f(s,u(s))ds. \end{aligned}$$

The proof is complete.

□

From Lemma 2.7 and Lemma 2.9, we know that $u(t)$ is a solution of the problem (1.2) if and only if

$$u(t) = \int_0^1 G(t,s) \int_0^1 H_2(s,\tau) f(\tau, u(\tau)) d\tau ds. \tag{2.8}$$

We can define the operator $F : C[0,1] \to C[0,1]$ by

$$Fu(t) = \int_0^1 G(t,s) \int_0^1 H_2(s,\tau) f(\tau, u(\tau)) d\tau ds,$$

where $u \in C[0,1]$. Therefore solving (2.8) in $C[0,1]$ is equivalent to finding fixed points of the operator F.

3. Existence and uniqueness of solutions

In this section, first, we will use the following well-known contraction mapping theorem named also as the Banach fixed point theorem: Let E be a Banach space and S a nonempty closed subset of E. Assume $A : S \to S$ is a contraction, i.e., there is a λ $(0 < \lambda < 1)$ such that $\|Ax - Ay\| \leq \lambda \|x - y\|$ for all x, y in S. Then A has a unique fixed point in S.

Theorem 3.1. *We assume that the function $f(t,x)$ satisfies the following Lipschitz condition*
$(H3)$ *There is a constant $L > 0$ such that*

$$|f(t,x) - f(t,y)| \leq L|x - y| \text{ for all } t \in [0,1] \text{ and } x, y \in C[0,1].$$

If we have

$$\frac{L}{K_1 \Gamma(q)} \int_0^1 G(s,s)(1-s)^{q-1} ds < 1,$$

then the RLMBVP (1.1) has a unique solution in $C[0,1]$.

Proof . For $u_1, u_2 \in C[0,1]$ and $t \in [0,1]$, we have

$$\begin{aligned}
|(Au_1)(t) - (Au_2)(t)| &\leq \int_0^1 G(t,s) \int_0^1 |H_1(s,\tau)| \, |f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))| \, d\tau ds \\
&\leq \int_0^1 G(s,s) \int_0^1 \frac{(1-s)^{q-1}}{K_1 \Gamma(q)} L \, |u_1(\tau) - u_2(\tau)| \, d\tau ds \\
&\leq \left(\frac{L}{K_1 \Gamma(q)} \int_0^1 G(s,s)(1-s)^{q-1} ds \right) \|u_1 - u_2\|
\end{aligned}$$

by using Lemma 2.8, Lemma 2.6 and the condition $(H3)$. Hence, A is a contraction mapping and the theorem is proved. □

Theorem 3.2. *Suppose that* $(H3)$ *holds. Moreover,*

$$\frac{L}{K_2\Gamma(q)}\int_0^1 G(s,s)(1-s)^{q-1}ds < 1.$$

Then, the CMBVP (1.2) has a unique solution in $C[0,1]$.

Proof . The proof of Theorem 3.2 is very similar to that of Theorem 3.1 and therefore omitted. □

In the next theorem, the function $f(t,x)$ satisfies a Lipschitz condition on a subset of $C[0,1]$.

Theorem 3.3. *We assume that*
$(H4)$ *Let there exists a number* $r > 0$ *such that*

$$|f(t,x) - f(t,y)| \le L|x-y| \text{ for each } t \in [0,1],$$

for all x *and* y *in* $S = \{x \in C[0,1] : \|x\| \le r\}$, *where* $L > 0$ *is a constant which may depend on* r. *Also,*

$$\frac{L}{K_1\Gamma(q)}\int_0^1 G(s,s)(1-s)^{q-1}ds < 1.$$

Suppose that there exists nonnegative function $g \in C[0,1]$ *such that* $|f(t,u(t))| \le g(t)\,|u(t)|$ *and*

$$\frac{1}{K_1\Gamma(q)}\int_0^1 G(s,s)(1-s)^{q-1}\int_0^1 g(\tau)d\tau ds \le 1.$$

Then, the RLMBVP (1.1) has a unique solution $u \in C[0,1]$ *with* $|u(t)| \le r$, $\forall t \in [0,1]$.

Proof . Let us take $S = \{u \in C[0,1] : \|u\| \le r\}$. Obviously, S is a closed subset of $C[0,1]$. Let $A : C[0,1] \to C[0,1]$ be the operator defined by (2.6). For u_1 and u_2 in S, taking into account $(H4)$, in exactly the same way in the proof of Theorem 3.1 we can get $\|Au_1 - Au_2\| \le \lambda\|u_1 - u_2\|$, where $0 < \lambda < 1$.

It remains to show that A maps S into itself. If $u \in S$, then we obtain

$$\begin{aligned}
|(Au)(t)| &\le \int_0^1 G(t,s)\int_0^1 |H_1(s,\tau)|\,|f(\tau,u(\tau))|\,d\tau ds \\
&\le \int_0^1 G(s,s)\int_0^1 \frac{(1-s)^{q-1}}{K_1\Gamma(q)} g(\tau)\,|u(\tau)|\,d\tau ds \\
&\le \|u\|\,.
\end{aligned}$$

Since $\|Au\| \le r$, we have $A : S \to S$.

From the contraction mapping theorem, the RLMBVP (1.1) has a unique solution $u \in C[0,1]$ with $|u(t)| \le r$, $\forall t \in [0,1]$.

□

Theorem 3.4. *We assume that $(H4)$ holds. Also,*

$$\frac{L}{K_2\Gamma(q)}\int\limits_0^1 G(s,s)(1-s)^{q-1}ds<1.$$

Suppose that there exists nonnegative function $g\in C[0,1]$ such that $|f(t,u(t))|\le g(t)\,|u(t)|$ and

$$\frac{1}{K_2\Gamma(q)}\int\limits_0^1 G(s,s)(1-s)^{q-1}\int\limits_0^1 g(\tau)d\tau ds\le 1.$$

Then, the CMBVP (1.2) has a unique solution $u\in C[0,1]$ with $|u(t)|\le r$, $\forall t\in[0,1]$.

Proof . The proof of Theorem 3.4 is very similar to that of Theorem 3.3 and therefore omitted. □

To get an existence theorem without uniqueness of solution, we will apply the following Leray-Schauder nonlinear alternative theorem.

Theorem 3.5. *[11] Let E be a real Banach space and Ω be a bounded open subset of E, $0\in\Omega$, $F:\overline{\Omega}\to E$ be a completely continuous operator. Then either there exist $x\in\partial\Omega$, $\lambda>1$ such that $F(x)=\lambda x$, or there exists a fixed point $x^*\in\overline{\Omega}$.*

For convenience, we introduce the following notation. Let

$$B_1=\frac{1}{K_1\Gamma(q)}\int\limits_0^1 G(s,s)(1-s)^{q-1}\int\limits_0^1 g(\tau)d\tau ds,$$

$$B_2=\frac{1}{K_2\Gamma(q)}\int\limits_0^1 G(s,s)(1-s)^{q-1}\int\limits_0^1 g(\tau)d\tau ds,$$

$$D_1=\frac{1}{K_1\Gamma(q)}\int\limits_0^1 G(s,s)(1-s)^{q-1}\int\limits_0^1 h(\tau)d\tau ds,$$

$$D_2=\frac{1}{K_2\Gamma(q)}\int\limits_0^1 G(s,s)(1-s)^{q-1}\int\limits_0^1 h(\tau)d\tau ds.$$

Theorem 3.6. *Suppose that there exist nonnegative functions $g,h\in C[0,1]$ such that $|f(t,u(t))|\le g(t)\,|u(t)|+h(t)$ and $B_1<1$. Then the RLMBVP (1.1) has at least one solution $u\in C[0,1]$.*

Proof . Since $B_1<1$ and $D_1>0$, $r:=D_1(1-B_1)^{-1}>0$. Let us take $\Omega=\{u\in C[0,1]:\|u\|<r\}$. By Arzela-Ascoli theorem, it is easy to check that $A:\overline{\Omega}\to C[0,1]$ is completely continuous operator.

If we take $u \in \partial\Omega$, $\lambda > 1$ such that $Au = \lambda u$, then

$$\begin{aligned}
\lambda D_1(1-B_1)^{-1} &= \lambda r = \lambda\|u\| = \|Au\| = \max_{t\in[0,1]} |Au(t)| \\
&\leq \max_{t\in[0,1]} \int_0^1 G(t,s) \int_0^1 |H_1(s,\tau)|\,|f(\tau,u(\tau))|\,d\tau ds \\
&\leq \int_0^1 G(s,s) \int_0^1 \frac{(1-s)^{q-1}}{K_1\Gamma(q)} \left(g(\tau)\,|u(\tau)| + h(\tau)\right) d\tau ds \\
&\leq B_1\,\|u\| + D_1 = B_1 r + D_1 = D_1(1-B_1)^{-1}.
\end{aligned}$$

Hence we obtain $\lambda \leq 1$, this contradicts $\lambda > 1$. By Theorem 3.5, the operator A has a fixed point in $\overline{\Omega}$. Thus, the RLMBVP (1.1) has at least one solution $u \in \overline{\Omega}$.
□

Theorem 3.7. *Suppose that there exist nonnegative functions $g, h \in C[0,1]$ such that $|f(t,u(t))| \leq g(t)\,|u(t)| + h(t)$ and $B_2 < 1$. Then the CMBVP (1.2) has at least one solution $u \in C[0,1]$.*

Proof . The proof of Theorem 3.7 is very similar to that of Theorem 3.6 and therefore omitted.
□

Example 3.8. *Consider the following nonlinear RLMBVP*

$$\begin{cases} -{}^R D_{0^+}^{\frac{11}{2}}(u(t)) + 2 + t^2 u(t) = 0, \;\; t \in [0,1], \\ u''(0) = u'''(0) = u^{(4)}(0) = 0, \;\; u''(1) = u''(\frac{1}{2}), \\ u(0) - u'(0) = u'(\frac{1}{2}), \\ 2u(1) + 3u'(1) = u'(\frac{1}{2}). \end{cases} \tag{3.1}$$

Taking $a = b = a_1 = b_1 = \gamma_1 = 1$, $\xi_1 = \frac{1}{2}$, $c = 2$, $d = m = 3$, $n = 6$, $\alpha = \frac{11}{2}$ and $q = \frac{7}{2}$, we obtain $K_1 = 1 - \frac{1}{4\sqrt{2}}$ and $G(s,s) = \frac{1}{4}(-s^2 + s + 2)$. Since we have

$$|f(t,x) - f(t,y)| = |t^2 x(t) - t^2 y(t)| \leq |x(t) - y(t)|, \quad \forall t \in [0,1],$$

we take $L = 1$. Also, we obtain

$$\frac{L}{K_1\Gamma(q)} \int_0^1 G(s,s)(1-s)^{q-1} ds \approx 0,055875 < 1.$$

Hence, by Theorem 3.1, the RLMBVP (3.1) has a unique solution in $C[0,1]$.

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