



The Arrow Domination in Graphs

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(Communicated by Madjid Eshaghi Gordji)

Abstract

The arrow domination is introduced in this paper with its inverse as a new type of domination. Let G be a finite graph, undirected, simple and has no isolated vertex, a set D of $V(G)$ is said an arrow dominating set if $|N(w) \cap (V - D)| = i$ and $|N(w) \cap D| \geq j$ for every $w \in D$ such that i and j are two non-equal positive integers. The arrow domination number $\gamma_{ar}(G)$ is the minimum cardinality over all arrow dominating sets in G . Essential properties and bounds of arrow domination and its inverse when $i = 1$ and $j = 2$ are proved. Then, arrow domination number is discussed for several standard graphs and other graphs that formed by join and corona operations.

Keywords: Dominating set, Arrow dominating set, Arrow domination number.
2010 MSC: 05C69

1. Introduction

Let G be a graph, the number of edges incident on $x \in G$ means $deg(x)$. Open neighborhood of x is $N(x) = \{w \in V, wx \in E\}$ and closed neighborhood is the set $N[x] = N(x) \cup \{x\}$. A set D is said dominating set if $N[D] = V$. Also, $\gamma(G)$ is order of the smallest dominating set in G . For basic definitions in graph theory see [7, 13] and for domination see [8, 12]. There are several types of domination models in graphs discussed different conditions such as [1, 2, 3, 4, 5, 6, 9, 10, 11, 15]. Here, the arrow domination in graphs is introduced depending on the numbers of the adjacent and dominated vertices. Some bounds are given and several properties are proved. Then, the arrow domination number is evaluated for some standard graphs and graphs that formed by join and corona operations.

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2. Bounds and Properties

In this section, the definition of arrow domination is introduced and its properties are proved.

Definition 2.1. Let G be a finite graph, undirected, simple and without isolated vertex, a set D of $V(G)$ is an arrow dominating set, if $|N(w) \cap V - D| = i$ and $|N(w) \cap D| \geq j$ for every $w \in D$ where i and j are positive integers such that $i \neq j$.

Definition 2.2. An arrow dominating set D is minimal if there is no arrow dominating subset in it. The smallest minimal arrow dominating set is minimum and its order is arrow domination number $\gamma_{ar}(G)$. Such set is referred as γ_{ar} -set. For example see Fig. 1.

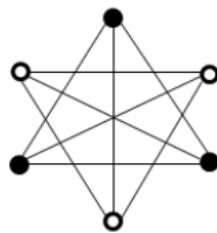


Figure 1: Minimum arrow dominating set

Remark 2.3. If G is a graph with $\gamma_{ar}(G)$, then:

1. $|V(G)| \geq 4$.
2. $\delta(G) \geq 1$ and $\Delta(G) \geq 3$.
3. $\gamma_{ar}(G) \geq 3$.
4. $deg(v) \geq 3 \forall v \in D$.
5. $N(v) \cap D \neq \phi$ for all $v \in D$.
6. $N(v) \cap V - D \neq \phi$ for all $v \in D$.
7. $\gamma_{ar}(G) \geq \gamma(G)$.

Remark 2.4. For any graph G with $\Delta(G) \leq 2$, then G has no arrow dominating set.

Remark 2.5. If G be a disconnected graph with a component that is isomorphic to K_1 , K_2 , or K_3 , then G has no arrow dominating set.

Remark 2.6. Let $G(n, m)$ be a graph in which there is a vertex of degree $n - 1$, then \overline{G} has no arrow dominating set.

Proposition 2.7. For any graph G with arrow dominating set D , then every end vertex don't belongs to D .

Proof . According to Remark 2.3, since $deg(u) = 1$ for any end vertex u . \square

Proposition 2.8. In any graph G with arrow dominating set D , every support vertex that adjacent with one end vertex is belongs to D .

Proof . According to Remark 2.3, every end vertex don't belongs to D . Then, the support vertex must be in D . \square

Proposition 2.9. *For any graph G having support vertex that adjacent with two or more end vertices. Then, G hasn't arrow dominating set.*

Proof . If D is a γ_{ar} -set in G . Let v be a support vertex adjacent with two end vertices u_1 and u_2 . If $v \in D$, then v dominates at least two vertices u_1 and u_2 which is contradict the arrow domination. If $v \notin D$, then since $u_1, u_2 \notin D$ by Remark 2.3, then D don't dominates u_1 and u_2 which is contradiction since D is γ_{ar} -set. \square

Remark 2.10. *If G has arrow dominating set and r end vertices, then $\gamma_{ar}(G) \leq n - r$.*

Lemma 2.11. *[14] If the degree of each vertex in G is at least two, then G has a cycle.*

Proposition 2.12. *For any graph G with γ_{ar} -set D , then $G[D]$ has a cycle.*

Proof . By Remark 2.3 and Lemma 2.11. \square

Theorem 2.13. *Every arrow dominating set is minimal arrow dominating set.*

Proof . Let D is any arrow dominating set of a graph G . Let D isn't minimal, thus there is $u \in D$ and $D - \{u\}$ is minimal. Let w is dominated by u , if $w \in Pn[u, D]$, then w hasn't any neighborhood in $D - \{u\}$, so w don't dominated by $D - \{u\}$ which is contradiction. Now, if w is dominated by another vertex $t \in D$. Since u has at least two neighborhoods x_1, x_2 in D , then both x_1 and x_2 dominate v and another vertex from $V - D$ which is contradiction. Hence, D is minimal. \square

Theorem 2.14. *Let $G(n, m)$ be a graph having an arrow dominating set D and $\gamma_{ar}(G)$, then:*

$$2\gamma_{ar}(G) \leq m \leq \binom{n}{2} + \gamma_{ar}^2(G) + (1 - n)\gamma_{ar}(G).$$

Proof . Since every vertex of D dominates only one vertex, then there is $|D| = \gamma_{ar}(G)$ edges between $V - D$ and D . There are two cases:

Case 1: If $G[D]$ and $G[V - D]$ are null graphs. Since $deg(v) = 2$ at least in $G[D]$. Then, $G[D]$ is a cycle graph with size $\gamma_{ar}(G)$. Therefore, $m \geq 2\gamma_{ar}(G)$.

Case 2: It is clear if $G[D]$ and $G[V - D]$ are completes. \square

Theorem 2.15. *Let $G(n, m)$ is a graph with $\gamma_{ar}(G)$, then:*

$$\lceil \frac{n}{2} \rceil \leq \gamma_{ar}(G) \leq n - 1$$

Proof . Let D be the γ_{ar} -set of G . Since every x of D dominates only one vertex, then the order of any arrow dominating set must be $\frac{n}{2}$ if $u \in Pn[v, D]$ for every $u \in V - D$ and every $v \in D$ where v dominates u . But, if there are two vertices or more in D dominate the same vertex in $V - D$, then $|D| > \frac{n}{2}$. Thus, $\gamma_{ar}(G) \geq \lceil \frac{n}{2} \rceil$.

The upper bound proved depending on the fact, $V - D \neq \phi$, then it must be having one vertex at least. Thus, $\gamma_{ar}(G) \leq n - 1$. \square

Corollary 2.16. *For any $\gamma_{ar}(G)$:*

1. $\gamma_{ar}(G) \geq \lceil \frac{n}{\delta(G)+1} \rceil$.
2. $\gamma_{ar}(G) \geq \lceil \frac{n}{\Delta(G)-1} \rceil$.

Proof . Let D be a γ_{ar} -set of G . To prove (1): since $\delta(G) \geq 1$ by Remark 2.3, then $\delta(G) + 1 \geq 2$. Since $\gamma_{ar}(G) \geq \lceil \frac{n}{2} \rceil$ by Theorem 2.15, then $\lceil \frac{n}{\delta+1} \rceil \leq \lceil \frac{n}{2} \rceil \leq \gamma_{ar}(G)$. To prove (2): since $\Delta(G) \geq 3$ by Remark 2.3, then $\Delta(G) - 1 \geq 2$. Thus, $\lceil \frac{n}{\Delta-1} \rceil \leq \lceil \frac{n}{2} \rceil \leq \gamma_{ar}(G)$. \square

3. Arrow Domination in Some Graphs

The arrow domination number is proved here for standard graphs and some graphs constructed by join and corona operations.

Remark 3.1. According to Remark 2.4, the path graph P_n and cycle graph C_n don't having arrow domination for all n .

Proposition 3.2. Let K_n be a complete graph, then $\gamma_{ar}(K_n) = n - 1$ iff $n \geq 4$.

Proof . Since $V - D$ contains only one vertex. \square

Theorem 3.3. Let W_n be a wheel graph where $n \geq 3$, then $\gamma_{ar}(W_n) = n$.

Proof . Since wheel graph $W_n = C_n + K_1$, let $D = V(C_n)$ where every $v \in D$ dominates the vertex of K_1 and adjacent with $n - 1$ vertices in D . Hence, D is γ_{ar} -set and $\gamma_{ar}(W_n) = n$. \square

Proposition 3.4. Star graph S_n hasn't arrow dominating set for all n .

Proof . By Proposition 2.9. \square

Theorem 3.5. Let $K_{n,m}$ be a complete bipartite graph, then $\gamma_{ar}(K_{n,m}) = n + m - 2$ if and only if $n, m \geq 3$.

Proof . Let β_1 and β_2 be the vertices sets of $K_{n,m}$, where $|\beta_1| = n$ and $|\beta_2| = m$. Since all vertices of any arrow dominating set are not isolated in $G[D]$, then $D \neq \beta_1$ and $D \neq \beta_2$. So, D contains vertices from both sets β_1 and β_2 .

Case i: For $m \geq 3$ and $n = 2$. If D contains one vertex u from β_1 and two vertices w, t from β_2 , then w, t dominates one vertex from β_1 and adjacent with only u which is contradiction. In the same way when $m = 2$ and $n \geq 3$ or $n = m = 2$. Thus, $K_{n,m}$ has no arrow dominating set.

Case j: For $m, n \geq 3$, suppose that D contains all β_1 vertices unless one vertex and all vertices of β_2 unless one vertex. Then, for every $v \in D$ dominates one vertex and adjacent with two vertices or more from D . Hence, D is arrow dominating set. Since when we remove any vertex from D get some vertices dominate more than one vertex. Thus, D is minimum and $\gamma_{ar}(K_{n,m}) = n + m - 2$. \square

Proposition 3.6. The complement path graph $\overline{P_n}$ has no arrow dominating set for all n .

Proof . It is clear, $\Delta(\overline{P_4}) = 2$. For $n = 5$, there are three vertices v_2, v_3, v_4 of degree two don't belong to D . So, if $D = \{v_1, v_5\}$, then every v_i of D has only one neighborhood in D and dominate two vertices. For $n \geq 6$, every dominating set D has either a vertex that dominates two or more vertices or a vertex don't dominate any vertex. Hence, there is no arrow dominating set in $\overline{P_n}$. \square

Theorem 3.7. The complement cycle graph $\overline{C_n}$ has an arrow dominating set if and only if $n = 6$ where $\gamma_{ar}(\overline{C_6}) = 3$.

Proof . It is clear, $\Delta(\overline{C_4}) = 1$ and $deg(v) = 2$ for every v in $\overline{C_5}$, then there is no arrow dominating set for $n \leq 5$. If $n = 6$, let $D = \{v_1, v_3, v_5\}$, then D is a γ_{ar} -set of order three every vertex in which dominates only one and adjacent with exactly two vertices. If $n \geq 7$, then $\overline{C_n}$ has no arrow dominating set for the same cause of $\overline{P_n}$ ($n \geq 6$) in the above proposition. \square

Proposition 3.8. *The complement bipartite graph $\overline{K_{n,m}}$ has an arrow dominating set if and only if $n, m \geq 4$ where $\gamma_{ar}(\overline{K_{n,m}}) = n + m - 2$.*

Proof . Since $\overline{K_{n,m}} = K_n \cup K_m$, then $\gamma_{ar}(\overline{K_{n,m}}) = n + m - 2$ according to Proposition 3.2. \square

Remark 3.9. *The complements of wheel graph $\overline{W_n}$ and complete graph $\overline{K_n}$ are without arrow dominating sets.*

Theorem 3.10. *If G be a graph of order n , then:*

1. $\gamma_{ar}(G \odot K_1) = n$ if and only if $\deg(v) \geq 2$ for every $v \in G$.
2. $G \odot K_2$ has no arrow dominating set.
3. $\gamma_{ar}(G \odot K_m) = nm$ iff $m \geq 3$.
4. $G \odot \overline{K_m}$ has no arrow dominating set for $m \geq 2$.

Proof . 1. Let $D = V(G)$. Since $\deg(v) \geq 2$ in G , then every vertex in D adjacent with two or more vertices from D and dominates exactly one vertex of one K_1 copy. Thus, D is a γ_{ar} -set of order n .
 2. There are n copies of K_2 every vertex of them adjacent with one vertex from G and has degree two, so $v \notin D, \forall v \in K_2$. Therefore, if $G \odot K_2$ has arrow dominating set D , it must be contains all vertices of G . But in this case every vertex of D dominates two vertices which is contradiction. Thus, $G \odot K_2$ has no arrow dominating set.
 3. Since every vertex of G adjacent with three or more vertices of one copy of K_m , then let $D = V(K_m)$ where every vertex in it dominates exactly one vertex from G and adjacent with $m - 1$ vertices. Thus, D is a γ_{ar} -set of order nm .
 4. According to Proposition 2.9. \square

Proposition 3.11. *Let G and \acute{G} be two null graphs of orders n and m respectively, then $\gamma_{ar}(G + \acute{G}) = n + m - 2$ iff $n, m \geq 3$.*

Proof . By Theorem 3.5 where $G + \acute{G} = K_{n,m}$. \square

Theorem 3.12. *For any two graphs \acute{G} and \ddot{G} of orders n and m ; ($n, m \geq 3$) respectively, then $n + m - 2 \leq \gamma_{ar}(\acute{G} + \ddot{G}) \leq n + m - 1$.*

Proof . Suppose that \acute{G} and \ddot{G} two null graphs to prove the lower bound. Thus, $\acute{G} + \ddot{G} = K_{n,m}$ and the arrow domination number equals $n + m - 2$ according to Theorem 3.5. Now, to prove the upper bound let \acute{G} and \ddot{G} two complete graphs. Then, their addition gives a complete graph K_{n+m} with arrow domination number equals $n + m - 1$ according to Proposition 3.2. \square

Proposition 3.13. *Let \acute{G} and \ddot{G} are two graphs having arrow dominating sets, then $\gamma_{ar}(\acute{G} \cup \ddot{G}) = \gamma_{ar}(\acute{G}) + \gamma_{ar}(\ddot{G})$.*

Proof . Suppose that D be a γ_{ar} -set of \acute{G} and \acute{D} be a γ_{ar} -set of \ddot{G} . Then, the γ_{ar} -set of $\acute{G} \cup \ddot{G}$ equals the union of D and \acute{D} . \square

4. The Inverse Arrow Domination

The inverse arrow domination is defined here and its properties are studied.

Definition 4.1. Let G be a graph having γ_{ar} -set. A set D^{-1} of $V - D$ is inverse arrow dominating set of G , if D^{-1} is arrow dominating set.

Definition 4.2. A set D^{-1} is minimal, if there is no arrow dominating set in it. Also, said minimum if it has the least order. The inverse arrow domination number $\gamma_{ar}^{-1}(G)$ is the order of the minimum inverse arrow dominating sets in G . Such set is referred as γ_{ar}^{-1} -set.

Remark 4.3. Let G be a graph with $\gamma_{ar}^{-1}(G)$. Then:

1. $|V(G)| \geq 6$.
2. $\gamma_{ar}^{-1}(G) \geq 3$.
3. $\gamma_{ar}^{-1}(G) \geq \gamma_{ar}(G)$.

Remark 4.4. Let $G(n, m)$ be a graph with $\gamma_{ar}(G)$. If $\gamma_{ar}(G) > \frac{n}{2}$, then G has no inverse arrow dominating set.

Proposition 4.5. Every graph with odd order has no inverse arrow dominating set.

Proof . Let G be a graph of order odd number n . Suppose that G having a γ_{ar} -set D and $\gamma_{ar}(G)$. Since $\gamma_{ar}(G) \geq \lceil \frac{n}{2} \rceil$ by Theorem 2.15 and since n odd number, then $\gamma_{ar}(G) > \frac{n}{2}$. Hence, G has no inverse arrow dominating set according to Remark 4.4. \square

Proposition 4.6. Let G be a graph having a γ_{ar} -set D . If there is a vertex $v \in V - D$ which is dominated by two or more vertices. Then, G has no inverse arrow dominating set.

Proof . Suppose that v is dominated by x_1 and x_2 in D . If $v \in D^{-1}$, then it dominates x_1 and x_2 which is contradiction. So $v \notin D^{-1}$. But x_1 and x_2 having no neighborhood in $V - D$ unless v . Then, x_1 and x_2 are not dominated by any inverse dominating set. Thus, G has no inverse arrow dominating set. \square

Proposition 4.7. Let G be a graph having a γ_{ar} -set D . If there is a vertex $u \in V - D$ with $\deg(u) < 2$ in $G[V - D]$. Then, G has no inverse arrow dominating set.

Proof . Since u is an isolated vertex or end vertex in $G[V - D]$, then $u \notin D^{-1}$ since it has less than two neighborhoods in $V - D$. Thus, $|V - D| < \frac{n}{2}$ and there is no inverse arrow dominating set. \square

Proposition 4.8. Let G be a graph having a γ_{ar} -set D . If there is an end vertex in G or a vertex with degree two, then G has no inverse arrow dominating set.

Proof . Let w be a vertex in G with degree one or two, then w don't belongs to any arrow dominating set. Hence, if G has an arrow dominating set, it has no inverse arrow dominating set. \square

Theorem 4.9. Let G be a graph having a γ_{ar} -set D . Then, G has an inverse arrow dominating set if and only if:

1. $\gamma_{ar}(G) = \frac{n}{2}$.
2. $|N(x) \cap D| = 1$ for all $x \in V - D$.

3. $\deg(u) \geq 2$ in $G[V - D]$ for all $u \in V - D$.

Proof . Suppose that the three conditions hold to prove that G has an inverse arrow dominating set. By condition (1), $|V - D| = \frac{n}{2}$ and G of even order. Let $D^{-1} = V - D$, then by condition (2), every $u \in D^{-1}$ dominates only one vertex from D . By condition (3), every $u \in D^{-1}$ has at least two neighborhoods in D^{-1} . Thus, D^{-1} is a minimum inverse arrow dominating set in G .

Now, we prove the converse by contradiction. Suppose that G has an inverse arrow dominating set D^{-1} , then if $\gamma_{ar}(G) > \frac{n}{2}$, this gives a contradiction with Remark 4.4. If there is a vertex $x \in V - D$ and $|N(x) \cap D| = 2$, then $x \notin D^{-1}$ since it will dominates two vertices. So, $|D^{-1}| < \frac{n}{2}$ which is contradict Remark 4.3. In similar way, we get a contradiction with Proposition 4.8 if there is a vertex $x \in V - D$ with degree one or two. \square

Corollary 4.10. $D^{-1} = V - D$ and $\gamma_{ar}^{-1}(G) = \gamma_{ar}(G)$ For any graph G .

Proof . It is clear, depending on Theorem 4.9. \square

Proposition 4.11. For any graph G with $\gamma_{ar}^{-1}(G)$, then G has no γ_{ar} -required vertex.

Proof . Since G has inverse arrow dominating set, then $D^{-1} = V - D$ and $\gamma_{ar}^{-1}(G) = \gamma_{ar}(G)$ according to Corollary 4.10. Then, every γ_{ar} -required vertex belongs to both D and D^{-1} which is contradict the fact $D \cap D^{-1} = \phi$. \square

Proposition 4.12. Let G be a graph having arrow dominating set, then:

1. $K_n, W_n, K_{n,m}$ and $\overline{K_{n,m}}$ have no inverse arrow dominating set.
2. $\overline{C_n}$ has an inverse dominating set if and only if $n = 6$.

Proof . It is clear $\gamma_{ar}^{-1}(\overline{C_6}) = 3$ by Theorem 3.7 and the other graphs has no inverse arrow dominating sets depending on Remark 4.4. \square

5. Conclusion

A new model of vertices domination “arrow domination” is introduced with its inverse model. More properties, bounds and applications are studied.

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