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An overview of Bayesian prediction of future record statistics using upper record ranked set sampling scheme

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Abstract

Two sample prediction is considered for a one-parameter exponential distribution. In practical experiments using sampling methods based on different schemes is crucial. This paper addresses the problem of Bayesian prediction of record values from a future sequence, based on an upper record ranked set sampling scheme. First, under an upper record ranked set sample (RRSS) and different values of hyperparameters, point predictions have been studied with respect to both symmetric and asymmetric loss functions. These predictors are compared in the sense of their mean squared prediction errors. Next, we have derived two prediction intervals for future record values. Prediction intervals are compared in terms of coverage probability and expected length. Finally, a simulation study is performed to compare the performances of the predictors. The real data set is also analyzed for an illustration of the findings.

Keywords: Record values; Prediction; Mean squared prediction error; Loss function; Coverage probability; Record ranked set sampling scheme.
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1. Introduction

The purpose of statistical prediction is to infer the values of future statistics based on available observations. Predicting future record statistics is one of the important problems in real-life situations. This is why many authors attracted to research on it. For instance, we would be interested

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in predicting the degree of temperature or amount of snowfall in the future when the present record will be broken. For more details see, [1, 2, 3] and the references contained therein. Prediction problems discussed in the literature can be broadly divided into two types, referred to as one-sample and two-sample prediction. Here, the prediction of records based on records (two-sample prediction) has been addressed. Many authors have studied prediction problems. For example, Raqab and Balakrishnan [4] studied prediction intervals for future records, Ahmadi and Mirmostafaee [5] considered the same problem in a parametric setting and obtained prediction intervals for order statistics as well as for the mean life time from a future sample based on observed usual records from an exponential distribution. Ahmadi and Balakrishnan [6] addressed the question' How can one predict future records (order statistics) from an independent Y-sequence based on order statistics (records) from X-sequence? and derived several nonparametric prediction intervals. MirMostafaee and Ahmadi [7] considered point prediction of future order statistics from an exponential distribution, Salehi et al. [8] considered the prediction of order statistics and record values based on an ordered ranked set sample. Kaminsky and Nelson [9] considered the prediction of order statistics in one-sample as well as two-sample cases, and obtained linear point predictors and prediction intervals based on samples from location-scale families. For some other discussion in this regard, one may also refer to Vock and Balakrishnan [10], and Asgharzadeh et al. [11]. The Bayesian approach, as an alternative to the classical approach, is in statistical inference. Its principle is to incorporate the information in the parameters' history through a prior distribution assuming, a known form of distribution. The parameters of a prior distribution called hyperparameters. In the Bayesian inference, the performance of the predictor depends on the prior distribution and also on the loss function used. A symmetric loss as SEL function is found in different fields. The symmetric nature of this function gives equal weight to overestimation as well as underestimation, while in the estimation of parameters of lifetime model, overestimation may be more severe than underestimation or vice-versa. An asymmetric loss function, is also useful. For example, in the estimation of reliability and failure rate function, an overestimation is usually much more serious than underestimate. In this article, we use general entropy loss function, because of different choices of a parameter value involved in the loss function can produce different symmetric and asymmetric loss functions. The first aim of this paper is to study the bahavior of the predictors from a future sequence using an upper record ranked set sample (RRSS) as an informative sample with respect to both symmetric and asymmetric loss functions for different values of hyperparameters. Our second aim of this paper is to discuss the problem of Bayesian prediction intervals for records from a future sequence. There are some experiments where have been done sequentially, and only record-breaking data are observed. These types of data have been used in a wide variety of practical experiments, such as oil and mining surveys, quality control, hydrology, sports achievements, seismology, the strength of materials, economics, industry, and climatology. Now, we give a brief description of records. Let $\{X_i, i \geq 1\}$ be an infinite sequence of independent and identically distributed (i.i.d) random variables with absolutely continuous cumulative distribution function (cdf) $F(x;\theta)$ and corresponding probability density function (pdf) $f(x;\theta)$ where θ is possibly a vector real-valued parameter. An observation X_j is called an upper record if $X_i < X_j$ for every $i < j$, i.e., X_j is an upper record value if its value exceeds all previous observations. An analogous definition can be given for lower record value. Some key references are [12, 13, 14]. In practical experiments, obtaining observations for the variable is costly and time consuming. In such situations, considering suitable sampling schemes, in order to reduce the cost and increase efficiency are crucial. The record ranked set sample (RRSS) scheme, as an alternative method for generating record-breaking data, has been formally proposed by Salehi and Ahmadi [15]. Among the authors who worked on this scheme, Salehi and Ahmadi [16] considered the estimation of stress and strength using upper RRSS from the exponential distribution. They also, with the collaboration of

Dey [17], made a comparison between RRSS scheme and the ordinary record statistics in estimating the unknown parameter of the proportional hazard rate model. They showed that the RRSS scheme out-performs the ordinary record statistics in the frequentist/Bayesian point and interval estimation under that family of distributions. Eskandarzadeh et al. [18] obtained information measures for the RRSS scheme. Paul and Thomas [19] proposed concomitant the RRSS for situations that measuring of the variable of interest is costly or even impossible. Hassan et al. [20] considered Bayesian comparison of the RRSS and the ordinary records based on generalized inverted exponential distribution. Safaryian et al. [21] proposed some improved estimators including the preliminary test estimator as well as stein-type shrinkage estimator for stress-strength reliability using record ranked set sampling scheme. Recently, Sadeghpour et al. [22] considered the estimation of stress and strength reliability using lower record ranked set sampling scheme under the generalized exponential distribution. For formal definition of upper RRSS, suppose we have n independent random sequences where the ith sequence sampling is finished when the ith upper record is observed. The only observations available for analysis are the last upper record value in each sequence. This process is called, the RRSS scheme because it is designed based on the plan of RSS defined by McIntyre [23]. Let us denote the last upper record for the ith sequence in this plan by $R_{i,i}$ then $\mathbf{R} = (R_{1,1}, R_{2,2}, \cdots, R_{n,n})^\top$ will be an upper RRSS of size n. The following observational procedure illustrates this plan

$$\begin{array}{llllll} 1: & \underline{R_{(1)1}} & & & & \rightarrow R_{1,1} = R_{(1)1} \\ 2: & R_{(1)2} & \underline{R_{(2)2}} & & & \rightarrow R_{2,2} = R_{(2)2} \\ & \vdots & \vdots & \vdots & \vdots & \vdots \\ n: & R_{(1)n} & R_{(2)n} & \cdots & \underline{R_{(n)n}} & \rightarrow R_{n,n} = R_{(n)n} \end{array} \tag{1.1}$$

where $R_{(i)j}$ is the ith ordinary upper record in the jth sequence. Notice that, here $R_{i,i}$'s are independent random variables but not necessarily ordered with probability one. However, from Lemma 2.1 in [15], it can be seen that $R_{i,i}$'s have stochastic orders in probability, i.e. for $i < j$; $P(R_{i,i} < R_{j,j}) > \frac{1}{2}$. Thus, if $R_{i,i}$'s are upper records then using the marginal density of ordinary records the joint pdf of the elements of $\mathbf{R}$ readily follows as (see, [2])

$$f_{\mathbf{R}}(\mathbf{r};\theta) = \prod_{i=1}^{n} \frac{\{-\log \bar{F}(r_{i,i};\theta)\}^{i-1}}{(i-1)!} f(r_{i,i};\theta), \quad \theta \in \Theta, \tag{1.2}$$

where $\bar{F}(.) = 1 - F(.)$, $\mathbf{r} = (r_{1,1}, r_{2,2}, \cdots, r_{n,n})^\top$ is the observed vector of $\mathbf{R}$ and Θ is the parameter space. Substituting $\bar{F}$ by F gives that of the lower RRSS. In RRSS scheme, a specific statistic of a future sequence is predicted based on an observed sample. These two samples are supposed to be independent and called *future samples* and *informative samples*, respectively. More precisely, we intend to predict ordinary upper record statistics arising from a future sequence based on an observed upper RRSS through the Bayesian viewpoint. Throughout this paper, suppose that X has an exponential distribution with mean $\frac{1}{\theta}$ denoted by $X \sim Exp(\theta)$ if its cdf has the form

$$F(x;\theta) = 1 - e^{-\theta x}, \quad x > 0, \ \theta > 0. \tag{1.3}$$

This distribution playes a important role in many reliability analysis, and lifetime applications. The rest of the study consists of six sections. Two Bayesian prediction intervals for upper future record values using RRSS are presented in Section 2. Also, in Section 3, we derive several Bayes point predictors with respect to both symmetric and asymmetric loss functions. In Section 4, for different values of hyperparameters, we use a Monte Carlo simulation study to investigate and compare the performance of the predictors based on the RRSS scheme. Also, a real data on the daily heat degree is analyzed in Section 5. Finally a conclusion is presented in Section 6.

2. Bayesian prediction interval for future record values

This section deals with the problem of Bayesian prediction interval. At initial, suppose $\mathbf{r} = (r_{1,1}, r_{2,2}, \cdots, r_{n,n})^\top$ be the observation of random vector $\mathbf{R} = (R_{1,1}, R_{2,2}, \cdots, R_{n,n})^\top$, an upper RRSS of size n from $Exp(\theta)$. Then, from (1.2), the likelihood function of θ given the observed data $\mathbf{r}$ can be written as

$$L(\theta|\mathbf{r}) \propto \theta^N e^{-\theta \sum_{i=1}^{n} r_{i,i}}, \tag{2.1}$$

where $N = \frac{n(n+1)}{2}$ and $t = \sum_{i=1}^{n} r_{i,i}$ is the observed value of $T = \sum_{i=1}^{n} R_{i,i}$. The joint density of $\mathbf{R}$ belongs to a full rank exponential family and so T is a complete sufficient statistic for θ and plays a key role in all the results that can be achieved. In the Bayesian process we need a prior distribution for the parameter of θ. Because θ is nonnegative, a natural the choice for the priors of θ would be to assume that its density is of the following forms

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \qquad \theta > 0. \tag{2.2}$$

The hyperparameters $\alpha(> 0)$ and $\beta(> 0)$ are chosen to reflect the prior knowledge about θ, $\Gamma(.)$ denotes the complete gamma function. By using Bayes theorem, we get the posterior distribution of θ given $\mathbf{r}$ as follows

$$\Pi(\theta|\mathbf{r}) = \frac{(\beta+t)^{N+\alpha}}{\Gamma(N+\alpha)} \theta^{N+\alpha-1} e^{-\theta(\beta+t)}. \tag{2.3}$$

In other words, $\theta|\mathbf{R} \sim Gamma(N+\alpha, \beta+T)$. Suppose that Y_s is s-th upper record value from a future sequence. Then the Bayesian predictive density function of Y_s given $\mathbf{r}$ is given by

$$f^*_{Y_s}(y|\mathbf{r}) = \int_\theta f_{Y_s}(y|\theta)\Pi(\theta|\mathbf{r})d\theta.$$

For the exponential distribution, the conditional density of Y_s is gamma distribution with parameters s and θ. By using this fact and (2.3) we obtain

$$f^*_{Y_s}(y|\mathbf{r}) = \frac{1}{B(s, N+\alpha)y} p(y)^s (1-p(y))^{N+\alpha}, \quad y > 0, \tag{2.4}$$

where $p(y) = \frac{y}{\beta+t+y}$. Clearly, the Bayesian predictive density function is free of θ. We will use it as a criterion for predicting future upper record values. Based on Bayesian predictive density function the following results hold:

(i) For $s \geq 1$, the random variable $Y_s|\mathbf{r}$ with the Bayesian predictive density function $f^*_{Y_s}(y|\mathbf{r})$ is identical in distribution with $(\beta+T)\frac{W}{1-W}$ where $W \sim Beta(s, N+\alpha)$, and also W and T are independent.

(ii) $\frac{N+\alpha}{s(\beta+T)} Y_s|\mathbf{r} \sim F_{2s,2(N+\alpha)}$.

A predictor can be either a point or an interval predictor. In many practical data-analytic situations we are interested in using the observations from an initial sample to construct an interval that will have a present probability of containing some statistic based on a future sample of observations from the same underlying distribution. Such an interval is called a prediction interval for the statistic of interest. In the following, we will present two Bayesian prediction intervals for Y_s. We start with the survival method.

2.1. Survival method

In order to construct Bayesian prediction interval based on the survival method, we first find $\bar{F}^*_{Y_s}(y|\mathbf{r})$. From (2.4) we have

$$\bar{F}^*_{Y_s}(y|\mathbf{r}) = \int_y^\infty \frac{1}{B(s, N+\alpha)z} p(z)^s (1-p(z))^{N+\alpha} dz, \tag{2.5}$$

where $\bar{F}^*_{Y_s}(y|\mathbf{r})$ indicates the Bayesian predictive survival function for Y_s. Thus, a $100(1-\gamma)\%$ Bayesian prediction interval for Y_s is such that

$$P(L(\mathbf{r}) < Y_s < U(\mathbf{r})|\mathbf{r}) = 1-\gamma,$$

or equivalently

$$\bar{F}^*_{Y_s}(L(\mathbf{r})|\mathbf{r}) = 1-\frac{\gamma}{2}, \quad \bar{F}^*_{Y_s}(U(\mathbf{r})|\mathbf{r}) = \frac{\gamma}{2},$$

where $L(\mathbf{r})$ and $U(\mathbf{r})$ are the lower and upper limits satisfying

$$L(\mathbf{r}) = \frac{B_{1-\frac{\gamma}{2}}(s, N+\alpha)}{1-B_{1-\frac{\gamma}{2}}(s, N+\alpha)}(\beta+T) \;\&\; U(\mathbf{r}) = \frac{B_{\frac{\gamma}{2}}(s, N+\alpha)}{1-B_{\frac{\gamma}{2}}(s, N+\alpha)}(\beta+T)$$

where $B_\gamma(c_1, c_2)$ represent the 100γ-th percentile of $Beta(c_1, c_2)$ distribution.

2.2. The highest posterior density (HPD) method

Now, we construct a HPD interval for Y_s. From (2.4) it is clearly concluded that $f^*_{Y_s}(y|\mathbf{r})$ is continuous and uni-modal pdf, then a $100(1-\gamma)\%$ HPD interval for Y_s is given by (ζ_1, ζ_2) where ζ_1 and ζ_2 are the solutions the following equations

$$\int_{\zeta_1}^{\zeta_2} f^*_{Y_s}(y|\mathbf{r})dy = 1-\gamma \;,\; 0<\gamma<1,$$

$$f^*_{Y_s}(\zeta_1|\mathbf{r}) = f^*_{Y_s}(\zeta_2|\mathbf{r}),$$

or equivalently, we need to solve the following two equations:

$$\int_{\zeta_1}^{\zeta_2} \frac{1}{B(s, N+\alpha)y} p(y)^s (1-p(y))^{N+\alpha} dy = 1-\gamma \;, \quad 0<\gamma<1,$$

$$\left(\frac{\zeta_1}{\zeta_2}\right)^{s-1} = \left(\frac{\beta+t+\zeta_1}{\beta+t+\zeta_2}\right)^{N+\alpha+s}.$$

The Monte Carlo simulation has to be used here. We have illustrated in Sections 4 and 5 these results. Now, we consider the case where $s=1$. In this case, the Bayesian predictive density function in (2.4) for y is strictly decreasing function, $(d/dy)f^*_{Y_1}(y|\mathbf{r}) = -\frac{N+\alpha+1}{\beta+t+y} < 0$. It follows that $100(1-\gamma)\%$ Bayesian prediction interval for Y_1 based on the HPD method, is given by $(0, (\beta+T)\{\gamma^{-\frac{1}{N+\alpha}}-1\})$.

3. Bayesian point prediction for future record values

In this section, we discuss how to predict future record values based on an upper RRSS. We obtain Bayes point predictor for record values from a future sequence with respect to both symmetric, and asymmetric loss functions such as general entropy loss (GEL) function, precautionary loss (PL) function, squared error loss (SEL) function, weighted squared error loss (WSEL) function and linear-exponential (LINEX) loss function. The loss function we considered for Bayes predictor is the GEL function of the form

$$L_1(\theta,\hat{\theta}) = q\big[\big(\frac{\hat{\theta}}{\theta}\big)^p - p\ln\big(\frac{\hat{\theta}}{\theta}\big) - 1\big]; \quad q>0;\ p\neq 0,$$

where $\hat{\theta}$ is an estimate of θ. This loss function is a generalization of the entropy loss function. It was proposed by Calabria and Pulcini [24] and its minimum occurs at $\hat{\theta}=\theta$. Because the value of q does not play any role in the optimization of the loss function, so without loss of generality, we can assume $q=1$. The Bayes point predictor for Y_s under GEL function is of the form

$$\hat{Y}_s^{(BG)} = \big(E_{f^*}(Y_s^{-p}|\mathbf{r})\big)^{-\frac{1}{p}}, \tag{3.1}$$

provided that expectation exist and is finite. The proper choice for p is a challenging task for an analyst because of it reflects the asymmetry of the loss function. Note that

- If we put $p=-2$ in (3.1), it provides the Bayes point predictor under an asymmetric loss as PL function of the form $L_2(\theta,\hat{\theta}) = \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}$. Under the PL function, the Bayes point predictor for Y_s is $(E_{f^*}(Y_s^2|\mathbf{r}))^{\frac{1}{2}}$ given by the following

$$\hat{Y}_s^{(BP)} = \big(\frac{(s-1)(s-2)}{(N+\alpha)(N+\alpha+1)}\big)^{\frac{1}{2}}(\beta+T), \quad s>2. \tag{3.2}$$

Now, we can find the mean squared prediction error for $\hat{Y}_s^{(BP)}$ as follows

$$E\big(\hat{Y}_s^{(BP)} - Y_s\big)^2 = \frac{(s-1)(s-2)}{(N+\alpha)(N+\alpha+1)}\big(\frac{N}{\theta^2} + (\beta+\frac{N}{\theta})^2\big) - \frac{2s}{\theta}\big(\frac{(s-1)(s-2)}{(N+\alpha)(N+\alpha+1)}\big)^{\frac{1}{2}}(\beta+\frac{N}{\theta}) + \frac{s+s^2}{\theta^2}. \tag{3.3}$$

- When $p=-1$, the Bayes point predictor in (3.1) coincide with the Bayes point predictor under a symmetric loss as SEL function of the form $L_3(\theta,\hat{\theta}) = (\hat{\theta}-\theta)^2$. Under the SEL function, the Bayes point predictor for Y_s is $E_{f^*}(Y_s|\mathbf{r})$ given by the following

$$\hat{Y}_s^{(BS)} = \frac{s}{N+\alpha-1}(\beta+T), \quad s>0. \tag{3.4}$$

We can find the mean squared prediction error for $\hat{Y}_s^{(BS)}$ as follows

$$E\big(\hat{Y}_s^{(BS)} - Y_s\big)^2 = \frac{s^2}{(N+\alpha-1)^2}\big(\frac{N}{\theta^2} + (\beta+\frac{N}{\theta})^2\big) - \frac{2s^2}{\theta(N+\alpha-1)}(\beta+\frac{N}{\theta}) + \frac{s+s^2}{\theta^2}. \tag{3.5}$$

• In what follows when $p = 1$, the Bayes point predictor in (3.1) coincide with the Bayes point predictor under an asymmetric loss as WSEL function of the form $L_4(\theta, \hat{\theta}) = \frac{(\hat{\theta}-\theta)^2}{\theta}$. The Bayes point predictor for Y_s under the WSEL function may be defined as $(E_{f^*}(Y_s^{-1}|\mathbf{r}))^{-1}$. From (2.4) we obtain

$$E_{f^*}(Y_s^{-1}|\mathbf{r}) = \frac{N+\alpha}{(s-1)(\beta+T)},$$

therefore

$$\hat{Y}_s^{(BW)} = \frac{s-1}{N+\alpha}(\beta+T), \quad s>1. \tag{3.6}$$

We find

$$E\big(\hat{Y}_s^{(BW)} - Y_s\big)^2 = \frac{(s-1)^2}{(N+\alpha)^2}\big(\frac{N}{\theta^2} + (\beta+\frac{N}{\theta})^2\big) - \frac{2s(s-1)}{\theta(N+\alpha)}(\beta+\frac{N}{\theta}) + \frac{s+s^2}{\theta^2}. \tag{3.7}$$

Another usefull loss function is the linear-exponential (LINEX) loss function. The LINEX loss function is a convex but asymmetric loss function, and defined as $L_5(\theta, \hat{\theta}) = b[e^{a(\hat{\theta}-\theta)} - a(\hat{\theta}-\theta) - 1]$, where $b > 0$ is the scale parameter and $a \neq 0$ is the shape parameter. Without loss of generality, it can be assumed that $b = 1$. A positive value of a is used when the overestimation is more serious than an underestimation while a negative values of a is vice-versa. For a close to zero, this loss function is approximately SEL and therefore almost symmetric (see, [25, 26]). Under the LINEX loss function, the Bayes point predictor for Y_s is

$$\hat{Y}_s^{(BL)} = -\frac{1}{a}\ln E_{f^*}(e^{-aY_s}|\mathbf{r}) = -\frac{1}{a}\ln\big[\int_0^\infty e^{-ay} f^*_{Y_s}(y|\mathbf{r})dy\big], \tag{3.8}$$

provided that expectation exist and is finite. The Bayes point predictor under LINEX loss function can not be obtained in explicit form. It must be solved by an appropriate numerical method. Moreover, the following may be noted:

(i) Under the modified SEL function of the form $L_6(\theta, \hat{\theta}) = (\frac{\hat{\theta}-\theta}{\theta})^2$, the Bayes point predictor for Y_s is, $\hat{Y}_s^{(BM)} = \frac{E_{f^*}(Y_s^{-1}|\mathbf{r})}{E_{f^*}(Y_s^{-2}|\mathbf{r})} = \frac{s-2}{N+\alpha+1}(\beta+T), \quad s>2.$

(ii) Under the zero-one loss function of the form

$$L_7(\theta, \hat{\theta}_i) = \begin{cases} 0 & \theta \in \Theta_i, \\ 1 & \theta \in \Theta_j, \end{cases}$$

for $i \neq j$, the Bayes point predictor for Y_s is the mode of the Bayesian predictive density function (2.4). So, $\hat{Y}_s^{(BZ)} = \frac{s-1}{N+\alpha+1}(\beta+T), \quad s>1.$

4. Simulation study

In order to monitor the performance of the Bayes predictors obtained in the preceding sections, a Monte Carlo simulation is carried out. Next, we use the following Algorithm 1, for the Bayes predictors based on the RRSS scheme.

•Algorithm 1

Step 1. For given values of the hyperparameters α and β we generate θ from the prior distribution (2.2).

Step 2. For generated θ an upper RRSS of size n, say $\mathbf{r} = (r_{1,1}, r_{2,2}, \cdots, r_{n,n})^\top$, is generated from (1.1).
Step 3. For given $\mathbf{r}$ in **Step 2**, compute the Bayes point predictors with respect to both symmetric and asymmetric loss functions in (3.2), (3.4), (3.6) and (3.8).
Step 4. Compute (for 1000 iterations), the values of $MSPEs$ for all the Bayes point predictors obtained in **Step 3**. $MSPE$ represents the estimate of mean squared prediction error.
Step 5. Compute EL and CP.

The acronyms EL and CP stand for the average of expected length and coverage probability, respectively. Let (A, B) be a Bayesian prediction interval for Y_s. Moreover, (A_i, B_i), $i = 1, ..., 1000$, observed values of lower and upper bound of the proposed prediction interval. Thus

$$EL = \frac{1}{1000}\sum_{i=1}^{1000}(B_i - A_i) \quad \& \quad CP = \frac{1}{1000}\sum_{i=1}^{1000} I(A_i \leq Y_s \leq B_i)$$

where $I(.)$ is the indicator function. In this simulation study, we consider $s = 2, 4, 6, 7$, and upper record values with sizes $n = 2(1)5$. Also, we use $p = -2, -1, 1$ in the GEL function. As mentioned in the Section 2, these different choices allowed us to compare the performance of the Bayes predictors under different loss functions. The values of the parameter in LINEX loss function is considered to be $a = 1, 2$. The value of the model parameter is considered to be $\theta = 1.17$. For Bayesian prediction computing, we consider different hyperparameters. In first case, we use $(\alpha = 1, \beta = 1)$ and in the second case, we take $(\alpha = 1.5, \beta = 1.5)$, which are denoted by cases P-I and P-II, respectively. Also, we assumed that the case $\alpha = \beta = 0$ i.e. Jeffrey's noninformative prior, since we do not have any prior information. The corresponding results of the Bayes point predictors are reported in Tables 1-3. The values in parentheses are MSPEs. For $s = 2$ in PL function, we obtained invalid point predictors. These cases are indicated by a dash (-). We observe the following points from the numerical results in Tables 1-3:

• All the MSPEs obtained are decreasing with respect to the sample sizes of upper record values, n, when all other parameters are kept fixed and increasing with respect to s. For more details, see Figures 1(a), 1(b) and 1(c).
• The MSPEs does not have a constant bahavior. But when n increases, predictors under the SEL function (which is a symmetric loss function) almost performs better than the WSEL and PL functions (which are asymmetric loss functions). For more details, see Figures 2(a), 2(b) and 2(c).
• Also, in LINEX loss function the MSPEs are increasing in a and s.
• The Bayes predictors based on informative priors perform much better than the Bayes predictors based on noninformative prior in terms of MSPEs.
• The MSPEs decreases with the joint increase of the hyperparameters (α and β), which is reasonable. Notice that, from (2.2) the priors have the same means. However, the variance of case P-II is smaller than the others. For this reason, case P-II works better than others in terms of MSPE.

Table 1: Bayes point predictors for Y_s and MSPEs for noninformative prior.

	$p=-2$ (**PL**)				$p=-1$ (**SEL**)			
$n\downarrow$	$s=2$	$s=4$	$s=6$	$s=7$	$s=2$	$s=4$	$s=6$	$s=7$
2	-	1.86 (6.55)	3.39 (11.26)	4.15 (14.22)	2.62 (4.35)	5.25 (14.51)	7.87 (30.48)	9.19 (40.64)
3	-	1.95 (5.71)	3.57 (8.94)	4.37 (10.89)	2.07 (2.26)	4.13 (6.15)	6.20 (11.67)	7.23 (15.04)
4	-	2.00 (5.38)	3.65 (7.86)	4.47 (9.35)	1.90 (1.85)	3.81 (4.48)	5.71 (7.90)	6.66 (9.91)
5	-	2.03 (5.10)	3.70 (7.28)	4.54 (8.53)	1.83 (1.69)	3.67 (3.85)	5.50 (6.49)	6.42 (7.98)

	$p=1$ (**WSEL**)			
$n\downarrow$	$s=2$	$s=4$	$s=6$	$s=7$
2	0.87 (2.42)	2.62 (5.81)	4.37 (11.13)	5.25 (14.51)
3	0.86 (2.29)	2.58 (4.72)	4.31 (8.10)	5.17 (10.16)
4	0.85(2.25)	2.57 (4.28)	4.28 (6.89)	5.14(8.42)
5	0.85 (2.23)	2.57 (4.06)	4.28 (6.29)	5.14 (7.55)

	LINEX $a=1$				**LINEX** $a=2$			
$n\downarrow$	$s=2$	$s=4$	$s=6$	$s=7$	$s=2$	$s=4$	$s=6$	$s=7$
2	1.27 (0.41)	2.32 (1.91)	3.23 (4.89)	3.66 (7.04)	0.96 (0.65)	1.75 (3.06)	2.45 (7.74)	2.76 (11.05)
3	1.25 (0.33)	2.38 (1.50)	3.41 (3.76)	3.91 (5.37)	0.98 (0.59)	1.85 (2.63)	2.65 (6.49)	3.02 (9.18)
4	1.24 (0.29)	2.41 (1.28)	3.51 (3.16)	4.04 (4.48)	0.98 (0.56)	1.90 (2.41)	2.76 (5.82)	3.17 (8.17)
5	1.24 (0.27)	2.43 (1.16)	3.57 (2.80)	4.12 (3.94)	0.98 (0.54)	1.93(2.29)	2.83 (5.42)	3.26 (7.56)

Table 2: Bayes point predictors for Y_s and their MSPEs for case P-I.

	$p=-2$ (**PL**)				$p=-1$ (**SEL**)			
$n\downarrow$	$s=2$	$s=4$	$s=6$	$s=7$	$s=2$	$s=4$	$s=6$	$s=7$
2	-	1.99 (5.69)	3.62 (8.95)	4.44 (10.93)	2.42 (2.86)	4.83 (8.55)	7.25 (17.06)	8.46 (22.38)
3	-	2.02 (5.35)	3.69 (8.04)	4.51 (9.63)	2.06 (2.05)	4.11 (5.28)	6.17 (9.71)	7.20 (12.37)
4	-	2.04 (5.13)	3.72 (7.43)	4.56 (8.76)	1.91 (1.78)	3.83 (4.22)	5.74 (7.33)	6.70 (9.13)
5	-	2.06 (5.00)	3.75 (7.05)	4.60 (8.21)	1.85 (1.66)	3.69 (3.75)	5.54 (6.26)	6.46 (7.67)

	$p=1$ (**WSEL**)			
$n\downarrow$	$s=2$	$s=4$	$s=6$	$s=7$
2	0.91 (2.35)	2.72 (4.68)	4.53 (8.20)	5.44 (10.37)
3	0.89 (2.23)	2.64 (4.32)	4.41 (7.13)	5.29 (8.80)
4	0.87 (2.21)	2.61 (4.10)	4.35 (6.47)	5.22 (7.83)
5	0.87 (2.20)	2.60 (3.96)	4.33 (6.07)	5.19 (7.24)

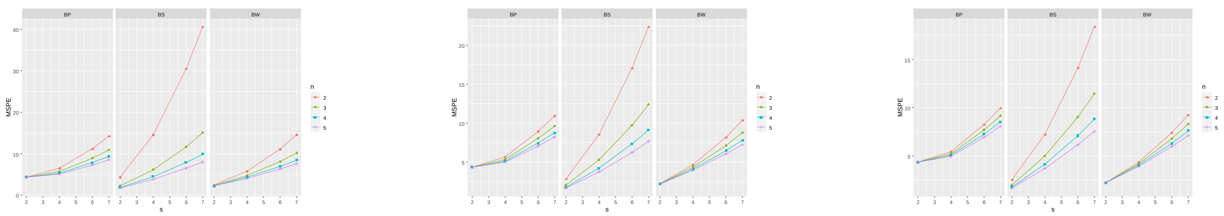
	LINEX $a=1$				**LINEX** $a=2$			
$n\downarrow$	$s=2$	$s=4$	$s=6$	$s=7$	$s=2$	$s=4$	$s=6$	$s=7$
2	1.31 (0.27)	2.44 (1.34)	3.46 (3.54)	3.93 (5.16)	1.01 (0.54)	1.87 (2.55)	2.64 (6.49)	2.99 (9.29)
3	1.28 (0.27)	2.44 (1.26)	3.52 (3.18)	4.04 (4.57)	1.00 (0.54)	1.90 (2.41)	2.74 (5.94)	3.13 (8.41)
4	1.26 (0.26)	2.44 (1.16)	3.57 (2.87)	4.11 (4.07)	1.00 (0.53)	1.93 (2.30)	2.81 (5.54)	3.23 (7.77)
5	1.25 (0.25)	2.45 (1.09)	3.61 (2.63)	4.17 (3.71)	1.00 (0.52)	1.94(2.22)	2.86 (5.26)	3.30 (7.33)

Table 3: Bayes point predictors for Y_s and their MSPEs for case P-II.

	$p=-2$ (**PL**)				$p=-1$ (**SEL**)			
$n\downarrow$	$s=2$	$s=4$	$s=6$	$s=7$	$s=2$	$s=4$	$s=6$	$s=7$
2	-	2.03 (5.42)	3.71 (8.26)	4.54 (9.96)	2.36 (2.54)	4.71 (7.25)	7.07 (14.14)	8.25 (18.40)
3	-	2.05 (5.21)	3.73 (7.70)	4.57 (9.17)	2.05 (1.97)	4.10 (4.99)	6.15 (9.05)	7.18 (11.47)
4	-	2.06 (5.06)	3.75 (7.25)	4.60 (8.52)	1.92 (1.76)	3.83 (4.12)	5.75 (7.10)	6.71 (8.82)
5	-	2.07 (4.95)	3.77 (6.94)	4.62 (8.06)	1.85 (1.65)	3.70 (3.70)	5.55 (6.16)	6.48 (7.54)

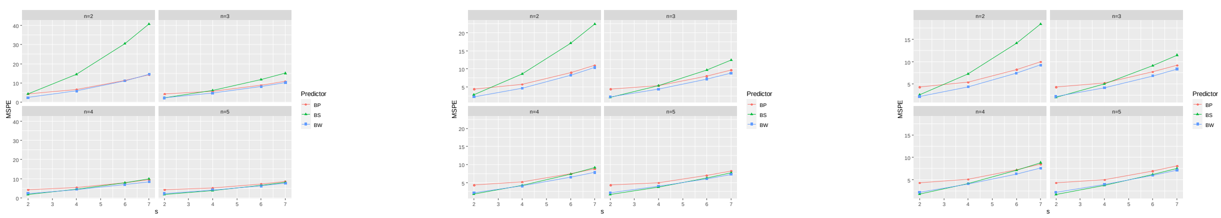
	$p=1$ (**WSEL**)			
$n\downarrow$	$s=2$	$s=4$	$s=6$	$s=7$
2	0.92 (2.20)	2.75 (4.37)	4.58 (7.41)	5.50 (9.26)
3	0.89 (2.20)	2.67 (4.18)	4.45 (6.78)	5.33 (8.32)
4	0.88 (2.20)	2.63 (4.03)	4.38 (6.30)	5.25 (7.59)
5	0.87 (2.19)	2.61 (3.92)	4.35 (5.97)	5.21 (7.11)

	LINEX							
	$a=1$				$a=2$			
$n\downarrow$	$s=2$	$s=4$	$s=6$	$s=7$	$s=2$	$s=4$	$s=6$	$s=7$
2	1.33 (0.24)	2.48 (1.18)	3.53 (3.16)	4.01 (4.62)	1.02 (0.51)	1.91 (2.41)	2.70 (6.11)	3.07 (8.74)
3	1.28 (.25)	2.46 (1.17)	3.56 (2.97)	4.09 (4.27)	1.01 (0.52)	1.92 (2.33)	2.77 (5.73)	3.18 (8.11)
4	1.27 (0.25)	2.46 (1.11)	3.60 (2.75)	4.15 (3.90)	1.00 (0.52)	1.94 (2.25)	2.83 (5.42)	3.26 (7.60)
5	1.26 (0.24)	2.46 (1.06)	3.62 (2.56)	4.19 (3.60)	0.99 (0.52)	1.95(2.19)	2.87 (5.19)	3.31 (7.23)



(a) Noninformative prior. (b) Case P-I. (c) Case P-II.

Figure 1: Plot the MSPEs for $s=2,4,6,7$ and $n=2(1)5$.



(a) Noninformative prior. (b) Case P-I. (c) Case P-II.

Figure 2: The Behavior of the MSPEs under PL, SEL and WSEL functions.

In Tables 4-6 according to the survival and HPD methods the values of EL and CP are reported. Here, the values in parentheses referes to the CPs. The following points are extracted from Tables 4- 6:

• The ELs and CPs improve when n gets large thus the ELs decreases with increasing n, also when n increases the CPs become closer to prediction coefficient of 0.95 in most cases.

• The Bayesian prediction intervals based on informative priors as compared to the Bayesian prediction intervals based on noninformative prior, acts very good, in terms of ELs and CPs.

• In more details, the ELs on the basis of the HPD method are smaller than those of survival method and everywhere the CPs of the HPD method is more than survival method. Note that these Bayesian intervals improve which was expected as the hyperparameters (α and β) increase jointly. So the Bayesian prediction intervals based on case P-II has a better performance than others.

Table 4: The values of EL and CP for Y_s according to the survival and HPD methods for noninformative prior, $\gamma = 0.05$.

	SURVIVAL				**HPD**			
$n\downarrow$	$s=2$	$s=4$	$s=6$	$s=7$	$s=2$	$s=4$	$s=6$	$s=7$
2	10.44 (0.937)	18.37 (0.942)	26.09 (0.939)	29.92 (0.938)	6.88 (0.959)	12.99 (0.960)	18.73 (0.958)	21.44 (0.959)
3	6.80 (0.942)	11.12 (0.943)	15.14 (0.946)	17.11 (0.947)	5.51 (0.958)	9.62 (0.957)	13.08 (0.959)	14.81 (0.958)
4	5.80 (0.942)	9.09 (0.947)	12.03 (0.951)	13.44 (0.950)	5.02 (0.960)	8.17 (0.959)	10.91 (0.958)	12.23 (0.952)
5	5.37 (0.946)	8.21 (0.951)	10.67 (0.951)	11.84 (0.952)	4.68 (0.961)	7.53 (0.962)	9.92 (0.960)	11.01 (0.961)

Table 5: The values of EL and CP for Y_s according to the survival and HPD methods for case P-I, $\gamma = 0.05$.

	SURVIVAL				**HPD**			
$n\downarrow$	$s=2$	$s=4$	$s=6$	$s=7$	$s=2$	$s=4$	$s=6$	$s=7$
2	8.79 (0.946)	14.97 (0.945)	20.90 (0.946)	23.82 (0.946)	6.56 (0.961)	11.71 (0.962)	16.52 (0.960)	19.04 (0.960)
3	6.58 (0.948)	10.60 (0.947)	14.30 (0.947)	16.10 (0.948)	5.47 (0.960)	9.19 (0.961)	12.59 (0.962)	14.14 (0.961)
4	5.76 (0.947)	8.97 (0.949)	11.82 (0.952)	13.18 (0.951)	5.00 (0.960)	8.04 (0.962)	10.76 (0.962)	12.05 (0.963)
5	5.36 (0.950)	8.19 (0.954)	10.61 (0.953)	11.76 (0.953)	4.66 (0.961)	7.51 (0.964)	9.85 (0.963)	10.94 (0.965)

Table 6: The values of EL and CP for Y_s according to the survival and HPD methods for case P-II, $\gamma = 0.05$.

	SURVIVAL				**HPD**			
$n\downarrow$	$s=2$	$s=4$	$s=6$	$s=7$	$s=2$	$s=4$	$s=6$	$s=7$
2	8.31 (0.949)	13.97 (0.950)	19.36 (0.949)	22.01 (0.951)	6.40 (0.959)	11.21 (0.960)	15.86 (0.963)	18.01 (0.964)
3	6.49 (0.949)	10.39 (0.950)	13.96 (0.951)	15.71 (0.952)	5.45 (0.963)	9.08 (0.964)	12.32 (0.965)	13.89 (0.964)
4	5.74 (0.950)	8.91 (0.952)	11.72 (0.953)	13.06 (0.952)	4.98 (0.964)	8.03 (0.963)	10.70 (0.971)	12.02 (0.972)
5	5.37 (0.953)	8.18 (0.9554)	10.58 (0.955)	11.72 (0.954)	4.70 (0.970)	7.52 (0.972)	9.84 (0.975)	10.92 (0.974)

5. Real data analysis

We illustrate the performances of the predictors by considering a real data set (say Y) on the daily heat degree (in degrees celsius) of the month of January for three years 2009, 2014, 2018 of Nova Scotia Province and BACCARO PT Station for government Canada. (the given data are available at this address: first, log in to the website: www.climate.weather.gc.ca and click on the Historical Data. The web site of Historical Data will then appear). The Weibull distribution was one of the best models fitted on Y based on suggestions of Easy Fit software. In more detail, based on the maximum likelihood approach, we have $Y \sim Weibull(a = 5.4, b = 33.3)$ with the pdf of $f(y) = \frac{a}{b} y^{\frac{1}{b}-1} e^{-ay^{\frac{1}{b}}}$. From the above expression, it is evident that $X = aY^{\frac{1}{b}}$ has exponential distribution. The p-value of the goodness of fit test Kolmogorov-Smirnov of the one-parameter exponential distribution on the converted data, X, is 0.997 which supports the adequacy of the fitting. From this transformed data set, it can be shown that $\mathbf{r} = (5.946, 5.935, 5.959)$, so we have $n = 3$ (or $N = 6$). From $T = \sum_{i=1}^{n} R_{i,i}$ we also have $T = 17.84$. It is observed that $R_{i,i}$'s are not necessarily ordered, as mentioned earlier in the introduction section. Hence, the results obtained in the preceding sections can be applied to on X and $\mathbf{r}$. Since the mean squared prediction errors of the point predictors are a function of θ, we estimated the values of the mean squared prediction errors by substituting their respective the Bayes estimator of θ. In which case from (2.3), the Bayes estimator of θ under PL function, SEL function and WSEL function, are derived as

$$\hat{\theta}^{(BP)} = (E(\theta^2|\mathbf{r}))^{\frac{1}{2}} = \frac{((N+\alpha)(N+\alpha+1))^{\frac{1}{2}}}{\beta+T} = \begin{cases} 0.36 & \text{for } \alpha=\beta=0, \\ 0.40 & \text{for } \alpha=\beta=1, \\ 0.41 & \text{for } \alpha=\beta=1.5, \end{cases}$$

$$\hat{\theta}^{(BS)} = E(\theta|\mathbf{r}) = \frac{N+\alpha}{\beta+T} = \begin{cases} 0.34 & \text{for } \alpha=\beta=0, \\ 0.37 & \text{for } \alpha=\beta=1, \\ 0.39 & \text{for } \alpha=\beta=1.5, \end{cases}$$

$$\hat{\theta}^{(BW)} = (E(\theta^{-1}|\mathbf{r}))^{-1} = \frac{N+\alpha-1}{\beta+T} = \begin{cases} 0.28 & \text{for } \alpha=\beta=0, \\ 0.32 & \text{for } \alpha=\beta=1, \\ 0.34 & \text{for } \alpha=\beta=1.5, \end{cases}$$

respectively. All the above Bayes estimators are uniqe for the parameter of θ, and so are admissible. The Bayes estimator of θ under LINEX loss function is

$$\hat{\theta}^{(BL)} = -\frac{1}{a}\ln E(e^{-a\theta}|\mathbf{r}) = -\frac{1}{a}\ln\Big[\int_0^\infty e^{-a\theta}\Pi(\theta|\mathbf{r})d\theta\Big], \tag{5.1}$$

provided that $E(e^{-a\theta}|\mathbf{r})$ exist and is finite. The integral (5.1) does not have an explicit solution. It must be solved by an appropriate numerical method. Clearly, the mean squared prediction error under LINEX loss function also does not have a closed form. To this end, to evaluate the performance of predictors, the bootstrap method is employed.

Finally, we predict the upper record values ($s = 3, 4, 6, 7$), and their MSPEs as well as the Bayesian prediction intervals. Tables 7-12 summarize the results from a real data set. Note that in Tables 7-9, the values in parentheses are the MSPEs. Further, in Tables 10-12, the first entry is the Bayesian prediction interval and the second entry is the EL.

Table 7: Bayes point predictors for Y_s and their MSPEs for noninformative prior based on a real data.

	$p=-2$ (**PL**)				$p=-1$ (**SEL**)			
$n\downarrow$	$s=3$	$s=4$	$s=6$	$s=7$	$s=3$	$s=4$	$s=6$	$s=7$
3	3.89 (46.56)	6.74 (59.54)	12.31 (93.32)	15.08 (113.72)	10.70 (48.80)	14.27 (74.97)	21.41 (142.16)	24.98 (183.18)

	$p=1$ (**WSEL**)			
$n\downarrow$	$s=3$	$s=4$	$s=6$	$s=7$
3	5.95 (59.41)	8.92 (82.75)	14.87 (142.16)	17.84 (178.23)

	LINEX							
	$a=1$				$a=2$			
$n\downarrow$	$s=3$	$s=4$	$s=6$	$s=7$	$s=3$	$s=4$	$s=6$	$s=7$
3	3.92 (28.16)	5.10 (51.91)	7.30 (124.18)	8.33 (173.53)	2.76 (44.51)	3.59 (80.69)	5.15 (187.77)	5.89 (259.35)

Table 8: Bayes point predictors for Y_s and their MSPEs for case P-I based on a real data.

	$p=-2$ (**PL**)				$p=-1$ (**SEL**)			
$n\downarrow$	$s=3$	$s=4$	$s=6$	$s=7$	$s=3$	$s=4$	$s=6$	$s=7$
3	3.56 (40.71)	6.17 (52.45)	11.26 (81.45)	13.79 (98.79)	9.42 (32.85)	12.56 (48.74)	18.84 (87.93)	21.98 (111.23)

	$p=1$ (**WSEL**)			
$n\downarrow$	$s=3$	$s=4$	$s=6$	$s=7$
3	5.38 (48.48)	8.07 (66.76)	13.46 (111.14)	16.15 (137.22)

	LINEX							
	$a=1$				$a=2$			
$n\downarrow$	$s=3$	$s=4$	$s=6$	$s=7$	$s=3$	$s=4$	$s=6$	$s=7$
3	3.56 (22.86)	4.64 (41.98)	6.67 (99.88)	7.64 (139.30)	2.54 (35.61)	3.32 (64.48)	4.78 (149.80)	5.48 (206.77)

Table 9: Bayes point predictors for Y_s and their MSPEs for case P-II based on a real data.

	$p=-2$ (**PL**)				$p=-1$ (**SEL**)			
$n\downarrow$	$s=3$	$s=4$	$s=6$	$s=7$	$s=3$	$s=4$	$s=6$	$s=7$
3	3.42 (38.30)	5.93 (49.54)	10.83 (77.08)	13.27 (93.12)	8.93 (28.46)	11.90 (41.72)	17.85 (73.93)	20.83 (92.87)

	$p=1$ (**WSEL**)			
$n\downarrow$	$s=3$	$s=4$	$s=6$	$s=7$
3	5.16 (44.51)	7.74 (61.22)	12.89 (101.24)	15.47 (124.54)

	LINEX							
	$a=1$				$a=2$			
$n\downarrow$	$s=3$	$s=4$	$s=6$	$s=7$	$s=3$	$s=4$	$s=6$	$s=7$
3	3.42 (20.88)	4.46 (38.29)	6.42 (90.89)	7.38 (126.64)	2.46 (32.33)	3.21 (58.51)	4.63 (135.79)	5.31 (187.39)

Table 10: 95% prediction intervals for Y_s and their ELs according to the survival and HPD methods for noninformative prior based on a real data.

	SURVIVAL				**HPD**			
$n\downarrow$	$s=3$	$s=4$	$s=6$	$s=7$	$s=3$	$s=4$	$s=6$	$s=7$
3	(1.66,33.26)	(2.83,41.77)	(5.44,58.47)	(6.82,66.73)	(0.40,25.11)	(1.38,30.27)	(3.49,43.13)	(4.94,59.04)
	31.6	38.94	53.03	59.91	24.71	28.89	39.64	54.10

Table 11: 95% prediction intervals for Y_s and their ELs according to the survival and HPD methods for case P-I based on a real data.

	SURVIVAL				**HPD**			
$n\downarrow$	$s=3$	$s=4$	$s=6$	$s=7$	$s=3$	$s=4$	$s=6$	$s=7$
3	(1.52,28.27)	(2.61,35.37)	(5.04,49.26)	(6.33,56.12)	(0.90,18.36)	(2.38,27.78)	(3.84,35.60)	(3.99,46.16)
	26.75	32.76	44.22	49.79	17.46	25.40	31.76	42.17

Table 12: 95% prediction intervals for Y_s and their ELs according to the survival and HPD methods for case P-II based on a real data.

	SURVIVAL				HPD			
$n\downarrow$	$s=3$	$s=4$	$s=6$	$s=7$	$s=3$	$s=4$	$s=6$	$s=7$
3	(1.47,26.42)	(2.52,32.99)	(4.87,45.85)	(6.12,52.19)	(0.45,17.51)	(1.98,25.40)	(3.15,29.70)	(4.32,38.76)
	24.95	30.47	40.98	46.07	17.06	23.42	26.55	34.44

According to the results of Tables 7-12, the predictors based on cases P-I and P-II performs better than the case noninformative prior. More precisely, we conclude that the case P-II is better than the others. Clearly all the MSPEs are increasing in s. As noticed in the simulation study, here too, the HPD method has a better performance than the survival method. Thus, the obtained results in this section confirm the results of the previous one.

6. Conclusion

In this paper, we considered prediction of record values from a future sequence, when the only observed data are upper RRSS's. By considering the parent distribution as exponential, we have derived several Bayes point predictors with respect to both symmetric and asymmetric loss functions. We have also compared the point predictors in terms of the MSPEs. Results showed that the MSPEs are decreasing with respect to the sample sizes of upper record values, n, and the predictors under the SEL function improve when n gets large as compared to the WSEL and PL functions. Next, our study shows that Bayesian prediction interval obtained based on the HPD method has further coverage probability and shorter expected length as compared to the survival method. Therefore, we suggest that whenever n is large then Bayesian prediction approach based on the RRSS scheme should be used. Also, the performance of predictors based on case P-II are superior than others. Because Bayesian inference is sensitive to the choice of hyperparameters for informative priors, care must be taken in the selection of values. Finally, the efficiency of some of the obtained results is illustrated throughout using real data. The results in the real data section confirm the results of the simulation section.

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