Abstract
In this paper, using fixed point methods, we prove the fuzzy orthogonally $*$-$n$-derivation on orthogonally fuzzy $C^*$-algebra for the functional equation

$$f\left(\frac{\mu x + \mu y}{2} + \mu w\right) + f\left(\frac{\mu x + \mu w}{2} + \mu y\right) + f\left(\frac{\mu y + \mu w}{2} + \mu x\right) = 2\mu f(x) - 2\mu f(y) - 2\mu f(w).$$

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1. Introduction
The stability problem of functional equations originated from the question of Ulam [20] concerning the stability of group homomorphisms. Hyers [10] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ theorem was generalized by Th.M. Rassias [17] for linear mappings by considering an unbounded Cauchy difference. Park et al. proved stability homomorphisms and derivations in Banach algebras, Banach ternary algebras, $C^*$-algebras, Lie $C^*$-algebras and $C^*$-ternary algebras [11, 13, 16]. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem [4, 8, 9, 12, 18].

In the following, we review the basic definitions of orthogonally sets [2, 6] and the definition of fuzzy normed spaces [2, 13], which can be consider the main definition of our paper.
Definition 1.1. Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be an binary relation. If $\perp$ satisfies the following condition
\[ \exists x_0; (\forall y; y \perp x_0) \text{ or } (\forall y; x_0 \perp y), \]
then $X$ is called an orthogonally set (briefly O-set). We denote this O-set by $(X, \perp)$.

Definition 1.2. Let $(X, \perp)$ be an orthogonally space. A function $f : X \to X$ is called $\perp$-preserving, if $x \perp y$, then $f(x) \perp f(y)$ for all $x, y$.

Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is called orthogonally sequence (briefly O-sequence) if
\[ (\forall n; x_n \perp x_{n+1}) \text{ or } (\forall n; x_{n+1} \perp x_n). \]

Definition 1.3. Let $(X, \perp, d)$ be an orthogonally metric space (i.e., $(X, \perp)$ is an O-set and $(X, d)$ is a metric space), then $f : X \to X$ is $\perp$-continuous at $a \in X$ if for each O-sequence $\{a_n\}_{n \in \mathbb{N}}$ in $X$, $a_n \to a$, implies $f(a_n) \to f(a)$. Also, $f$ is $\perp$-continuous on $X$ if $f$ is $\perp$-continuous at each $a \in X$.

It is obvious to see that every continuous mapping is $\perp$-continuous.

Definition 1.4. Let $(X, \perp, d)$ be an orthogonally metric space, then $X$ is orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent.

Every complete metric space is O-complete, but the converse is not true.

Definition 1.5. Let $(X, \perp, d)$ be an orthogonally metric space and $0 < \lambda < 1$. A mapping $f : X \to X$ is said to be orthogonality contraction with Lipschitz constant $\lambda$ if
\[ d(fx, fy) \leq \lambda d(x, y) \text{ if } x \perp y. \]

Definition 1.6. Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies the following conditions:
1. $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.7. [3] Let $(X, d, \perp)$ be an O-complete generalized metric space and $0 \leq \lambda < 1$. Let $T : X \to X$ be $\perp$-preserving, $\perp$-continuous and $\perp$-$\lambda$-contraction. Consider the “O-sequence of successive approximations with initial element $x_0$”: $x_0, T(x_0), T^2(x_0), \ldots, T^n(x_0), \ldots$. Then, either $d(T^n(x_0), T^{n+1}(x_0)) = \infty$ for all $n \geq 0$, or there exists a positive integer $n_0$ such that $d(T^n(x_0), T^{n+1}(x_0)) < \infty$ for all $n \geq n_0$. If the second alternative holds, then
i) the O-sequence of $\{T^n(x_0)\}$ is convergent to a fixed point $x^*$ of $T$.
ii) $x^*$ is the unique fixed point of $T$ in $X^* = \{y \in X : d(T^n(x_0), y) < \infty\}$.
iii) $d(y, x^*) \leq \frac{1}{1-\lambda} d(y, T(y))$ for all $y \in X^*$.

In the following, we use the definition of fuzzy normed spaces to investigate a fuzzy version of the Hyers-Ulam stability for the functional equation in the fuzzy normed algebra setting [1, 3, 14, 19].
Definition 1.8. Let $X$ be a vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on $X$ if

1. $N(x, t) = 0$ for all $x \in X$ and $t \in \mathbb{R}$ with $t \leq 0$;
2. $x = 0$ if and only if $N(x, t) = 1$ for all $x \in X$ and $t > 0$;
3. $N(cx, t) = N(x, \frac{t}{c})$ for all $x \in X$ and $c \neq 0$;
4. $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$ for all $x, y \in X$ and $s, t \in \mathbb{R}$;
5. $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t \to \infty} N(x, t) = 1$ for all $x \in X$ and $t \in \mathbb{R}$;
6. $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair $(X, N)$ is called a fuzzy normed vector space.

Definition 1.9. Let $(X, \perp, N)$ be an orthogonally fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ or converges if there exists $x \in X$ such that

$$
\lim_{n \to \infty} N(x_n - x, t) = 1
$$

for all $t > 0$. In this case, $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n \to \infty} x_n = x$.

Definition 1.10. Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is called Cauchy if, for each $\epsilon > 0$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy normed vector space is said to be complete and the complete fuzzy normed vector space is called a fuzzy Banach space. We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces $X$ and $Y$ is continuous at a point $x_0 \in X$ if, for each sequence $\{x_n\}$ converging to $x_0 \in X$, the sequence $f(x_n)$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on $X$.

Definition 1.11. A fuzzy normed algebra $(X, N)$ is a fuzzy normed space $(X, N)$ with the algebraic structure such that

1. $N(xy, ts) \geq N(x, t)N(y, s)$ for all $x, y \in X$ and $t, s > 0$.

Every normed algebra $(X, ||\cdot||)$ defines a fuzzy normed algebra $(X, N)$, where $N$ is defined by

$$
N(x, t) = \frac{t}{t + ||x||}
$$

for all $t > 0$. This space is called the induced fuzzy normed algebra.

Definition 1.12. Let $(X, N)$ and $(Y, N)$ be fuzzy normed algebras.
1. A $C$-linear mapping $f : X \to Y$ is called a homomorphism if

$$
f(xy) = f(x)f(y)
$$

for all $x, y \in X$.

2. An $C$-linear mapping $f : X \to X$ is called a derivation if

$$
f(xy) = f(x)y + xf(y)
$$

for all $x, y \in X$. 

Definition 1.13. Let \((U, N)\) be a fuzzy Banach algebra. Then an involution on \(U\) is a mapping 
\[ u \mapsto u^* \] 
for all \(u \in U\) which satisfies the following:
(a) \(u^{**} = u\) for any \(u \in U\);
(b) \((\alpha u + \beta v)^* = \alpha u^* + \beta v^*\);
(c) \((uv)^* = v^* u^*\) for any \(u, v \in U\).

If, in addition, \(N(u^* u, t) = N(u, t)N(u, s)\) and \(N(u^*, t) = N(u, t)\) for all \(u \in U\) and \(t, s > 0\), then \(U\) is a fuzzy C*-algebra.

2. Stability of \(*\)-\(n\)-derivation in orthogonally fuzzy \(C^*\)-algebras

Throughout this section, assume that \((A, \|\cdot\|_1, \perp_1)\) with norm \(N_A\) and \(a \perp_1 b\) if \(ab^* = b^* a = 0\) is an orthogonally fuzzy \(C^*\)-algebras. For any mapping \(f : A \to A\), we define
\[
\Delta_\mu f(x, y, w) := f\left(\frac{\mu x + \mu y}{2} + \mu w\right) + f\left(\frac{\mu x + \mu w}{2} + \mu y\right) + f\left(\frac{\mu y + \mu w}{2} + \mu x\right) - 2\mu f(x) - 2\mu f(y) - 2\mu f(w) \tag{2.1}
\]
for all \(\mu \in T^1 := \{v \in \mathbb{C} : |v| = 1\}\) and \(x, y, w \in A\) with \(x \perp y, y \perp w\) and \(w \perp x\). Note that a \(C\)-linear mapping \(\delta : A \to A\) is called a fuzzy \(C^*\)-algebra derivation on fuzzy \(C^*\)-algebra if \(\delta\) satisfies the following
\[
\delta(xy) = y\delta(x) + x\delta(y) \tag{2.2}
\]
and
\[
\delta(x^*) = \delta(x)^* \tag{2.3}
\]
for all \(x, y \in A\). We are going to investigate the generalized Hyers-Ulam stability of orthogonally fuzzy \(C^*\)-algebra derivation on orthogonally fuzzy \(C^*\)-algebra for the functional equation
\[
\Delta_\mu f(x, y, w) := 0. \tag{2.4}
\]

For a given mapping \(f : A \to A\), we define
\[
D(z_1, z_2, \ldots, z_n) := f(z_1 z_2 \ldots z_n) - f(z_1)z_2z_3\ldots z_n - z_1 f(z_2)z_3\ldots z_n - \ldots - z_1 z_2 \ldots z_{n-1} f(z_n) \tag{2.5}
\]
for all \(z_i \in A, z_i \perp z_{i+1}\).

Theorem 2.1. Let \(f : A \to A\) be a mapping for which there are functions \(\varphi : A^n \to [0, \infty)\) such that there exists an \(L < \frac{1}{2}\) with
\[
\varphi\left(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_n}{2}\right) \leq \frac{L \varphi(x_1, x_2, \ldots, x_n)}{2} \tag{2.6}
\]
\[
N_A(\Delta_\mu f(x_1, x_2, x_3), t) \geq \frac{t}{t + \varphi(x_1, x_2, x_3, \ldots, 0)} \tag{2.7}
\]
\[
N_A(D(x_1, x_2, \ldots, x_n), t) \geq \frac{t}{t + \varphi(x_1, x_2, \ldots, x_n)} \tag{2.8}
\]
for all $g, h \in \mathbb{T}$ and $x \in A$, with $x \perp x_{i+1}$ and $t > 0$. Then there exists a unique fuzzy orthogonally $*$-$n$-derivation $\delta : A \to A$ such that

$$N_A(f(x) - \delta(x), t) \geq \frac{(6 - 6L)t}{(6 - 6L)t + \varphi(x, x, \ldots, x)}$$

for all $x \in A$ and $t > 0$.

**Proof.** Putting $\mu = 1$, $x = x_1 = x_2 = x_3$ and $x_4 = \ldots = x_n = 0$ in (2.9), we have

$$N_A(3f(2x) - 6f(x), t) \geq \frac{t}{t + \varphi(x, x, \ldots, x)}$$

for all $x \in A$. So

$$N_A(f(x) - 2f\left(\frac{x}{2}\right), t) \geq \frac{3t}{3t + \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \ldots, 0\right)} \geq \frac{3t}{3t + \frac{L}{2} \phi(x, x, \ldots, 0)}$$

for all $x \in A$. Consider the set $X := \{g : A \to A\}$ and define the generalized metric $d$ on $X$, by

$$d(g, h) = \inf\{\mu \in \mathbb{R}^+ : N_A(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, x, \ldots, 0)} \}, \quad \forall x \in A, t > 0\}.$$

Now, we put the orthogonality relation $\perp$ on $X$ as follows

$$h \perp g \iff h(x) \perp g(x) \text{ or } g(x) \perp h(x)$$

for all $x \in A$ and $g, h \in X$. It is easy to show that $(X, d, \perp)$ is an O-complete generalized metric space.

Now, we consider the linear mapping $T : X \to X$ defined by $Tg(x) = 2g\left(\frac{x}{2}\right)$ for all $x \in A$. Let $g, h \in X$ with $g \perp h$ be such that $d(g, h) = \epsilon$. Then $N_A(g(x) - h(x), \epsilon t) \geq \frac{t}{t + \varphi(x, x, \ldots, 0)}$ for all $x \in A$ and $t > 0$. Hence

$$N_A(Tg(x) - Th(x), \epsilon t) = N_A(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), \epsilon t)$$

$$= N_A(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{\epsilon t}{2})$$

$$\geq \frac{\epsilon t}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \ldots, 0\right)$$

$$\geq \frac{\epsilon t}{2} + \frac{L \phi(x, x, x, \ldots, 0)}{2} = \frac{t}{t + \varphi(x, x, \ldots, 0)}$$

for all $x \in A$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Tg, Th) \leq L \epsilon$. Hence we see that

$$d(Tg, Th) \leq L d(g, h)$$

for all $g, h \in X$ with $g \perp h$, that is, $T$ is a strictly contractive self-mapping of $X$ with the Lipschitz constant $L$. Now, we show that $T$ is $\perp$-continuous. To this end, let $\{g_n\}_{n \in \mathbb{N}}$ be an O-sequence with
\(g_n \perp g_{n+1}\) or \(g_{n+1} \perp g_n\) in \((X, d, \perp)\) for all \(n \in \mathbb{N}\), which convergent to \(g \in X\) and let \(\epsilon > 0\) be given. Then there exists \(N \in \mathbb{N}\) and \(k \in \mathbb{R}^+\) with \(k < \epsilon\) such that

\[
N_A(g_n(x) - g(x), kL) \geq \frac{t}{t + \varphi(x, x, x, \ldots, 0)}
\]

for all \(x \in A\) and \(n \geq N\) and so

\[
N_A(2g_n(x) - 2g\left(\frac{x}{2}\right), kL) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \ldots, 0\right)}
\]

for all \(x \in A\) and \(n \geq N\). By inequality (2.10) and the definition of \(T\), we get

\[
N_A(Tg_n(x) - Tg(x), kL) \geq \frac{t}{t + \varphi(x, x, x, \ldots, 0)}
\]

for all \(x \in A\) and \(n \geq N\). Hence

\[
d(T(g_n), T(g)) \leq kL < \epsilon
\]

for all \(n \geq N\). It follows that \(T\) is \(\perp\)-continuous. It follows from (2.12) that

\[
N_A(f(x) - 2f\left(\frac{x}{2}\right), \frac{Lt}{2}) \geq \frac{3t}{3t + \varphi(x, x, x, \ldots, 0)}
\]

(2.13)

for all \(x \in A\) and \(t > 0\). This implies that \(d(f, Tf) \leq \frac{L}{6}\). By Theorem 4.7, there exists a mapping \(\delta : A \rightarrow A\) satisfying the following:

- \(\delta\) is a fixed point of \(T\), that is,

\[
\delta\left(\frac{x}{2}\right) = \frac{\delta(x)}{2}
\]

(2.14)

for all \(x \in A\). The mapping \(\delta\) is a unique fixed point of \(T\) in the set \(Y = \{h \in X : d(g, h) < \infty\}\). This implies that \(\delta\) is a unique mapping satisfying equation (2.10) such that there exists \(\mu \in (0, \infty)\) satisfying

\[
N_A(g(x) - H(x), \mu t) \geq \frac{t}{t + \varphi(x, x, x, \ldots, 0)}
\]

(2.15)

for all \(x \in X\) and \(t > 0\).

- \(d(T^n f, \delta) \rightarrow 0\) as \(n \rightarrow \infty\). This implies the equality

\[
N - \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right) = \delta(x)
\]

(2.16)

for all \(x \in X\).

- \(d(f, \delta) \leq d(f, Tf)\) with \(f \in X\), which implies the inequality \(d(f, \delta) \leq \frac{L}{6-6L}\). This implies that the inequality (2.22) holds. It follows from equations (2.17) and (2.18) that

\[
N_A\left(\delta\left(\frac{\mu x_1 + \mu x_2}{2} + \mu x_3\right), \delta\left(\frac{\mu x_1 + \mu x_3}{2} + \mu x_2\right) + \delta\left(\frac{\mu x_2 + \mu x_3}{2} + \mu x_1\right)\right)
\]

\[
- 2\mu\delta(x_1) - 2\mu\delta(x_2) - 2\mu\delta(x_3), t)
\]

\[
= N - \lim_{n \rightarrow \infty} 2^n f\left(\frac{\mu x_1 + \mu x_2}{2n+1} + \mu x_3\right) + 2^n f\left(\frac{\mu x_1 + \mu x_3}{2n} + \mu x_2\right) + 2^n f\left(\frac{\mu x_2 + \mu x_3}{2n} + \mu x_1\right)
\]

\[
- 2^{n+1} \mu f\left(\frac{x_1}{2n}\right) - 2^{n+1} \mu f\left(\frac{x_2}{2n}\right) - 2^{n+1} \mu f\left(\frac{x_3}{2n}\right), t)
\]

\[
\geq \lim_{n \rightarrow \infty} \frac{t}{2^n} + \varphi\left(\frac{x_1}{2n}, \frac{x_2}{2n}, \frac{x_3}{2n}, \ldots, 0\right) \geq \lim_{n \rightarrow \infty} \frac{t}{2^n} + \frac{L^n}{2^n} \varphi(x_1, x_2, x_3, \ldots, 0) \rightarrow 1
\]
for all $\mu \in \mathbb{T}, x_1, x_2, \ldots, x_n \in A$ and $t > 0$. Hence

$$\delta\left(\frac{\mu x_1 + \mu x_2}{2} + \mu x_3\right) + \delta\left(\frac{\mu x_1 + \mu x_3}{2} + \mu x_2\right) + \delta\left(\frac{\mu x_2 + \mu x_3}{2} + \mu x_1\right) - 2\mu\delta(x_1) - 2\mu\delta(x_2) - 2\mu\delta(x_3) = 0$$

for all $x_1, x_2, \ldots, x_n \in A$. So the mapping $\delta : A \to A$ is additive and $C$-linear. It follows from equations (2.8) that

$$N_A(2^n f \left(\frac{x_1 x_2 \ldots x_n}{2^n}\right) - 2^n f \left(\frac{x_1}{2^n}\right)x_2 x_3 \ldots x_n - x_1 2^n f \left(\frac{x_2}{2^n}\right)x_3 \ldots x_n - \ldots
-x_1 x_2 \ldots x_{n-1} 2^n f \left(\frac{x_n}{2^n}\right), 2^n t) \geq \frac{t}{l + \varphi(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \ldots, \frac{x_n}{2^n})}$$

for all $x_1, x_2, \ldots, x_n \in A$. Then

$$N_A(2^n f \left(\frac{x_1 x_2 \ldots x_n}{2^n}\right) - 2^n f \left(\frac{x_1}{2^n}\right)x_2 x_3 \ldots x_n - x_1 2^n f \left(\frac{x_2}{2^n}\right)x_3 \ldots x_n - \ldots
-x_1 x_2 \ldots x_{n-1} 2^n f \left(\frac{x_n}{2^n}\right), t) \geq \frac{t}{2^n l} + \frac{L''}{2^n} \varphi(x_1, x_2, \ldots, x_n) \to 1 \text{ when } n \to \infty$$

for all $x_1, x_2, \ldots, x_n \in A$ and $t > 0$. So $\delta(x_1 x_2 \ldots x_n) - \delta(x_1)x_2 x_3 \ldots x_n - x_1 \delta(x_2)x_3 \ldots x_n - \ldots
-x_1 x_2 \ldots x_{n-1}\delta(x_n) = 1$ for all $x_1, x_2, \ldots, x_n \in A$ and $t > 0$. It follows from equation (2.8) that

$$N_A(2^n f \left(\frac{x^*}{2^n}\right) - 2^n f \left(\frac{x}{2^n}\right)^*), 2^n t) \geq \frac{t}{l + \varphi(\frac{x}{2^n}, 0, \ldots, 0)} \quad (2.17)$$

for all $x \in A$ and $t > 0$. Then

$$N_A(2^n f \left(\frac{x^*}{2^n}\right) - 2^n f \left(\frac{x}{2^n}\right)^*), t) \geq \frac{t}{2^n l} + \varphi(\frac{x}{2^n}, 0, \ldots, 0) \geq \frac{t}{2^n l} + \frac{L''}{2^n} \varphi(x, 0, \ldots, 0) \quad (2.18)$$

for all $x \in A$ and $t > 0$. Since $\lim_{n \to \infty} \frac{t}{2^n l} = 1$ for all $x \in A$ and $t > 0$, we get $N_A(\delta(\frac{x^*}{2^n}) - \delta(\frac{x}{2^n})^*, t) = 1$ for all $x \in A$ and $t > 0$. Thus $\delta(\frac{x^*}{2^n}) = \delta(\frac{x}{2^n})^*$ for all $x \in A$.

**Corollary 2.2.** Let $\theta, p$ be non-negative real numbers with $0 < p < 1$. Suppose that $f : A \to A$ is a mapping, such that

$$N_A(\Delta_{\mu}f(x_1, x_2, x_3), t) \geq \frac{t}{l + \theta(\|x_1\|_A^p + \|x_2\|_A^p + \|x_3\|_A^p)} \quad (2.19)$$

$$N_A(D(x_1, x_2, \ldots, x_n), t) \geq \frac{t}{l + \theta(\|x_1\|_A^p \|x_2\|_A^p \ldots \|x_n\|_A^p)} \quad (2.20)$$

$$N_A(f(x^*) - f(x)^*, t) \geq \frac{t}{l + \theta(\|x\|_A^p)} \quad (2.21)$$
for all $\mu \in \mathbb{T}$, $x_1, x_2, \ldots, x_n \in A$ and $t > 0$ with and $x_i \perp x_{i+1}$. Then there exists a unique fuzzy orthogonally $*-n$-derivation $\delta : A \to A$ such that

$$N_A(f(x) - \delta(x), t) \geq \frac{t}{t + \theta(\|x\|_A^p)}$$

(2.22)

for all $x \in A$ and $t > 0$.

**Proof.** The proof follows from Theorem 2.1 by taking

$$\varphi(x_1, x_2, x_3, t) := \theta(\|x_1\|_A^p + \|x_2\|_A^p + \|x_3\|_A^p),$$

$$\varphi(x_1, x_2, \ldots, x_n, t) := \frac{t}{t + \theta(\|x_1\|_A^p \|x_2\|_A^p \cdots \|x_n\|_A^p)}$$

for all $x_1, x_2, x_3 \in A$ with $x_i \perp x_{i+1}$. Then we can choose $L = 2^{-p}$ and so the desired conclusion follows. □

**References**


