On Subclass of Analytic Univalent Functions Defined By Fractional Differ-integral Operator I

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Abstract

In this paper, we studied and introduced a new subclass of analytic univalent functions defined by differ–integral operator. We obtain distortion bounds, extreme points, and some theorem of this subclass.

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1. Introduction

Let $RHB$ denoted the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (n \in \{1, 2, \ldots\}) \quad (1)$$

Which are analytic and univalent functions in the unit disk: $U = \{ z : |z| < 1 \}$. $RHB$, we define the subclass of $RH$ consisting of the functions defined by the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0, n \in \{1, 2, \ldots\}) \quad (2)$$

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Let \( g \in RH \) and \( f \in RH \) if

\[
g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \quad (b_n \geq 0, n \in \{1, 2, \ldots\}) \quad (3)
\]

Then the Hadamard product or (Convolution) Defined by

\[
(f \ast g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n = (f \ast g)(z) \quad (4)
\]

**Definition 1:** A function \( f \in RH \) is said to be in the class \( RH_\mu^{\psi, \beta, \eta, A} \) if and only if satisfies the condition:

\[
\left| \begin{array}{c}
z[H_{0,z}^{\mu,\nu,\eta} f(z)]'' + \theta (1 - [H_{0,z}^{\mu,\nu,\eta} f(z)]') \\
\psi(1 - \beta) + z A [H_{0,z}^{\mu,\nu,\eta} f(z)]' + \theta (1 - [H_{0,z}^{\mu,\nu,\eta} f(z)])
\end{array} \right| < 1 \quad (5)
\]

where \( 0 \leq \beta < 1, A \geq 0, \theta \geq 0, -\infty < \mu < 1, \nu < 2, \eta \in R, \psi > 0 \),
and \( H_{0,z}^{\mu,\nu,\eta} \) is the fractional differintegral operator of order \( \mu \) \((-\infty < \mu < 1) \) (see Goyal and Prajapat [1]). For this operator if \( H_{0,z}^{\mu,\nu,\eta} : W(n) \rightarrow W(n) \) \((6)\), then

\[
H_{0,z}^{\mu,\nu,\eta} f(z) = z - \sum_{k=n+1}^{\infty} R_k(\mu, \nu, \eta) a_k z^k \quad (7)
\]

where \( R_k(\mu, \nu, \eta) = G(\mu, \nu, \eta) M(\mu, \nu, \eta, k) \) \((8)\) and

\[
G(\mu, \nu, \eta) = \frac{(1 - \nu)(1 - \mu + |\eta|)}{(1 - \nu + |\eta|)}, \quad M(\mu, \nu, \eta, k) = \frac{\Gamma(k+1)(1 - \nu + \eta)_k}{(1 - \nu)_k(1 - \mu + |\eta|)_k} \quad (9)
\]

Throughout the paper

\[
(a)_n = \prod_{k=1}^{n} a + k - 1 or = (a + 1)(a + 2)...(a + n + 1) \quad (10)
\]

is the factorial function, or if \( a > 0 \), then \( (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \) \((11)\) (where \( \Gamma \) is Euler’s Gamma function).

For \( z \neq 0 \), (1.5) may be expressed as

\[
H_{0,z}^{\mu,\nu,\eta} f(z) = \left\{ \begin{array}{l}
\frac{\Gamma(2 - \nu)(2 - \mu + n)}{\Gamma(2 - \mu + \eta)} z^\nu J_0^{\mu,\nu,\eta} f(z); 0 \leq \mu < 1 \\
\frac{\Gamma(2 - \nu)(2 - \mu + n)}{\Gamma(2 - \nu + \eta)} z^\nu I_0^{\mu,\nu,\eta} f(z); -\infty \leq \mu < 0
\end{array} \right. \quad (12)
\]

where \( J_0^{\mu,\nu,\eta} f(z) \) is the fractional derivative operator of order \( \mu (0 \leq \mu < 1) \), while \( I_0^{\mu,\nu,\eta} f(z) \) is the fractional integral operator of order \(-\mu (-\infty < \mu < 0)\) introduced and studied by Saigo ([4],[5]).

It may be worth noting that, by choosing \(-\infty < \mu = \nu < 1\) the operator \( H_{0,z}^{\mu,\nu,\eta} f(z) \) becomes

\[
H_{0,z}^{\mu,\nu,\eta} f(z) = H_{0,z}^{\mu} f(z) = \Gamma(2 - \mu) z^\mu D_z^\mu f(z) \quad (13)
\]
Where $D^\mu f(z)$ is respectively, the fractional integral operator of order $-\mu(-\infty < \mu < 0)$ and fractional derivative operator of order $\mu(0 \leq \mu < 1)$ considered by Owa[3] and defined by Liouville[2]. Further if $\mu = \nu = 0$, then

$$H^{0,0}_{0,z} f(z) = f(z) \quad (14)$$

and for $\mu \to 1^-$ and $\nu = 1$

$$\lim_\mu \to 1^- H^{\mu,1}_{0,z} f(z) = z f'(z) \quad (15)$$

**Theorem 1:** A function $f \in RH^\mu_{\psi}(\psi, \beta, \theta, \eta, A)$ if and only if

$$\sum_{n=2}^{\infty} R_n(\mu, \nu, \eta)(n(1 + A) - (1 - A + 2\theta)) a_n \leq (1 - \beta) \psi \quad (16)$$

where $0 \leq \beta < 1, A \geq 0, \theta \geq 0, -\infty < \mu < 1, \nu < 2, \eta \in R, \psi > 0$.

**Proof.** Assume that the inequality (16) holds true and let $|z| = 1$, we have

$$\left| z \left[ H^{0,0}_{0,z} f(z) \right]^\nu + \theta \left[ (1 - H^{0,0}_{0,z} f(z))' \right] \right| \left( H^{0,0}_{0,z} f(z) + \theta (1 - H^{0,0}_{0,z} f(z))' \right) < 1$$

So

$$\left| \sum_{n=2}^{\infty} \frac{\left( 1 - \beta \right) \psi + z \left[ A(-\sum_{n=2}^{\infty} R_n(\mu, \nu, \eta)(n(1 + A) - (1 - A + 2\theta)) a_n z^{n-1} \right] + \theta (\sum_{n=2}^{\infty} R_n(\mu, \nu, \eta)a_n z^{n-1}) \right| < 1$$

$$\left| \sum_{n=2}^{\infty} \frac{\left( 1 - \beta \right) \psi + \left[ -\sum_{n=2}^{\infty} A R_n(\mu, \nu, \eta)(n(1 + A) - (1 - A + 2\theta)) a_n z^{n-1} \right] + (\sum_{n=2}^{\infty} \theta R_n(\mu, \nu, \eta)a_n z^{n-1}) \right| < 1$$

$$\left| \sum_{n=2}^{\infty} \frac{\left( 1 - \beta \right) \psi - \sum_{n=2}^{\infty} R_n(\mu, \nu, \eta) [A(n(1 + A) - (1 - A + 2\theta))] a_n z^{n-1} \right| < 1$$

$$\left| \sum_{n=2}^{\infty} \frac{\left( 1 - \beta \right) \psi - \sum_{n=2}^{\infty} R_n(\mu, \nu, \eta) [A(n(1 + A) - (1 - A + 2\theta))] a_n z^{n-1} \right| < 1$$

Since $|z| = 1$, we get

$$\sum_{n=2}^{\infty} R_n(\mu, \nu, \eta)[n(1 + A) - (1 - A + 2\theta)] a_n \leq (1 - \beta) \psi$$
Conversely, suppose that is in the class \( RH_\varphi^\mu(\psi, \beta, \theta, \eta, A) \)

\[
\left| \frac{z \left[ H_{0,z}^{\mu,\nu} f(z) \right]'' + \theta (1 - \left[ H_{0,z}^{\mu,\nu} f(z) \right]' )}{(1 - \beta) + z A \left[ H_{0,z}^{\mu,\nu} f(z) \right]'' + \theta (1 - \left[ H_{0,z}^{\mu,\nu} f(z) \right]' )} \right| < 1
\]

For all \( z \), we have \( |Re(z)| \leq |z| \), since

\[
Re \left\{ \frac{\sum_{n=2}^{\infty} R_n(\mu, \nu, \eta)(n - 1 - \theta) a_n z^{n-1}}{(1 - \beta) \psi - \sum_{n=2}^{\infty} R_n(\mu, \nu, \eta) [A(n - 1) - \theta] a_n z^{n-1}} \right\} < 1
\]

Choose the value of \( z \) on the real axis and let \( z \to 1 \) we get

\[
\sum_{n=2}^{\infty} R_n(\mu, \nu, \eta)[n(1 + A) - (1 - A + 2\theta)] a_n \leq (1 - \beta) \psi
\]

**Corollary 2:** Let \( f \in RH_\varphi^\mu(\psi, \beta, \theta, \eta, A) \) then

\[
a_n \leq \frac{(1 - \beta) \psi}{R_n(\mu, \nu, \eta)[n(1 + A) - (1 - A + 2\theta)]}
\]

**Theorem 3:** Let the function \( f \) defined by (2) be in the class \( RH_\varphi^\mu(\psi, \beta, \theta, \eta, A) \). Then

\[
\left| f(z) \right| \leq \frac{(1 - \beta) \psi}{R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]} r^2
\]

The equality in (17) is attained by the function \( f \) given by

\[
f(z) = z - \frac{(1 - \beta) \psi}{R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]} z^2
\]

**Proof.** Since the function \( f \) defined by (2) in the \( RH_\varphi^\mu(\psi, \beta, \theta, \eta, A) \), we have from theorem 1

\[
\sum_{n=2}^{\infty} a_n \leq \frac{(1 - \beta) \psi}{R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]}
\]

Thus

\[
\left| f(z) \right| = \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n
\]

\[
\left| f(z) \right| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n
\]

\[
\left| f(z) \right| \leq r + r^2 \sum_{n=2}^{\infty} a_n
\]

\[
\left| f(z) \right| \leq r + \frac{(1 - \beta) \psi}{R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]} r^2
\]
Similarly

\[|f(z)| \geq |z| + \sum_{n=2}^{\infty} a_n |z|^n\]

\[|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n\]

\[|f(z)| \geq r - r^2 \sum_{n=2}^{\infty} a_n\]

\[|f(z)| \geq r - \frac{(1 - \beta)\psi}{R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]^2}\]

This completes the proof.

**Theorem 4:** Let \( f_1(z) = z \) and \( f_n(z) = z - (1 - \beta)\psi R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]^n \) Then \( f \) is in the class \( RH^\mu_\nu(\psi, \beta, \theta, \eta, A) \) if and only if can be expressed in the form \( f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z) \) where \( \sigma_n \geq 1, \sum_{n=1}^{\infty} \sigma_n = 1 \) or \( 1 = \sigma_1 + \sum_{n=2}^{\infty} \sigma_n \).

**Proof.** Assume that

\[ f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z) \]

\[ f(z) = \sigma_1 f_1(z) + \sum_{n=2}^{\infty} \sigma_n f_n(z) \]

\[ f(z) = \sigma_1 z + \sum_{n=2}^{\infty} \sigma_n \left( z - \frac{(1 - \beta)\psi}{R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]^n} \right) \]

\[ f(z) = \sigma_1 z + \sum_{n=2}^{\infty} \sigma_n z - \sum_{n=2}^{\infty} \sigma_n \frac{(1 - \beta)\psi}{R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]^n} \]

\[ f(z) = z \left( \sigma_1 + \sum_{n=2}^{\infty} \sigma_n \right) - \sum_{n=2}^{\infty} \sigma_n \frac{(1 - \beta)\psi}{R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]^n} \]

\[ f(z) = z - \sum_{n=2}^{\infty} \sigma_n \frac{(1 - \beta)\psi}{R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]^n} \]

From theorem 1, \( \sum_{n=2}^{\infty} R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)] a_n \leq (1 - \beta)\psi \). Then

\[ \sum_{n=2}^{\infty} \frac{R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]}{(1 - \beta)\psi} \sigma_n \frac{(1 - \beta)\psi}{R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]} \]

\[ \sum_{n=2}^{\infty} \sigma_n = 1 - \sigma_1 \leq 1 \]

Conversely, suppose that \( f \in RH^\mu_\nu(\psi, \beta, \theta, \eta, A) \) implies from theorem 1

\[ a_n \leq \frac{(1 - \beta)\psi}{R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)]} \]
Setting

\[ \sigma_n = \frac{R_n(\mu, \nu, \eta)(2(1+A) - (1 - A + 2\theta))}{(1 - \beta)\psi} a_n \]
\[ a_n = \frac{R_n(\mu, \nu, \eta)(2(1+A) - (1 - A + 2\theta))}{(1 - \beta)\psi} \sigma_n \]

\[ f(z) = z - \sum_{n=2}^{\infty} a_n z^n \]
\[ f(z) = z - \sum_{n=2}^{\infty} \frac{(1 - \beta)\psi}{R_n(\mu, \nu, \eta)(2(1+A) - (1 - A + 2\theta))} \sigma_n z^n \]

From

\[ f_n(z) = z - \frac{(1 - \beta)\psi}{R_n(\mu, \nu, \eta)(2(1+A) - (1 - A + 2\theta))} z_n = z - f_n(z) \]
\[ f(z) = z - \sum_{n=2}^{\infty} \sigma_n (z - f_n(z)) \]
\[ f(z) = z - \sum_{n=2}^{\infty} \sigma_n z + \sum_{n=2}^{\infty} \sigma_n f_n(z) \]
\[ f(z) = z(1 - \sum_{n=2}^{\infty} \sigma_n) + \sum_{n=2}^{\infty} \sigma_n f_n(z) \]
\[ f(z) = f_1 \sigma_1 + \sum_{n=2}^{\infty} \sigma_n f_n(z) \]
\[ f(z) = \sum_{n=2}^{\infty} \sigma_n f_n(z) \]

This complete the proof.

Now, we shall prove that the class \( RH_{\mu}^{\nu}(\psi, \theta, \eta, A) \) is closed under arithmetic mean. Let the function \( f_r(r = 2, 3, ..., m) \) define by

\[ f_r(z) = z - \sum_{n=2}^{\infty} a_{n,r} z^n \] (18)

**Theorem 5:** Let the function defined \( f \) by (2) be in the class \( RH_{\mu}^{\nu}(\psi, \theta, \eta, A) \). For every \( r = 2, 3, ..., m \), then the arithmetic mean of \( f_r(r = 2, 3, ..., m) \) is defined by

\[ g(z) = z - \sum_{n=2}^{\infty} c_n z^n , \quad (c_n \geq 2, n \geq 2, n \in \mathbb{N}) \]

Also belong to the class \( RH_{\mu}^{\nu}(\psi, \theta, \eta, A) \), where \( c_n = \frac{1}{m} \sum_{r=2}^{m} a_{n,r} \).
Proof. Since \( f_r \in RH^\mu_\nu(\psi, \beta, \theta, \eta, A) \), then from theorem 1 we get

\[
\sum_{n=2}^{\infty} R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)] a_n \leq (1 - \beta)\psi \quad (19)
\]

\[
\sum_{n=2}^{\infty} R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)] c_n
\]

\[
\sum_{n=2}^{\infty} R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)] \left[ \frac{1}{m} \sum_{r=2}^{m} a_{n,r} \right]
\]

\[
\frac{1}{m} \sum_{r=2}^{m} \left[ \sum_{n=2}^{\infty} R_n(\mu, \nu, \eta)[2(1 + A) - (1 - A + 2\theta)] a_{n,r} \right]
\]

by (19)

\[
\leq \frac{1}{m} \sum_{r=2}^{m} (1 - \beta)\psi
\]

\[
(1 - \beta)\psi \frac{1}{m} m
\]

\[
\leq (1 - \beta)\psi
\]

This complete the proof.

References


