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On Subclass of Analytic Univalent Functions Defined By Fractional Differ-integral Operator I

Rafid Habib Buti[a*], Ameen Khlaif Sachit[b]

[a] *Department of Mathematics and Computer Applications College of Sciences, University of Al-Muthanna, Samawa, Iraq.*
[b] *Babil Education Directorate ,Babil , Iraq.*

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Abstract

In this paper , we studied and introduced a new subclass of analytic univalent functions defined by differ – integral operator . We obtain distortion bounds, extreme points , and some theorem of this subclass .

Keywords: Univalent Function , Fractional Calculus , Differ-integral Operator .
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1. Introduction

Let RHB denoted the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (n \in \{1, 2, ..\}) \quad (1)$$

Which are analytic and univalent functions in the unit disk: $U = \{z : |z| < 1\}$. RHB , we define the subclass of RH consisting of the functions defined by the form :

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0, n \in \{1, 2, ..\}) \quad (2)$$

*Corresponding Author: Rafid Habib Buti
Email address: Rafidhbuti@yahoo.com, Ameenaltaa198315@gmail.com (Rafid Habib Buti[a*], Ameen Khlaif Sachit[b])

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Let $g \in RH$ and $f \in RH$ if

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, (b_n \geq 0, n \in \{1, 2, ..\}) \quad (3)$$

Then the Hadamard product or (Convolution) Defined by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n = (f * g)(z) \quad (4)$$

Definition 1: A function $f \in RH$ is said to be in the class $RH_{\nu}^{\mu}(\psi, \beta, \theta, \eta, A)$ if and only if satisfies the condition :

$$\left| \frac{z[H_{0,z}^{\mu,\nu,\eta} f(z)]'' + \theta(1 - [H_{0,z}^{\mu,\nu,\eta} f(z)]')}{\psi(1 - \beta) + zA[H_{0,z}^{\mu,\nu,\eta} f(z)]'' + \theta(1 - [H_{0,z}^{\mu,\nu,\eta} f(z)]')} \right| < 1 \quad (5)$$

where $0 \leq \beta < 1, A \geq 0, \theta \geq 0, -\infty < \mu < 1, \nu < 2, \eta \in R, \psi > 0$.
and $H_{0,z}^{\mu,\nu,\eta}$ is the fractional differintegral operator of order μ $(-\infty < \mu < 1)$ (see Goyal and Prajapat [1]). For this operator if $H_{0,z}^{\mu,\nu,\eta} : W(n) \to W(n) \quad (6)$, then

$$H_{0,z}^{\mu,\nu,\eta} f(z) = z - \sum_{k=n+1}^{\infty} R_k(\mu, \nu, \eta) a_k z^k, \ (a_k \geqslant 0, n \in \mathbb{N} = \{1, 2, 3, ...\}, z \in U) \quad (7)$$

where $R_k(\mu, \nu, \eta) = G(\mu, \nu, \eta) M(\mu, \nu, \eta, k) \quad (8)$ and

$$G(\mu, \nu, \eta) = \frac{(1 - \nu)(1 - \mu + |\eta)}{(1 - \nu + |\eta)}, M(\mu, \nu, \eta, k) = \frac{\Gamma(k + 1)(1 - \nu + \eta)_k}{(1 - \nu)_k (1 - \mu + |\eta)_k} \quad (9)$$

Throughout the paper

$$(a)_n = \prod_{k=1}^{n} a + k - 1 or = (a + 1)(a + 2)...(a + n + 1) \quad (10)$$

is the factorial function , or if $a > 0$, then $(a)_n = \dfrac{\Gamma(a + n)}{\Gamma(a)}$ (11) (where Γ is Euler‘s Gamma function).
For $z \neq 0$, (1.5) may be expressed as

$$H_{0,z}^{\mu,\nu,\eta} f(z) = \begin{cases} \dfrac{\Gamma(2 - \nu)\Gamma(2 - \mu + n)}{\Gamma(2 - \nu + \eta)} z^{\nu} J_{0,z}^{\mu,\nu,\eta} f(z); 0 \leq \mu < 1 \\ \dfrac{\Gamma(2 - \nu)\Gamma(2 - \mu + n)}{\Gamma(2 - \nu + \eta)} z^{\nu} I_{0,z}^{-\mu,\nu,\eta} f(z); -\infty \leq \mu < 0 \quad (12) \end{cases}$$

where $J_{0,z}^{\mu,\nu,\eta} f(z)$ is the fractional derivative operator of order $\mu(0 \leq \mu < 1)$, while $I_{0,z}^{-\mu,\nu,\eta} f(z)$is the fractional integral operator of order $-\mu(-\infty < \mu < 0)$ introduced and studied by Saigo ([4],[5]).
It may be worth noting that, by choosing $-\infty < \mu = \nu < 1$ the operator $H_{0,z}^{\mu,\nu,\eta} f(z)$ becomes

$$H_{0,z}^{\mu,\nu,\eta} f(z) = H_z^{\mu} f(z) = \Gamma(2 - \mu) z^{\mu} D_z^{\mu} f(z) \quad (13)$$

Where $D_z^{\mu} f(z)$ is respectively, the fractional integral operator of order $-\mu(-\infty < \mu < 0)$ and fractional derivative operator of order $\mu(0 \le \mu < 1)$ considered by Owa[3] and defined by Liouville[2]. Further if $\mu = \nu = 0$, then

$$H_{0,z}^{0,0,\eta} f(z) = f(z) \quad (14)$$

and for $\mu \to 1^-$ and $\nu = 1$

$$\lim_{\mu \to 1^-} H_{0,z}^{\mu,1,\eta} f(z) = z f^{'}(z) \quad (15)$$

Theorem 1: A function $f \in RH_{\nu}^{\mu}(\psi, \beta, \theta, \eta, A)$ if and only if

$$\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)[n(1+A) - (1-A+2\theta)]a_n \le (1-\beta)\psi \quad (16)$$

where $0 \le \beta < 1, A \ge 0, \theta \ge 0, -\infty < \mu < 1, \nu < 2, \eta \in R, \psi > 0$.
Proof . Assume that the inequality (16) holds true and let $|z| = 1$, we have

$$\left| \frac{z\left[H_{0,z}^{\mu,\nu,\eta} f(z)\right]^{''} + \theta(1 - \left[H_{0,z}^{\mu,\nu,\eta} f(z)\right]^{'})}{(1-\beta) + zA\left[H_{0,z}^{\mu,\nu,\eta} f(z)\right]^{''} + \theta(1 - \left[H_{0,z}^{\mu,\nu,\eta} f(z)\right]^{'})} \right|$$

So

$$\left| \frac{z[-\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)(n-1)a_n z^{n-2}] + \theta(\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)a_n z^{n-1})}{(1-\beta)\psi + z\left[A(-\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)(n-1)a_n z^{n-2})\right] + \theta(\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)a_n z^{n-1})} \right| < 1$$

$$\left| \frac{-\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)(n-1)a_n z^{n-1} + \sum_{n=2}^{\infty} \theta R_n(\mu,\nu,\eta)a_n z^{n-1}}{(1-\beta)\psi + [-\sum_{n=2}^{\infty} AR_n(\mu,\nu,\eta)(n-1)a_n z^{n-1}] + (\sum_{n=2}^{\infty} \theta R_n(\mu,\nu,\eta)a_n z^{n})} \right| < 1$$

$$\left| \frac{\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)(n-1-\theta)a_n z^{n-1}}{(1-\beta)\psi - \sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)\left[A(n-1)-\theta\right]a_n z^{n-1}} \right| < 1$$

$$\left| \sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)(n-(1+\theta))a_n z^{n-1} \right| < \left| (1-\beta)\psi - \sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)\left[A(n-1)-\theta\right]a_n z^{n-1} \right|$$

$$\left| \sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)(n-(1+\theta))a_n z^{n-1} \right| - \left| (1-\beta)\psi - \sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)\left[A(n-1)-\theta\right]a_n z^{n-1} \right|$$

$$\le \sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)(n-(1+\theta))a_n|z|^{n-1} - (1-\beta)\psi + \sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)\left[A(n-1)-\mu\right]a_n|z|^{n-1}$$

Since $|z| = 1$, we get

$$\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)[n(1+A) - (1-A+2\theta)]a_n \le (1-\beta)\psi$$

Conversely ,suppose that is in the class $RH_{\nu}^{\mu}(\psi,\beta,\theta,\eta,A)$

$$\left|\frac{z\left[H_{0,z}^{\mu,\nu,\eta}f(z)\right]^{''}+\theta(1-\left[H_{0,z}^{\mu,\nu,\eta}f(z)\right]^{'})}{(1-\beta)+zA\left[H_{0,z}^{\mu,\nu,\eta}f(z)\right]^{''}+\theta(1-\left[H_{0,z}^{\mu,\nu,\eta}f(z)\right]^{'})}\right|<1$$

For all z , we have$|Re(z)|\leq|z|$, since

$$Re\left\{\frac{\sum_{n=2}^{\infty}R_n(\mu,\nu,\eta)(n-1-\theta)a_nz^{n-1}}{(1-\beta)\psi-\sum_{n=2}^{\infty}R_n(\mu,\nu,\eta)\left[A(n-1)-\theta\right]a_nz^{n-1}}\right\}<1$$

Choose the value of$z$ on the real axis and let $z\to 1$ we get

$$\sum_{n=2}^{\infty}R_n(\mu,\nu,\eta)[n(1+A)-(1-A+2\theta)]a_n\leq(1-\beta)\psi$$

Corollary 2 : Let$f\in RH_{\nu}^{\mu}(\psi,\beta,\theta,\eta,A)$ then

$$a_n\leq\frac{(1-\beta)}{R_n(\mu,\nu,\eta)[n(1+A)-(1-A+2\theta)]}$$

Theorem 3: Let the function f defined by (2) be in the class $RH_{\nu}^{\mu}(\psi,\beta,\theta,\eta,A)$. Then

$$r-\frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}r^2\leq|f(z)|\leq r+\frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}r^2,$$
$$0<|z|=r<1 \quad (17)$$

The equality in (17) is attained by the function f given by

$$f(z)=z-\frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}z^2$$

Proof . Since the function$f$ defined by (2) in the $RH_{\nu}^{\mu}(\psi,\beta,\theta,\eta,A)$, we have from theorem 1

$$\sum_{n=2}^{\infty}a_n\leq\frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}$$

Thus

$$|f(z)|=|z-\sum_{n=2}^{\infty}a_nz^n|\leq|z|+\sum_{n=2}^{\infty}a_n|z|^n$$

$$|f(z)|\leq|z|+|z|^2\sum_{n=2}^{\infty}a_n$$

$$|f(z)|\leq r+r^2\sum_{n=2}^{\infty}a_n$$

$$|f(z)|\leq r+\frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}r^2$$

Similarly

$$|f(z)| \geq |z| + \sum_{n=2}^{\infty} a_n |z|^n$$
$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n$$
$$|f(z)| \geq r - r^2 \sum_{n=2}^{\infty} a_n$$
$$|f(z)| \geq r - \frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]} r^2$$

This completes the proof .

Theorem 4: Let $f_1(z) = z$ and $f_n(z) = z - \frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]} z^n$ Then$f$ is in the class $RH_{\nu}^{\mu}(\psi,\beta,\theta,\eta,A)$ if and only if can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z)$ where $(\sigma_n \geq 1, \sum_{n=1}^{\infty} \sigma_n = 1 or 1 = \sigma_1 + \sum_{n=2}^{\infty} \sigma_n)$.

Proof . Assume that

$$f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z)$$
$$f(z) = \sigma_1 f_1(z) + \sum_{n=2}^{\infty} \sigma_n f_n(z)$$
$$f(z) = \sigma_1 z + \sum_{n=2}^{\infty} \sigma_n \left(z - \frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]} z^n \right)$$
$$f(z) = \sigma_1 z + \sum_{n=2}^{\infty} \sigma_n z - \sum_{n=2}^{\infty} \sigma_n \frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]} z^n$$
$$f(z) = z\left(\sigma_1 + \sum_{n=2}^{\infty} \sigma_n\right) - \sum_{n=2}^{\infty} \sigma_n \frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]} z^n$$
$$f(z) = z - \sum_{n=2}^{\infty} \sigma_n \frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]} z^n$$

From theorem 1, $\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]a_n \leq (1-\beta)\psi$.Then

$$\sum_{n=2}^{\infty} \left[\frac{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}{(1-\beta)\psi} \right] \sigma_n \frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}$$
$$\sum_{n=2}^{\infty} \sigma_n = 1 - \sigma_1 \leq 1$$

Conversely ,suppose that $f \in RH_{\nu}^{\mu}(\psi,\beta,\theta,\eta,A)$ implies from theorem 1

$$a_n \leq \frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}$$

Setting

$$\sigma_n = \frac{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}{(1-\beta)\psi}a_n$$
$$a_n = \frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}\sigma_n$$
$$f(z) = z-\sum_{n=2}^{\infty}a_n z^n$$
$$f(z) = z-\sum_{n=2}^{\infty}\frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}\sigma_n z^n$$

From

$$f_n(z) = z-\frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}z_n$$
$$\frac{(1-\beta)\psi}{R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]}z_n = z-f_n(z)$$
$$f(z) = z-\sum_{n=2}^{\infty}\sigma_n(z-f_n(z))$$
$$f(z) = z-\sum_{n=2}^{\infty}\sigma_n z+\sum_{n=2}^{\infty}\sigma_n f_n(z)$$
$$f(z) = z(1-\sum_{n=2}^{\infty}\sigma_n)+\sum_{n=2}^{\infty}\sigma_n f_n(z)$$
$$f(z) = f_1\sigma_1+\sum_{n=2}^{\infty}\sigma_n f_n(z)$$
$$f(z) = \sum_{n=2}^{\infty}\sigma_n f_n(z)$$

This complete the proof .

Now ,we shall prove that the class $RH_{\nu}^{\mu}(\psi,\beta,\theta,\eta,A)$ is closed under arithmetic mean . Let the function $f_r(r=2,3,...,m)$ define by

$$f_r(z) = z-\sum_{n=2}^{\infty}a_{n,r}z^n \qquad (18)$$

Theorem 5: Let the function defined f by (2) be in the class$RH_{\nu}^{\mu}(\psi,\beta,\theta,\eta,A)$. For every $(r=2,3,...,m)$,then the arithmetic mean of $f_r(r=2,3,...,m)$ is defined by

$$g(z) = z-\sum_{n=2}^{\infty}c_n z^n \quad , \quad (c_n \geq 2, n\geq 2, n\in\mathbb{N})$$

Also belong to the class $RH_{\nu}^{\mu}(\psi,\beta,\theta,\eta,A)$,where $c_n = \frac{1}{m}\sum_{r=2}^{m}a_{n,r}$.

Proof . Since $f_r \in RH_{\nu}^{\mu}(\psi,\beta,\theta,\eta,A)$,then from theorem 1 we get

$$\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]a_n \le (1-\beta)\psi \qquad (19)$$

$$\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]c_n$$

$$\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]\left[\frac{1}{m}\sum_{r=2}^{m} a_{n,r}\right]$$

$$\frac{1}{m}\sum_{r=2}^{m}\left[\sum_{n=2}^{\infty} R_n(\mu,\nu,\eta)[2(1+A)-(1-A+2\theta)]a_{n,r}\right]$$

by (19)

$$\le \frac{1}{m}\sum_{r=2}^{m}(1-\beta)\psi$$

$$(1-\beta)\psi\frac{1}{m}.m$$

$$\le (1-\beta)\psi$$

This complete the proof .

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