Int. J. Nonlinear Anal. Appl. 12 (2021) No. 1, 597-614
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.4865

# Ćirić Type multi-valued $\alpha_*$-$\eta_*$-$\theta$-Contractions on b-meric spaces with Applications

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*(Communicated by Madjid Eshaghi Gordji)*

**Abstract**

In this paper, we give sufficient conditions for the existence of solutions of a system of Volterra-type integral inclusion equations using new sort of multi-valued contractions, named as generalized multi-valued $\alpha_*$-$\eta_*$-$\theta$-contractions defined on $\alpha$-complete b-metric spaces. We give its relevance to fixed point results. We set up an example to elucidate our main results.

*Keywords:* fixed point, $\alpha$-complete b-metric space, $\alpha$ -continuous multi-valued mappings, triangular $\alpha$-orbital admissible, generalized multi-valued $\alpha_*$-$\eta_*$-$\theta$-contractions.
*2010 MSC:* 47H10, 54H25.

## 1. Introduction and Preliminaries

In 1989, Bakhtin [26] investigated the concept of b-metric spaces. However, Czerwik [29, 30] initiated study of fixed point of self-mappings in b-metric spaces and proved an analogue of Banach's fixed point theorem. Since then, numerous research articles have been published comprising fixed point theorems for various classes of single-valued andmultivalued operatorsin b-metric spaces, (see e.g., [1, 11, 12, 13, 14, 15, 19, 20, 21, 22, 25, 27, 37, 38]) and related references therein.

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*Received:* 26 June 2018 *Revised:* 6 November 2019

**Definition 1.1.** *[29] Let $\chi$ be a non-empty set and $s \geq 1$ ($s \in \mathbb{R}$). A function $\check{d}_b : \chi \times \chi \to [0, \infty)$ is said to be a b-metric, if for all $r, j, z \in \chi$,*

(i) $\check{d}_b(r, j) = 0 \Leftrightarrow r = j$;
(ii) $\check{d}_b(r, j) = \check{d}_b(j, r)$;
(ii) $\check{d}_b(r, j) \leq s\left[\check{d}_b(r, z) + \check{d}_b(z, j)\right]$.
The pair $(\chi, \check{d}_b)$ is called a $b$-metric space (with constant $s$).

**Example 1.2.** *[21] Let $H^p = \{f \in W(U) : \|f\|_{H^p} < \infty\}$, $p \in (0, 1)$ be $H^p$ space defined on the unit disk $U$, where $H(U)$ is the set of all holomorphic functions on $U$ and*

$$\|f\|_{H^p} = \sup_{0<r<1} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta \right)^{\frac{1}{p}}.$$

*Denote $\chi = H^p(U)$. Define a mapping $\check{d}_b : \chi \times \chi \to [0, \infty)$ by*

$$\check{d}_b(f, g) = \sup_{0<r<1} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|f\left(re^{i\theta}\right) - g\left(re^{i\theta}\right)\right|^p d\theta \right)^{\frac{1}{p}},$$

*for all $f, g \in X$. Then $(\chi, \check{d}_b)$ is a b-metric space with coefficient $s = 2^{\frac{1}{p}-1}$.*

**Definition 1.3.** *[36] Let $\check{T} : \chi \to \chi$ be a self-map and $\alpha : \chi \times \chi \to [0, +\infty)$. Then $\check{T}$ is said to be $\alpha$-admissible, if $\alpha(r, j) \geq 1 \Longrightarrow \alpha(\check{T}r, \check{T}j) \geq 1$.*

**Definition 1.4.** *[23] Let $\check{T} : \chi \to \chi$ be a self-map and $\alpha : \chi \times \chi \to [0, +\infty)$. Then $\check{T}$ is said to be triangular $\alpha$-admissible, if $\check{T}$ satisfies:*
*($\check{T}1$) $\alpha(r, j) \geq 1 \Longrightarrow \alpha(\check{T}r, \check{T}j) \geq 1$;*
*($\check{T}2$) $\alpha(r, u) \geq 1$ and $\alpha(u, j) \geq 1 \Longrightarrow \alpha(r, j) \geq 1$.*

**Definition 1.5.** *[35] Let $\check{T} : \chi \to \chi$ be a self-map and $\alpha : \chi \times \chi \to [0, +\infty)$. Then $\check{T}$ is said to be $\alpha$-orbital admissible if*
*($\check{T}3$) $\alpha(r, \check{T}r) \geq 1 \Longrightarrow \alpha(\check{T}r, \check{T}^2r) \geq 1$.*

**Definition 1.6.** *[35] Let $\check{T} : \chi \to \chi$ be a map and $\alpha : \chi \times \chi \to [0, +\infty)$. Then $\check{T}$ is said to be triangular $\alpha$-orbital admissible if $\check{T}$ satisfies ($\check{T}3$),*
*($\check{T}4$) $\alpha(r, j) \geq 1$ and $\alpha(j, \check{T}j) \geq 1 \Longrightarrow \alpha(r, \check{T}j) \geq 1$.*

**Definition 1.7.** *[6] Let $\check{T} : \chi \to \chi$ be a map and $\alpha, \eta : \chi \times \chi \to [0, +\infty)$. Then $\check{T}$ is said to be $\alpha$-orbital admissible with respect to $\eta$ if,*
*($\check{T}5$) $r \in \chi$, $\alpha(r, \check{T}r) \geq \eta(r, \check{T}r) \Longrightarrow \alpha(\check{T}r, \check{T}^2r) \geq \eta(\check{T}r, \check{T}^2r)$.*

**Definition 1.8.** *[6] Let $\check{T} : \chi \to \chi$ be a map and $\alpha, \eta : \chi \times \chi \to [0, +\infty)$. Then $\check{T}$ is said to be triangular $\alpha$-orbital admissible with respect to $\eta$ if $T$ satisfies ($\check{T}5$),*
*($\check{T}6$) $r, j \in \chi$, $\alpha(r, j) \geq \eta(r, j)$ and $\alpha(j, \check{T}j) \geq \eta(j, \check{T}j) \Longrightarrow \alpha(r, \check{T}j) \geq \eta(r, \check{T}j)$.*

**Definition 1.9.** *[24] Let $(\chi, \check{d}_b)$ be a b-metric space and $\alpha, \eta : \chi \times \chi \longrightarrow [0, \infty)$. Then $\chi$ is said to be $\alpha$-$\eta$-complete, if every Cauchy sequence $\{r_n\}$ in $\chi$ with $\alpha(r_n, r_{n+1}) \geq \eta(r_n, r_{n+1})$ for all $n \in \mathbb{N}$ converges in $\chi$.*

Recently, Jleli and Samet [8, 9] presented the notion of a $\theta$-contraction.

**Definition 1.10.** *Let $(\chi, \check{d})$ be a metric space. A map $\check{T} : \chi \longrightarrow \chi$ is called $\theta$-contraction, if there exists a constant $k \in (0, 1)$ and $\theta \in \Theta$ such that,*

$$r, j \in \chi, \check{d}\left(\check{T}r, \check{T}j\right) \neq 0 \Longrightarrow \theta\left(\check{d}\left(\check{T}r, \check{T}j\right)\right) \leq \left[\theta\left(\check{d}\left(r, j\right)\right)\right]^k,$$

*Where $\Theta$ is the set of functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ satisfying,*
*($\Theta1$) $\theta$ is non-decreasing;*
*($\Theta2$) for each sequence $\{\check{t}_n\} \subset (0, \infty)$,*

$$\lim_{n\to\infty} \theta(\check{t}_n) = 1 \text{ if and only if } \lim_{n\to\infty} \check{t}_n = 0^+;$$

*($\Theta3$) there exists $q \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{\check{t}\longrightarrow 0^+} \frac{\theta(\check{t})-1}{\check{t}^q} = \ell$.*

Jleli and Samet [8] established the following fixed point theorem.

**Theorem 1.11.** *[8] Let $(\chi, \check{d})$ be a complete metric space and $\check{T} : \chi \longrightarrow \chi$ be $\theta$-contraction. Then $\check{T}$ has a unique fixed point.*

As in [10], we denote by $\mu$ the family of all functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ satisfying the assertions ($\Theta1$), ($\Theta2$) and ($\Theta'3$), where
($\Theta'3$) means $\theta$ is continuous on $(0, \infty)$.
Note that ($\Theta3$) and ($\Theta'3$) are independent of each other [10].

**Example 1.12.** *[10] For all $t \in (0, \infty)$, consider*
$\phi_1(t) = e^t$, $\phi_4(t) = \cosh t$;
$\phi_2(t) = e^{\sqrt{te^t}}$, $\phi_5(t) = 1 + \ln(t + 1)$;
$\phi_3(t) = e^{\sqrt{t}}$, $\phi_6(t) = e^{te^t}$.

Then $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6 \in \mu$.

Very recently, Hussain et al. [5] defined a generalized $(\alpha, \eta)$-$\Theta$-contraction and extended the results of Jleli and Samet [8].

Let $(\chi, \check{d}_b)$ be a $b$-metric space. Let $CB_b(\chi)$ denote the set of all closed and bounded subsets of $\chi$. For $r \in \chi$ and $A, B \in CB_b(\chi)$, define

$$D_b(r, A) = \inf_{a\in A} \check{d}_b(r, a) \text{ and } D_b(A, B) = \sup_{a\in A} D_b(a, B).$$

Define a mapping $H_b : CB_b(\chi) \times CB_b(\chi) \longrightarrow [0, \infty)$ by

$$H_b(A, B) = \max\left\{\sup_{r\in A} D_b(r, B), \sup_{j\in B} D_b(j, A)\right\},$$

for each $A, B \in CB_b(\chi)$. Hence the map $H_b$ is called Hausdorff $b$-metric (induced by a $b$-metric space $(\chi, \check{d}_b)$).

**Lemma 1.13.** *[29] Let $(\chi, \check{d}_b)$ be a $b$-metric space. For any $A, B \in CB_b(\chi)$ and any $r, j \in \chi$, we have:*

(i) $D_b(r,B)\le \check{d}_b(r,b)$ for any $b\in B$;
(ii) $D_b(r,B)\le H_b(A,B)$;
(ii) $D_b(r,A)\le s\left[\check{d}_b(r,j)+D_b(j,B)\right]$.

**Lemma 1.14.** *[29] Let $(\chi,\check{d}_b)$ be a b-metric space, $A,\ B\in CB_b(\chi)$ and $M>1$. Then for all $a\in A$, there exists $b\in B$ such that $\check{d}_b\,(a,b)\le MH_b(A,B)$.*

**Definition 1.15.** *[33] Let $\check{T}:\chi\to CB_b(\chi)$ be a multi-valued mapping and $\alpha:\chi\times\chi\longrightarrow[0,+\infty)$. Then $\check{T}$ is said to be $\alpha_*$-admissible if $\alpha(r,j)\ge 1\Longrightarrow \alpha_*(\check{T}r,\check{T}j)\ge 1$, where*

$$\alpha_*(A,B)=\inf\left\{\alpha(r,j):r\in A,\ j\in B\right\}.$$

Now, we introduce the following definitions.

**Definition 1.16.** *Let $\hat{S},\check{T}:\chi\to CB_b\left(\chi\right)$ be two multi-valued maps and $\alpha,\eta:\chi\times\chi\to[0,+\infty)$ be two functions. We say that $\left(\hat{S},\check{T}\right)$ is triangular $\alpha_*$-$\eta_*$-admissible pair, if:*
*(i) $\alpha(r,j)\ge\eta(r,j)\Longrightarrow \alpha_*(\hat{S}r,\check{T}j)\ge\eta_*(\hat{S}r,\check{T}j)$ and $\alpha_*(\check{T}r,\hat{S}j)\ge\eta_*(\check{T}r,\hat{S}j)$, where*

$$\begin{aligned}\alpha_*(A,B)&=\inf\left\{\alpha(r,j):r\in A,\ j\in B\right\},\\ \eta_*(A,B)&=\inf\left\{\eta(r,j):r\in A,\ j\in B\right\};\end{aligned}$$

*(ii) $\alpha(r,u)\ge\eta(r,u)$ and $\alpha(u,j)\ge\eta(u,j)\Longrightarrow\alpha(r,j)\ge\eta(r,j)$.*

**Definition 1.17.** *Let $\hat{S},\check{T}:\chi\to CB_b\left(\chi\right)$ be two multi-valued maps and $\alpha,\eta:\chi\times\chi\to[0,+\infty)$ be functions. We asy that $\left(\hat{S},\check{T}\right)$ is an $\alpha_*$-$\eta_*$-orbital admissible pair, if,*
*(i) $\alpha_*(r,\hat{S}r)\ge\eta_*(r,\hat{S}r)$ and $\alpha_*(r,\check{T}r)\ge\eta_*(r,\check{T}r)\Longrightarrow\alpha_*(\hat{S}r,\check{T}^2r)\ge\eta_*(\hat{S}r,\check{T}^2r)$ and $\alpha_*(\check{T}r,\hat{S}^2r)\ge\eta_*(\check{T}r,\hat{S}^2r)$.*

**Definition 1.18.** *Let $\hat{S},\check{T}:\chi\to CB_b\left(\chi\right)$ be two multi-valued maps and $\alpha,\eta:\chi\times\chi\to[0,+\infty)$ be functions. Then $\left(\hat{S},\check{T}\right)$ is said to be triangular $\alpha_*$-$\eta_*$-orbital admissible pair, if:*
*(i) $\left(\hat{S},\check{T}\right)$ is $\alpha_*$-$\eta_*$-orbital admissible pair;*
*(ii) $\alpha(r,j)\ge\eta(r,j)$, $\alpha_*(j,\hat{S}j)\ge\eta_*(j,\hat{S}j)$ and $\alpha_*(j,\check{T}j)\ge\eta_*(j,\check{T}j)\Longrightarrow\alpha_*(r,\hat{S}j)\ge\eta_*(r,\hat{S}j)$ and $\alpha_*(r,\check{T}j)\ge\eta_*(r,\check{T}j)$.*

**Lemma 1.19.** *Let $\hat{S},\check{T}:\chi\to CB_b\left(\chi\right)$ such that $(\hat{S},\check{T})$ is triangular $\alpha_*$-$\eta_*$-orbital admissible pair. Assume that, there exists $r_0\in\chi$ such that $\alpha_*(r_0,\hat{S}r_0)\ge\min\left\{\eta_*\left(r_0,\hat{S}r_0\right),\eta_*\left(\hat{S}r_0,\check{T}\hat{S}r_0\right)\right\}$. Define the sequence $\{r_{\check{n}}\}$ in $\chi$ by $r_{2i+1}\in\hat{S}r_{2i}$ and $r_{2i+2}\in\check{T}r_{2i+1}$, where $i=0,1,2,\ldots$. Then for $\check{n},m\in\mathbb{N}\cup\{0\}$ with $m>\check{n}$, we have $\alpha\left(r_{\check{n}},r_m\right)\ge\eta\left(r_{\check{n}},r_m\right)$.*

**Definition 1.20.** *Let $(\chi,\check{d}_b)$ be a b-metric space. Let $\hat{S}:\chi\to CB_b(\chi)$ and $\alpha,\eta:\chi\times\chi\to[0,+\infty)$. Then $\hat{S}$ is said to be a multivalued $\alpha$-$\eta$-continuous on $(CB_b(\chi),H_b)$ if whenever $\{r_{\check{n}}\}$ is a sequence in $\chi$ with $\alpha(r_{\check{n}},r_{\check{n}+1})\ge\eta(r_{\check{n}},r_{\check{n}+1})$ for all $\check{n}\in\mathbb{N}$ and $r\in\chi$ such that $\lim\limits_{\check{n}\longrightarrow\infty}\check{d}_b(r_{\check{n}},r)=0$, then $\lim\limits_{\check{n}\longrightarrow\infty}H_b(\hat{S}r_{\check{n}},\hat{S}r)=0$.*

## 2. Fixed point results

First, inspired by Jleli and Samet [8, 9], we give the following definition.

**Definition 2.1.** *Let $s \geq 1$. We denote by $\Theta_s$ the set of all functions $\theta : (0, \infty) \longrightarrow (1, \infty)$, with the following properties:*
*($\Theta_s$1) $\theta$ is non-decreasing;*
*($\Theta_s$2) for each sequence $\{\check{t}_n\} \subset (0, \infty)$,*

$$\lim_{n\to\infty} \theta(\check{t}_n) = 1 \Leftrightarrow \lim_{n\to\infty} \check{t}_n = 0^+;$$

*($\Theta_s$3) there exists $q \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{\check{t}\longrightarrow 0^+} \frac{\theta(\check{t})-1}{\check{t}^q} = \ell$.*
*($\Theta_s$4) for each sequence $\{\check{t}_n\} \subset (0, \infty)$ such that $\theta(s\check{t}_n) \leq [\theta(\check{t}_{n-1})]^k$, for all $\check{n} \in \mathbb{N}$ then $\theta(s^n\check{t}_n) \leq [\theta(s^{n-1}\check{t}_{n-1})]^k$, for some $k \in (0, 1)$ and for all $\check{n} \in \mathbb{N}$.*

**Example 2.2.** *Let $\theta : (0, \infty) \longrightarrow (1, \infty)$ defined by $\theta(t) = e^{\sqrt{\check{t}}}$. Then clearly, $\theta$ satisfies ($\Theta_s$1)-($\Theta_s$4). Now we show only, ($\Theta_s$4). suppose that, for some $k \in (0, 1)$ and for all $n \in \mathbb{N}$, we $e^{\sqrt{s\check{t}_n}} \leq \left[\theta\left(e^{\sqrt{\check{t}_{n-1}}}\right)\right]^k$. Thus,*

$$\begin{aligned} e^{\sqrt{s^n\check{t}_n}} &= e^{\sqrt{s^{n-1}s\check{t}_n}} = \left[e^{\sqrt{s\check{t}_n}}\right]^{\sqrt{s^{n-1}}} \\ &\leq \left[\left(e^{\sqrt{\check{t}_{n-1}}}\right)^k\right]^{\sqrt{s^{n-1}}} = \left[e^{\sqrt{s^{n-1}\check{t}_{n-1}}}\right]^k, \end{aligned}$$

*hence ($\Theta_s$4) holds true. Note that also, $\theta(t) = e^{\check{t}} \in \Theta_s$.*

Now, we introduce the concept of generalized multi-valued $\alpha_*$–$\eta_*$-$\theta$-contractions as follows:

**Definition 2.3.** *Let $(\chi, \check{d}_b)$ be a b-metric space, and $\alpha, \eta : \chi \times \chi \longrightarrow [0, \infty)$ be two functions. Let $\hat{S}, \check{T} : \chi \longrightarrow CB_b(\chi)$ be two multi-valued maps. Then $(\hat{S}, T)$ is called a generalized multi-valued $\alpha_*$-$\eta_*$-$\theta$-contraction if for $r, j \in \chi$, with $\alpha(r, j) \geq \min\left\{\eta_*\left(r, \hat{S}r\right), \eta_*\left(j, \check{T}j\right)\right\}$ and $H_b\left(\hat{S}r, \check{T}j\right) > 0$, we have*

$$\theta\left(sH_b\left(\hat{S}r, \check{T}j\right)\right) \leq [\theta(M_s(r, j))]^k, \tag{2.1}$$

*where $\theta \in \Theta_s$, $k \in (0, 1)$ and*

$$M_s(r, j) = \max\left\{\check{d}_b(r, j), D_b\left(r, \hat{S}r\right), D_b\left(j, \check{T}j\right), \frac{D_b\left(r, \check{T}j\right) + D_b\left(j, \hat{S}r\right)}{2s}\right\}. \tag{2.2}$$

The following theorem is our main result.

**Theorem 2.4.** *Let $(\chi, \check{d}_b)$ be a b-metric space and $\alpha, \eta : \chi \times \chi \longrightarrow [0, \infty)$ be two functions. Let $\hat{S}, \check{T} : \chi \longrightarrow CB_b(\chi)$ be such that $(\hat{S}, T)$ is a generalized multi-valued $\alpha_*$-$\eta_*$-$\theta$-contraction. Suppose that,*

*(i)* $\left(X, \check{d}_b\right)$ *is an* $\alpha$-$\eta$-*complete b-metric space;*
*(ii)* $(\hat{S}, \check{T})$ *is triangular* $\alpha_*$-$\eta_*$-*orbital admissible pair;*
*(iii) there exists* $r_0 \in \chi$ *such that* $\alpha_*\left(r_0, \hat{S}r_0\right) \geq \min\left\{\eta_*\left(r_0, \hat{S}r_0\right), \eta_*\left(\hat{S}r_0, \check{T}\hat{S}r_0\right)\right\}$;
*(iv)* $\hat{S}$ *and* $\check{T}$ *are multi-valued* $\alpha$-$\eta$-*continuous.*
*Then* $\hat{S}$ *and* $\check{T}$ *have a common fixed point* $r^* \in \chi$.

**Proof .** Let $r_0 \in \chi$ be such that $\alpha_*\left(r_0, \hat{S}r_0\right) \geq \min\left\{\eta_*\left(r_0, \hat{S}r_0\right), \eta_*\left(\hat{S}r_0, \check{T}\hat{S}r_0\right)\right\}$. Choose $r_1 \in \hat{S}r_0$ such that

$$\alpha(r_0, r_1) \geq \min\left\{\eta_*\left(r_0, \hat{S}r_0\right), \eta_*\left(r_1, \check{T}r_1\right)\right\}$$

and $r_1 \neq r_0$. By (2.1) and Lemma 1.12, we have

$$0 < \theta\left(sD_b\left(r_1, \check{T}r_1\right)\right) \leq \theta\left(sH_b\left(\hat{S}r_0, \check{T}r_1\right)\right). \tag{2.3}$$

There exists $x_2 \in \check{T}x_1$ such that

$$\begin{aligned} 0 &\leq \theta\left(s\check{d}_b\left(r_1, r_2\right)\right) \leq \theta\left(sH_b\left(\hat{S}r_0, \check{T}r_1\right)\right) \\ &\leq \left[\theta\left(M_s\left(r_0, r_1\right)\right)\right]^k, \end{aligned}$$

which implies that

$$0 < \theta\left(s\check{d}_b\left(r_1, r_2\right)\right) \leq \left[\theta\left(M_b\left(r_0, r_1\right)\right)\right]^k, \tag{2.4}$$

where

$$\begin{aligned} M_s\left(r_0, r_1\right) &= \max\left\{\begin{array}{c} \check{d}_b\left(r_0, r_1\right), D_b\left(r_0, \hat{S}r_0\right), D_b\left(r_1, \check{T}r_1\right), \\ \frac{D_b\left(r_0, \check{T}r_1\right) + D_b\left(r_1, \hat{S}r_0\right)}{2s} \end{array}\right\} \\ &\leq \max\left\{\check{d}_b\left(r_0, r_1\right), \check{d}_b\left(r_0, r_1\right), \check{d}_b\left(r_1, r_2\right), \frac{D_b\left(r_0, \check{T}r_1\right) + \check{d}_b\left(r_1, r_1\right)}{2s}\right\} \\ &\leq \max\left\{\check{d}_b\left(r_0, r_1\right), \check{d}_b\left(r_1, r_2\right), \frac{D_b\left(r_0, \check{T}r_1\right)}{2s}\right\}. \end{aligned}$$

Since

$$\begin{aligned} \frac{D_b\left(r_0, \check{T}r_1\right)}{2s} &\leq \frac{s\left[\check{d}_b\left(r_0, r_1\right) + D_b\left(r_1, \check{T}r_1\right)\right]}{2s} \\ &\leq \frac{\left[\check{d}_b\left(r_0, r_1\right) + D_b\left(r_1, \check{T}r_1\right)\right]}{2} \leq \max\left\{\check{d}_b\left(r_0, r_1\right), D_b\left(r_1, \check{T}r_1\right)\right\}, \end{aligned}$$

then we get

$$M_s\left(r_0, r_1\right) \leq \max\left\{\check{d}_b\left(r_0, r_1\right), D_b\left(r_1, \check{T}r_1\right)\right\}.$$

If $\max\left\{\check{d}_b\left(r_0, r_1\right), D_b\left(r_1, \check{T}r_1\right)\right\} = D_b\left(r_1, \check{T}r_1\right)$, then from (2.4), we have

$$\theta\left(sD_b\left(r_1, \check{T}r_1\right)\right) \leq \left[\theta\left(D_b\left(r_1, \check{T}r_1\right)\right)\right]^k < \theta\left(D_b\left(r_1, \check{T}r_1\right)\right),$$

which is a contradiction. Therefore,

$$\max\left\{\check{d}_b\left(r_0, r_1\right), D_b\left(r_1, \check{T}r_1\right)\right\} = \check{d}_b\left(r_0, r_1\right).$$

By (2.4), we get that $\theta\left(s\check{d}_b\left(r_1,r_2\right)\right)<\theta\left(\check{d}_b\left(r_0,r_1\right)\right)$. Similarly, for $r_2\in\check{T}r_1$ and $r_3\in\hat{S}r_2$,

$$\begin{aligned}\theta\left(s\check{d}_b\left(r_2,r_3\right)\right) &\leq \theta\left(sD_b\left(r_2,\hat{S}r_2\right)\right)\\ &\leq \theta\left(sH_b\left(\check{T}r_1,\hat{S}r_2\right)\right)\\ &\leq \theta\left(\check{d}_b\left(r_1,r_2\right)\right),\end{aligned}$$

which implies that

$$\theta\left(s\check{d}_b\left(r_2,r_3\right)\right)\leq\theta\left(\check{d}_b\left(r_1,r_2\right)\right). \tag{2.5}$$

Continuing in this way, we define a sequence $\{r_n\}$ in $\chi$ such that $r_{2i+1}\in\hat{S}r_{2i}$ and $r_{2i+2}\in\check{T}r_{2i+1}$, $i=0,1,2,\ldots$.

Since $\alpha_*\left(r_0,\hat{S}r_0\right)\geq\min\left\{\eta_*\left(r_0,\hat{S}r_0\right),\eta_*\left(\hat{S}r_0,\check{T}\hat{S}r_0\right)\right\}$ and $(\hat{S},\check{T})$ is triangular $\alpha_*$-$\eta_*$-orbital admissible pair, so by using Lemma 1.19, we get

$$\alpha\left(r_n,r_{n+1}\right)\geq\eta\left(r_n,r_{n+1}\right), \text{ for all } n\in\mathbb{N}.$$

Then

$$\begin{aligned}0<\theta\left(s\check{d}_b\left(r_{2i+1},r_{2i+2}\right)\right) &\leq \theta\left(sH_b\left(\hat{S}r_{2i},\check{T}r_{2i+1}\right)\right)\\ &\leq \left[\theta\left(M_s\left(r_{2i},r_{2i+1}\right)\right)\right]^k,\end{aligned} \tag{2.6}$$

for all $i\in\mathbb{N}$, where

$$\begin{aligned}M_s\left(r_{2i},r_{2i+1}\right) &= \max\left\{\begin{array}{c}\check{d}_b\left(r_{2i},r_{2i+1}\right),D_b\left(r_{2i},\hat{S}r_{2i}\right),D_b\left(r_{2i+1},\check{T}r_{2i+1}\right),\\ \frac{D_b\left(r_{2i},\check{T}r_{2i+1}\right)+D_b\left(r_{2i+1},\hat{S}r_{2i}\right)}{2s}\end{array}\right\}\\ &\leq \max\left\{\begin{array}{c}\check{d}_b\left(r_{2i},r_{2i+1}\right),\check{d}_b\left(r_{2i},r_{2i+1}\right),\check{d}_b\left(r_{2i+1},r_{2i+2}\right),\\ \frac{D_b\left(r_{2i},\check{T}r_{2i+1}\right)}{2s}\end{array}\right\}\\ &\leq \max\left\{\check{d}_b\left(r_{2i},r_{2i+1}\right),\check{d}_b\left(r_{2i+1},r_{2i+2}\right),\frac{D_b\left(r_{2i},\check{T}r_{2i+1}\right)}{2s}\right\}.\end{aligned}$$

Since

$$\begin{aligned}\frac{D_b\left(r_{2i},\check{T}r_{2i+1}\right)}{2s} &\leq \frac{s\left[\check{d}_b\left(r_{2i},r_{2i+1}\right)+\check{d}_b\left(r_{2i+1},r_{2i+2}\right)\right]}{2s}\\ &\leq \frac{\left[\check{d}_b\left(r_{2i},r_{2i+1}\right)+\check{d}_b\left(r_{2i+1},r_{2i+2}\right)\right]}{2}\\ &\leq \max\left\{\check{d}_b\left(r_{2i},r_{2i+1}\right),\check{d}_b\left(r_{2i+1},r_{2i+2}\right)\right\},\end{aligned}$$

then we get

$$M_s\left(r_{2i},r_{2i+1}\right)\leq\max\left\{\check{d}_b\left(r_{2i},r_{2i+1}\right),\check{d}_b\left(r_{2i+1},r_{2i+2}\right)\right\}, \quad \forall\, i\geq 0.$$

If for some $i$, $\max\left\{\check{d}_b\left(r_{2i},r_{2i+1}\right),\check{d}_b\left(r_{2i+1},r_{2i+2}\right)\right\}=\check{d}_b\left(r_{2i+1},r_{2i+2}\right)$, then by (2.6) we have

$$\begin{aligned}1 &< \theta\left(\check{d}_b\left(r_{2i+1},r_{2i+2}\right)\right)\leq\left[\theta\left(\check{d}_b\left(r_{2i+1},r_{2i+2}\right)\right)\right]^k\\ &< \theta\left(\check{d}_b\left(r_{2i+1},r_{2i+2}\right)\right),\end{aligned}$$

which is a contradiction. Thus

$$\max\left\{\check{d}_b\left(r_{2i},r_{2i+1}\right),\check{d}_b\left(r_{2i+1},r_{2i+2}\right)\right\}=\check{d}_b\left(r_{2i},x_{r2i+1}\right)\quad\forall\ i\geq 0.$$

By (2.6), we get that

$$1<\theta\left(s\check{d}_b\left(r_{2i+1},r_{2i+2}\right)\right)\leq\left[\theta\left(\check{d}_b\left(r_{2i},r_{2i+1}\right)\right)\right]^k<\theta\left(\check{d}_b\left(r_{2i},r_{2i+1}\right)\right)\quad\forall\ i\geq 0.$$

This implies that

$$1<\theta\left(s\check{d}_b\left(r_{n+1},r_{n+2}\right)\right)\leq\left[\theta\left(\check{d}_b\left(r_n,r_{n+1}\right)\right)\right]^k<\theta\left(\check{d}_b\left(r_n,r_{n+1}\right)\right)\quad\forall\ n\geq 0.\tag{2.7}$$

From (2.7) and axiom $(\Theta_s 4)$, we have

$$1<\theta\left(s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)\right)\leq\left[\theta\left(s^{n-1}\check{d}_b\left(r_{n-1},r_n\right)\right)\right]^k<\theta\left(s^{n-1}\check{d}_b\left(r_{n-1},r_n\right)\right)\quad\forall\ n\geq 0.\tag{2.8}$$

Further,

$$\begin{aligned}1 &< \theta\left(s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)\right)=\theta\left(s^n\check{d}_b\left(\hat{S}r_n,\check{T}r_{n+1}\right)\right)\leq\left[\theta\left(s^{n-1}\check{d}_b\left(r_{n-1},r_n\right)\right)\right]^k\\ &= \left[\theta\left(s^{n-1}\check{d}_b\left(\hat{S}r_{n-2},\check{T}r_{n-1}\right)\right)\right]^k\leq\left[\theta\left(s^{n-2}\check{d}_b\left(r_{n-1},r_{n-2}\right)\right)\right]^{k^2}\\ &\leq \ldots\leq\left[\theta\left(\check{d}_b\left(r_0,r_1\right)\right)\right]^{k^n},\end{aligned}$$

Which implies,

$$1<\theta\left(s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)\right)\leq\left[\theta\left(\check{d}_b\left(r_0,r_1\right)\right)\right]^{k^n},\tag{2.9}$$

for all $n\in\mathbb{N}$. Taking the limit as $n\longrightarrow\infty$ in (2.9), since $\theta\in\Theta_s$, we have

$$\lim_{n\longrightarrow\infty}\theta\left(s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)\right)=1,$$

By $(\Theta_s 2)$, we get

$$\lim_{n\longrightarrow\infty}s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)=0.\tag{2.10}$$

From condition $(\Theta_s 3)$, there exist $q\in(0,1)$ and $\ell\in(0,\infty]$ such that

$$\lim_{n\longrightarrow\infty}\frac{\theta\left(s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)\right)-1}{\left[s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)\right]^q}=\ell.$$

Suppose that $\ell<\infty$. Let $W=\frac{\ell}{2}>0$. From the definition of the limit, there exists $n_0\geq 1$ such that

$$\left|\frac{\theta\left(s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)\right)-1}{\left[s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)\right]^q}-\ell\right|\leq W\quad\text{for all } n\geq n_0.$$

This implies

$$\frac{\theta\left(s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)\right)-1}{\left[s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)\right]^q}\geq\ell-W=W\quad\text{for all } n\geq n_0.$$

Then

$$n\left[s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)\right]^q\leq An\left[\theta\left(s^n\check{d}_b\left(r_{n+1},r_{n+2}\right)\right)-1\right]\quad\text{for all } n\geq n_0,$$

where $P = \frac{1}{W}$. Suppose now that $\ell = \infty$. Let $W > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \geq 1$ such that

$$\frac{\theta\left(s^n \check{d}_b\left(r_{n+1}, r_{n+2}\right)\right) - 1}{\left[s^n \check{d}_b\left(r_{n+1}, r_{n+2}\right)\right]^q} \geq W \quad \text{for all } n \geq n_0.$$

Which implies

$$n\left[s^n \check{d}_b\left(r_{n+1}, r_{n+2}\right)\right]^q \leq Pn\left[\theta\left(s^n \check{d}_b\left(r_{n+1}, r_{n+2}\right)\right) - 1\right] \quad \text{for all } n \geq n_0,$$

where $P = \frac{1}{W}$. Thus, in all cases, there exist $P > 0$ and $n_0 \geq 1$ such that

$$n\left[s^n \check{d}_b\left(r_{n+1}, r_{n+2}\right)\right]^q \leq Pn\left[\theta\left(s^n \check{d}_b\left(r_{n+1}, r_{n+2}\right)\right) - 1\right] \quad \text{for all } n \geq n_0.$$

By using (2.9), we get

$$n\left[s^n \check{d}_b\left(r_{n+1}, r_{n+2}\right)\right]^q \leq Pn\left(\left[\theta\left(d(r_0, r_1)\right)\right]^{k^n} - 1\right) \quad \text{for all } n \geq n_0. \tag{2.11}$$

Setting $n \longrightarrow \infty$ in the inequality (2.11), we get

$$\lim_{n \longrightarrow \infty} n\left[s^n \check{d}_b\left(r_{n+1}, r_{n+2}\right)\right]^q = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$s^n \check{d}_b\left(r_{n+1}, r_{n+2}\right) \leq \frac{1}{n^{\frac{1}{q}}} \text{ for all } n \geq n_1. \tag{2.12}$$

To prove $\{r_n\}$ is a Cauchy sequence, we use (2.12) and for $m > n \geq n_1$,

$$\begin{aligned} \check{d}_b\left(r_n, r_m\right) &\leq {}_{i=n}^{m-1} s^i \check{d}_b\left(r_i, r_{i+1}\right) \leq {}_{i=n}^{\infty} s^i \check{d}_b\left(r_i, r_{i+1}\right) \\ &\leq {}_{i=n}^{\infty} \frac{1}{i^{\frac{1}{q}}}. \end{aligned}$$

The convergence of the series ${}_{i=n}^{\infty} \frac{1}{i^{\frac{1}{q}}}$ entails $\lim_{n \longrightarrow \infty} \check{d}_b\left(r_n, r_m\right) = 0$. Thus $\{r_n\}$ is a Cauchy sequence. Since $\chi$ is an $\alpha$-$\eta$-complete b-metric space and $\alpha\left(r_n, r_{n+1}\right) \geq \eta\left(r_n, r_{n+1}\right)$, for all $n \in \mathbb{N}$, there exists $r^* \in \chi$ such that $\lim_{n \longrightarrow \infty} d\left(r_n, r^*\right) = 0$. This implies that $\lim_{i \longrightarrow \infty} \check{d}_b\left(r_{2i+1}, r^*\right) = 0$ and $\lim_{i \longrightarrow \infty} \check{d}_b\left(r_{2i+2}, r^*\right) = 0$. As $\check{T}$ is an $\alpha$-$\eta$-continuous multivalued mapping, so $\lim_{i \longrightarrow \infty} H_b\left(r_{2i+1}, r^*\right) = 0$. Thus

$$D_b\left(r^*, \check{T}r^*\right) = \lim_{i \longrightarrow \infty} D_b\left(r_{2i+2}, \check{T}r^*\right) \leq \lim_{i \longrightarrow \infty} H_b\left(\check{T}r_{2i+1}, \check{T}r^*\right) = 0.$$

Consequently, $r^* \in \check{T}r^*$. Similarly, $r^* \in \hat{S}r^*$. Therefore, $r^* \in \chi$ is a common fixed point of $\hat{S}$ and $\check{T}$. □

**Theorem 2.5.** *Let $\left(\chi, \check{d}_b\right)$ be a b-metric space, and $\alpha, \eta : \chi \times \chi \longrightarrow [0, \infty)$. Let $\hat{S}, \check{T} : \chi \longrightarrow CB_b(\chi)$ be such that $(\hat{S}, \check{T})$ is a generalized multi-valued $\alpha_*$-$\eta_*$-$\theta$-contraction. Suppose that,*
*(i) $\left(\chi, \check{d}_b\right)$ is an $\alpha$-$\eta$-complete b-metric space;*
*(ii) $(\hat{S}, \check{T})$ is triangular $\alpha_*$-$\eta_*$-orbital admissible pair;*
*(iii) there exists $r_0 \in \chi$ such that $\alpha_*\left(r_0, \hat{S}r_0\right) \geq \min\left\{\eta_*\left(r_0, \hat{S}r_0\right), \eta_*\left(\hat{S}r_0, \check{T}\hat{S}r_0\right)\right\}$;*
*(iv) if $\{r_n\}$ is a sequence in $\chi$ such that $\alpha\left(r_n, r_{n+1}\right) \geq \eta\left(r_n, r_{n+1}\right)$ for all $n \in \mathbb{N}$ and $r_n \longrightarrow r^* \in \chi$ as $n \longrightarrow \infty$, then either $\alpha_*\left(\hat{S}r_n, r^*\right) \geq \eta_*\left(\hat{S}r_n, r^*\right)$ or $\alpha_*\left(\check{T}r_{n+1}, r^*\right) \geq \eta_*\left(\check{T}r_{n+1}, r^*\right)$ holds for all $n \in \mathbb{N}$.*
*Then $\hat{S}$ and $\check{T}$ have a common fixed point $r^* \in \chi$.*

**Proof .** Let $r_0 \in \chi$ be such that $\alpha_*\left(r_0, \hat{S}r_0\right) \geq \min\left\{\eta_*\left(r_0, \hat{S}r_0\right), \eta_*\left(\hat{S}r_0, \check{T}\hat{S}r_0\right)\right\}$. As in proof of Theorem 2.4, we construct a sequence $\{r_n\}$ in $\chi$ defined by $r_{2i+1} \in \hat{S}r_{2i}$ and $r_{2i+2} \in \check{T}r_{2i+1}$, where $i \geq 0$, $\alpha(r_n, r_{n+1}) \geq \eta(r_n, r_{n+1})$, for all $n \in \mathbb{N}$ and $\{r_n\}$ converges to $r^* \in \chi$. Since $\alpha(r_n, x_{rn+1}) \geq \eta(r_n, r_{n+1})$ for all $n \in \mathbb{N}$ and $r_n \longrightarrow r^* \in \chi$ as $n \longrightarrow \infty$, by condition $(iv)$, either $\alpha_*\left(\hat{S}r_n, r^*\right) \geq \eta_*\left(\hat{S}r_n, r^*\right)$ or $\alpha_*\left(\check{T}r_{n+1}, r^*\right) \geq \eta_*\left(\check{T}r_{n+1}, r^*\right)$ holds all $n \in \mathbb{N}$. Thus,

$$\alpha(r_{n+1}, r^*) \geq \eta(r_{n+1}, r^*) \text{ or } \alpha(r_{n+2}, r^*) \geq \eta(r_{n+2}, r^*), \text{ holds for all } n \in \mathbb{N}.$$

Equivalently, there exists a subsequence $\{r_{n(k)}\}$ of $\{r_n\}$ such that

$$\alpha\left(r_{n(k)}, r^*\right) \geq \eta\left(r_{n(k)}, r^*\right) \text{ for all } k \in \mathbb{N} \tag{2.13}$$

From (2.13), we deduce that

$$\begin{aligned}\theta\left(D_b\left(r_{2n(k)+1}, \check{T}r^*\right)\right) &\leq \theta\left(D_b\left(\hat{S}r_{2n(k)}, \check{T}r^*\right)\right) \leq \theta\left(sH_b\left(\hat{S}r_{2n(k)}, \check{T}r^*\right)\right) \\ &\leq \left[\theta\left(M_s\left(r_{2n(k)}, r^*\right)\right)\right]^k.\end{aligned}$$

This implies that

$$\theta\left(D_b\left(r_{2n(k)+1}, \check{T}r^*\right)\right) \leq \left[\theta\left(M_s\left(r_{2n(k)}, r^*\right)\right)\right]^k < \theta\left(M_s\left(r_{2n(k)}, r^*\right)\right), \tag{2.14}$$

where

$$\begin{aligned}M_s\left(r_{2n(k)}, r^*\right) &= \max\left\{\begin{array}{c}\check{d}_b\left(r_{2n(k)}, r^*\right), D_b\left(r_{2n(k)}, \hat{S}r_{2n(k)}\right), D_b\left(r^*, \check{T}r^*\right), \\ \frac{D_b\left(r_{2n(k)}, \hat{S}r^*\right) + D_b\left(r^*, \check{T}r_{2n(k)}\right)}{2s}\end{array}\right\} \\ &\leq \max\left\{\begin{array}{c}\check{d}_b\left(r_{2n(k)}, r^*\right), \check{d}_b\left(r_{2n(k)}, r_{2n(k)+1}\right), D_b\left(r^*, \check{T}r^*\right), \\ \frac{D_b\left(r_{2n(k)}, \check{T}r^*\right) + D_b\left(r^*, \hat{S}r_{2n(k)}\right)}{2s}\end{array}\right\}.\end{aligned}$$

Suppose that $r^* \notin \check{T}r^*$, then $D_b\left(r^*, \check{T}r^*\right) > 0$. Taking the limit as $k \longrightarrow \infty$ in (2.14) and using the condition $(\Theta'3)$, we have

$$\theta\left(D_b\left(r^*, \check{T}r^*\right)\right) < \theta\left(D_b\left(r^*, \check{T}r^*\right)\right).$$

It is a contradiction. Hence $D_b\left(r^*, \check{T}r^*\right) = 0$, and so $r^* \in \check{T}r^*$. Similarly, we can show that $r^* \in \hat{S}r^*$. Thus $r^* \in \chi$ is a common fixed point of $\hat{S}$ and $\check{T}$. □

**Example 2.6.** *Let $\chi = [-1, 1]$ and define the function $\check{d}_b : \chi \times \chi \to [0, +\infty)$ by $\check{d}_b(r, j) = |r - j|^2$. Clearly, $(\chi, \check{d}_b)$ is a complete b-metric space with $s = 2$. Let $\theta(t) = e^t$, $t > 0$, then $\theta \in \Theta_s$. Define the mappings $\hat{S}, \check{T} : \chi \longrightarrow CB_b(\chi)$ by*

$$\check{T}r = \left\{\begin{array}{cc}\left[0, \frac{2r}{245}\right], & \text{if } r \in [-1, 0] \\ \{1\}, & \text{if } r \in (0, 1]\end{array}\right.,$$

*and*

$$\hat{S}r = \left\{\begin{array}{cc}\left[0, \frac{r}{300}\right], & \text{if } r \in [-1, 0] \\ \{1\}, & \text{if } r \in (0, 1]\end{array}\right..$$

*Moreover, define the functions* $\alpha, \eta : \chi \times \chi \longrightarrow [0, \infty)$ *by*

$$\alpha(r,j) = \begin{cases} 1, & \text{if } r,j \in [-1,0] \\ 0, & \text{otherwise.} \end{cases}$$

*and*

$$\eta(r,j) = \begin{cases} \frac{1}{5}, & \text{if } r,j \in [-1,0] \\ 3, & \text{otherwise.} \end{cases}$$

*If* $\{r_n\}$ *is a Cauchy sequence such that* $\alpha(r_n, r_{n+1}) \geq \eta(r_n, r_{n+1})$ *for all* $n \in \mathbb{N}$, *then* $\{r_n\} \subseteq [-1,0]$. *Since* $\left([-1,0], \check{d}_b\right)$ *is a complete b-metric space, then the sequence* $\{r_n\}$ *converges in* $[-1,0] \subseteq \chi$. *Thus* $\left(\chi, \check{d}_b\right)$ *is an* $\alpha$-$\eta$-*complete b-metric space. Let* $\alpha_*\left(r, \hat{S}r\right) \geq \eta_*\left(r, \hat{S}r\right)$ *and* $\alpha_*\left(r, \check{T}r\right) \geq \eta_*\left(r, \check{T}r\right)$. *So,* $r \in [-1,0]$ *and* $\hat{S}r$, $\check{T}r \in [-1,0]$. *Hence* $\hat{S}^2r = \hat{S}\left(\hat{S}r\right), \check{T}^2r = \check{T}\left(\check{T}r\right) \in [-1,0]$. *Then* $\alpha_*\left(\hat{S}r, \check{T}^2r\right) \geq \eta_*\left(\hat{S}r, \check{T}^2r\right)$ *and* $\alpha_*\left(\check{T}r, \hat{S}^2r\right) \geq \eta_*\left(\check{T}r, \hat{S}^2r\right)$. *Thus,* $\left(\hat{S}, \check{T}\right)$ *is* $\alpha_*$-$\eta_*$-*orbital admissible. Let* $r, j \in \chi$ *be such that* $\alpha(r,j) \geq \eta(r,j)$, $\alpha_*\left(j, \hat{S}j\right) \geq \eta_*\left(j, \hat{S}j\right)$ *and* $\alpha_*\left(j, \check{T}j\right) \geq \eta_*\left(j, \check{T}j\right)$. *Then we have* $r, j, \hat{S}j, \check{T}j \in [-1,0]$, *which implies that* $\alpha_*\left(r, \hat{S}j\right) \geq \eta_*\left(r, \hat{S}j\right)$ *and* $\alpha_*\left(r, \check{T}j\right) \geq \eta_*\left(r, \check{T}j\right)$. *Hence,* $\left(\hat{S}, \check{T}\right)$ *is triangular* $\alpha_*$-$\eta_*$-*orbital admissible pair. Let* $\{r_n\}$ *be a sequence such that* $r_n \longrightarrow r$ *as* $n \longrightarrow \infty$ *and* $\alpha(r_n, r_{n+1}) \geq \eta(r_n, r_{n+1})$ *for all* $n \in \mathbb{N}$. *Then* $\{r_n\} \subseteq [-1,0]$ *for all* $n \in \mathbb{N}$. *So* $\lim_{n \longrightarrow \infty} \check{T}r_n = \lim_{n \longrightarrow \infty} \left[0, \frac{2}{245}r_n\right] = \left[0, \frac{2}{245}r\right] = \check{T}r$. *Hence* $\check{T}$ *is a multi-vlued* $\alpha$-$\eta$-*continuous. Similarly, we can check that* $S$ *is a multi-vlued* $\alpha$-$\eta$-*continuous. Let* $r_0 = -\frac{1}{2}$. *Then*

$$\begin{aligned} \alpha_*\left(-\frac{1}{2}, \hat{S}\left(-\frac{1}{2}\right)\right) &= \alpha_*\left(-\frac{1}{2}, 0\right) = 1 \\ &\geq \min\left\{ \begin{array}{c} \eta_*\left(-\frac{1}{2}, \hat{S}\left(-\frac{1}{2}\right)\right), \\ \eta_*\left(\hat{S}\left(-\frac{1}{2}\right), \check{T}\left(\hat{S}\left(-\frac{1}{2}\right)\right)\right) \end{array} \right\} = \frac{1}{5}. \end{aligned}$$

*Let* $r, j \in \chi$ *be such that* $\alpha(r,j) \geq \min\left\{\eta_*\left(r, \hat{S}r\right), \eta_*\left(j, \check{T}j\right)\right\}$. *Then* $r, j \in [-1,0]$ *and* $H_b\left(\hat{S}r, \check{T}j\right) > 0$. *So*

$$\theta\left(sH_b\left(\hat{S}r, \check{T}j\right)\right) \leq \left[\theta\left(M_s(r,j)\right)\right]^k,$$

*where* $k \in \left(\frac{4}{5}, 1\right)$. *Hence all hypothese of Theorem 2.4 are satisfied. Thus,* $\hat{S}$ *and* $\check{T}$ *have a common fixed point.*

**Corollary 2.7.** *Let* $\left(\chi, \check{d}_b\right)$ *be a complete b-metric space, and* $\alpha, \eta : \chi \times \chi \longrightarrow [0, \infty)$. *Let* $\hat{S} : \chi \longrightarrow CB_b(\chi)$ *be such that* $\hat{S}$ *is a generalized multi-valued* $\alpha_*$-$\eta_*$-$\theta$-*contraction. Suppose that*
*(i)* $\left(\chi, \check{d}_b\right)$ *is an* $\alpha$-$\eta$-*complete b-metric space;*
*(ii)* $\hat{S}$ *is triangular* $\alpha_*$-$\eta_*$-*orbital admissible*;
*(iii) there exists* $r_0 \in \chi$ *such that* $\alpha_*\left(r_0, \hat{S}r_0\right) \geq \min\left\{\eta_*\left(r_0, \hat{S}r_0\right), \eta_*\left(\hat{S}r_0, \check{T}\hat{S}r_0\right)\right\}$;
*(iv) either* $\hat{S}$ *is a multi-valued* $\alpha$-$\eta$-*continuous or if* $\{r_n\}$ *is a sequence in* $\chi$ *such that* $\alpha(r_n, r_{n+1}) \geq \eta(r_n, r_{n+1})$ *for all* $n \in \mathbb{N}$ *and* $r_n \longrightarrow r^* \in \chi$ *as* $n \longrightarrow \infty$, *then either* $\alpha_*\left(\hat{S}r_n, r^*\right) \geq \eta_*\left(\hat{S}r_n, r^*\right)$ *or* $\alpha_*\left(\hat{S}r_{n+1}, r^*\right) \geq \eta_*\left(\hat{S}r_{n+1}, r^*\right)$ *holds for all* $n \in \mathbb{N}$.
*Then* $\hat{S}$ *has a fixed point* $r^* \in \chi$.

**Definition 2.8.** *Let* $(\chi, \check{d}_b)$ *be a b-metric space. Let* $\alpha, \eta : \chi \times \chi \longrightarrow [0, \infty)$ *and* $\hat{S}, \check{T} : \chi \longrightarrow CB_b(\chi)$ *be two multi-valued mappings. Then* $(\hat{S}, \check{T})$ *is said to be a multi-valued* $\alpha_*\eta_*$*-*$\theta$*-contraction mapping, if there exists* $\theta \in \Theta_s$ *such that for all* $r, j \in \chi$ *with* $\alpha(r, j) \geq \min\left\{\eta_*\left(r, \hat{S}r\right), \eta_*\left(j, \check{T}j\right)\right\}$, $(\hat{S}, \check{T})$ *satisfies:*

$$\theta\left(sH_b\left(\hat{S}r, \check{T}j\right)\right) \leq \left[\theta\left(\check{d}_b(r, j)\right)\right]^k, \ k \in (0, 1).$$

**Theorem 2.9.** *Let* $(\chi, \check{d}_b)$ *be a b-metric space, and* $\alpha, \eta : \chi \times \chi \longrightarrow [0, \infty)$. *Let* $\hat{S}, \check{T} : \chi \longrightarrow CB_b(\chi)$ *be such that* $(\hat{S}, \check{T})$ *is a multi-valued* $\alpha_*$*-*$\eta_*$*-*$\theta$*-contraction. Suppose that,*
*(i)* $(\chi, \check{d}_b)$ *is an* $\alpha$*-*$\eta$*-complete b-metric space;*
*(ii)* $(\hat{S}, \check{T})$ *is triangular* $\alpha_*$*-*$\eta_*$*-orbital admissible pair;*
*(iii) there exists* $r_0 \in \chi$ *such that* $\alpha_*\left(r_0, \hat{S}r_0\right) \geq \min\left\{\eta_*\left(r_0, \hat{S}r_0\right), \eta_*\left(\hat{S}r_0, \check{T}\hat{S}r_0\right)\right\}$;
*(iv) either* $\hat{S}$ *and* $\check{T}$ *are multi-valued* $\alpha$*-*$\eta$*-continuous or if* $\{r_n\}$ *is a sequence in* $\chi$ *such that* $\alpha(r_n, r_{n+1}) \geq \eta(r_n, r_{n+1})$ *for all* $n \in \mathbb{N}$ *and* $r_n \longrightarrow r^* \in \chi$ *as* $n \longrightarrow \infty$*, then either* $\alpha_*\left(\hat{S}r_n, r^*\right) \geq \eta_*\left(\hat{S}r_n, r^*\right)$ *or* $\alpha_*\left(\check{T}r_{n+1}, r^*\right) \geq \eta_*\left(\check{T}r_{n+1}, r^*\right)$ *holds for all* $n \in \mathbb{N}$.
*Then* $\hat{S}$ *and* $\check{T}$ *have a common fixed point* $r^* \in \chi$.

**Corollary 2.10.** *Let* $(\chi, \preceq)$ *be a partially ordered set and* $\hat{S}, \check{T} : \chi \longrightarrow \chi$. *Suppose that there exists a b-metric* $\check{d}_b$ *on* $\chi$ *such that* $(\chi, \check{d}_b)$ *is a complete b-metric space. Assume that,*
*(i) there exists* $\theta \in \Theta_s$ *such that*

$$\theta\left(sd\left(\hat{S}r, \check{T}j\right)\right) \leq \left[\theta\left(M_s(r, j)\right)\right]^k,$$

*where* $k \in (0, 1)$ *and*

$$M_s(r, j) = \max\left\{\check{d}_b(r, j), \check{d}_b\left(r, \hat{S}r\right), \check{d}_b(j, Sj), \frac{\check{d}_b\left(r, \check{T}j\right) + \check{d}_b\left(j, \hat{S}r\right)}{2s}\right\}$$

*for all* $r, j \in \chi$ *with* $r \preceq j$ *and* $\check{d}_b\left(\hat{S}r, \check{T}j\right) > 0$;
*(ii)* $\hat{S}$ *and* $\check{T}$ *are nondecreasing (that is, if for all* $r, j \in \chi$, $r \preceq j$ *implies* $\hat{S}r \preceq \hat{S}j$*);*
*(iii) there exists* $r_0 \in \chi$ *such that* $r_0 \preceq \hat{S}r_0$;
*(iv) either* $\hat{S}$ *and* $\check{T}$ *are continuous or if* $\{r_n\}$ *is a sequence in* $\chi$ *such that* $r_n \preceq r_{n+1}$ *for all* $n \in \mathbb{N}$ *and* $r_n \longrightarrow r^* \in \chi$ *as* $n \longrightarrow \infty$*, then either* $\hat{S}r_n \preceq r^*$ *or* $\check{T}r_{n+1} \preceq r^*$ *holds for all* $n \in \mathbb{N}$.
*Then* $\hat{S}$ *and* $\check{T}$ *have a common fixed point* $r^* \in \chi$.

Now, we deduce certain Suzuki-Samet type fixed point results.

**Theorem 2.11.** *Let* $(\chi, \check{d}_b)$ *be a complete b-metric space. Let* $\hat{S}, \check{T} : \chi \longrightarrow CB_b(\chi)$ *be two continuous multi-valued mappings. If for all* $r, j \in \chi$ *with*

$$\frac{1}{2}\min\left\{D_b\left(r, \hat{S}r\right), D_b\left(j, \check{T}j\right)\right\} \leq \check{d}_b(r, j),$$

*and* $H_b\left(\hat{S}r, \check{T}j\right) > 0$*, we have*

$$\theta\left(sH_b\left(\hat{S}r, \check{T}j\right)\right) \leq \left[\theta\left(M_s(r, j)\right)\right]^k,$$

*where* $\theta \in \Theta_s$. *Then* $\hat{S}$ *and* $\check{T}$ *have a common fixed point.*

**Proof .** Define $\alpha, \eta : \chi \times \chi \longrightarrow [0, \infty)$ by

$$\alpha(r, j) = \check{d}_b(r, j) \text{ and } \eta(r, j) = \frac{1}{2}\check{d}_b(r, j),$$

for all $r, j \in \chi$. Since $\frac{1}{2}\check{d}_b(r, j) \le \check{d}_b(r, j)$ for all $r, j \in \chi$, so $\eta(r, j) \le \alpha(r, j)$ for all $r, j \in \chi$. Hence the conditions (i), (iii) and (iv) of Theorem 2.2 hold. Since $\hat{S}$ and $\check{T}$ are continuous, $\hat{S}$ and $\check{T}$ are $\alpha$-$\eta$-continuous multi-vlued mappings. Let $\min\left\{\eta_*\left(r, \hat{S}r\right), \eta_*\left(r, \check{T}r\right)\right\} \le \alpha(r, j)$ with $H_b\left(\hat{S}r, \check{T}j\right) > 0$. Equivalently, if $\frac{1}{2}\min\left\{D_b\left(r, \hat{S}r\right), D_b\left(j, \check{T}y\right)\right\} \le \check{d}_b(r, j)$ with $H_b\left(\hat{S}r, \check{T}j\right) > 0$, then we have

$$\theta\left(sH_b\left(\hat{S}r, \check{T}j\right)\right) \le \left[\theta\left(M_s(r, j)\right)\right]^k.$$

That is, $(\hat{S}, \check{T})$ is a generalized multi-valued $\alpha_*$-$\eta_*$-$\theta$-contraction. Hence, all conditions of Theorem 2.2 hold. Thus $\hat{S}$ and $\check{T}$ have a common fixed point. □

**Theorem 2.12.** *Let $\left(\chi, \check{d}_b\right)$ be a complete b-metric space. Let $\hat{S}, \check{T} : \chi \longrightarrow CB_b(\chi)$. If for all $r, j \in \chi$ with*

$$\frac{1}{2(1+\pi)}\min\left\{D_b\left(r, \hat{S}r\right), D_b\left(j, \check{T}j\right)\right\} \le \check{d}_b(r, j),$$

*$\pi > 0$ and $H_b\left(\hat{S}r, \check{T}j\right) > 0$, we have*

$$\theta\left(sH_b\left(\hat{S}r, \check{T}j\right)\right) \le \left[\theta\left(M_s(r, j)\right)\right]^k,$$

*where $\theta \in \Theta_s$. Then $\hat{S}$ and $\check{T}$ have a common fixed point.*

**Proof .** The result follows from Theorem 2.3 by taking $\alpha, \eta : \chi \times \chi \longrightarrow [0, \infty)$ as

$$\alpha(r, j) = \check{d}_b(r, j) \text{ and } \eta(r, j) = \frac{1}{2(1+\pi)}\check{d}_b(r, j).$$

□

## 3. Application

we apply the result given by Theorem 2.4 to study the existence of a solution for a system of Volterra-type integral inclusions. For instance,

Consider the following system of Volterra-type integral inclusions:

$$r(t) \in \int_a^t \Gamma(t, s, r(s))ds + f(t) \text{ and } j(t) \in \int_a^t \Xi(t, s, j(s))ds + g(t) \tag{3.1}$$

where $\Gamma, \Xi : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow CVB(\mathbb{R})$, and $CVB(\mathbb{R})$ denotes the family of nonempty closed, convex and bounded subsets of $\mathbb{R}$( set of all real numbers). let $\chi = C([a, b], \mathbb{R})$ be the space of all continuous real valued functions on $[a, b]$. Note that $\chi$ is a complete $b$-metric space by considering $\check{d}_b(r, j) = \sup_{t\in[a,b]} |r(t) - j(t)|^2$ with $s = 2$. For each $r, j \in C([a, b], \mathbb{R})$, the operators $\Gamma(., ., x)$ and $\Xi(., ., y)$ are lower semi-continuous. Further, the functions $f$,$g : [a, b] \longrightarrow \mathbb{R}$ are continuous.

For the system of integrals inclusion given above, we can define multivalued operators $\hat{S}, \check{T}$ : $C([a,b],\mathbb{R}) \longrightarrow CB(C([a,b],\mathbb{R}))$ as follows:

$$\hat{S}r(t) = \left\{ u \in C([a,b],\mathbb{R}) : u \in \int_a^t \Gamma(t,s,r(s))ds + f(t),\ t \in [a,b] \right\},$$

and

$$\check{T}j(t) = \left\{ v \in C([a,b],\mathbb{R}) : v \in \int_a^t \Xi(t,s,j(s))ds + g(t),\ t \in [a,b] \right\}.$$

Let $r, j \in C([a,b],\mathbb{R})$ and denote $\Gamma_r := \Gamma(t,s,r(s))$ and $\Xi_j := \Xi(t,s,j(s))$, $t,s \in [a,b]$. Now for $\Gamma_r, \Xi_j : [a,b] \times [a,b] \longrightarrow CVB(\mathbb{R})$, by Michael's selection theorem, there exist continuous operators $\Upsilon_r, \Pi_j : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ with $\Upsilon_r(t,s) \in \Gamma_r(t,s)$ and $\Pi_j(t,s) \in \Xi_j(t,s)$ for all $t,s \in [a,b]$. This shows that $\int_a^t \Upsilon_r(t,s)ds + f(t) \in \hat{S}r(t)$ and $\int_a^t \Pi_j(t,s)ds + g(t) \in \check{T}j(t)$. Thus, the operators $\hat{S}r$ and $\check{T}j$ are nonempty. Since $g$, $\Upsilon_r$ and $\Pi_j$ are continuous on $[a,b]$ (resp. $[a,b] \times [a,b]$), their ranges are bounded and hence $\hat{S}r$ and $\check{T}j$ are bounded (i.e., $\hat{S}, \check{T} : \chi \longrightarrow CB_b(\chi)$).

**Theorem 3.1.** *Take $\chi = C([a,b],\mathbb{R})$. Consider the multivalued operators $\hat{S}, \check{T} : \chi \longrightarrow CB_b(\chi)$,*

$$\hat{S}r(t) = \left\{ u \in C([a,b],\mathbb{R}) : u \in \int_a^t \Gamma(t,s,r(s))ds + f(t),\ t \in [a,b] \right\},$$

*and*

$$\check{T}j(t) = \left\{ v \in C([a,b],\mathbb{R}) : v \in \int_a^t \Xi(t,s,j(s))ds + g(t),\ t \in [a,b] \right\},$$

*where $f, g : [a,b] \longrightarrow \mathbb{R}$ are continuous and $\Gamma, \Xi : [a,b] \times [a,b] \times \mathbb{R} \longrightarrow CVB(\mathbb{R})$ is such that for each $r \in C([a,b],\mathbb{R})$, the operators $\Gamma(.,.,r)$ and $\Xi(.,.,j)$ are lower semi-continuous.*

Assume that the following conditions hold:
(i) there exist a function $\xi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and a continuous mapping $\lambda : \chi \longrightarrow [0,\infty)$ such that for all $r, j \in \chi$, we have

$$H_b(\Gamma(t,s,r(s)), \Xi(t,s,j(s))) \leq \lambda(s) M_s(r,j) \text{ for all } t \in [a,b],$$

where

$$M_s(r,j) = \max \left\{ \begin{array}{c} \check{d}_b(r(s), j(s)), D_b(r(s), \Gamma(t,s,r(s))), \\ D_b(j(s), \Xi(t,s,j(s))), \frac{D_b(r(s),\Xi(t,s,j(s))) + D_b(j(s),\Gamma(t,s,r(s)))}{2s} \end{array} \right\};$$

(ii) there exists $r_0 \in C([a,b],\mathbb{R})$ such that for all $t \in [a,b]$,

$$\xi\left( r_0(t), \int_a^t \Gamma(t,s,r_0(s))ds + f(t) \right) \geq 0;$$

(iii) for all $t \in [a,b]$ and for all $r, j, z \in C([a,b],\mathbb{R})$,

$$\xi(r(t), j(t)) \geq 0 \text{ and } \xi(j(t), z(t)) \geq 0 \Longrightarrow \xi(r(t), z(t)) \geq 0;$$

(iv) for all $t \in [a,b]$ and for all $r, j \in C([a,b],\mathbb{R})$,

$$\xi(r(t), j(t)) \geq 0 \text{ implies } \xi\left( \int_a^t \Gamma(t,s,r(s))ds + f(t), \int_a^t \Xi(t,s,j(s))ds + g(t) \right) \geq 0;$$

(v) if a sequence $\{r_n\}$ in $C([a,b],\mathbb{R})$ with $\xi(r_n(t), r_{n+1}(t)) \geq 0$ for all $n \in \mathbb{N}$ and for all $t \in [a,b]$ such that $r_n \longrightarrow r \in C([a,b])$ as $n \longrightarrow \infty$, then there exists a subsequence $\left\{r_{n(k)}\right\}$ of $\{r_n\}$ such that $\xi\left(r_{n(k)}(t), r(t)\right) \geq 0$ for all $k \in \mathbb{N}$ and for all $t \in [a,b]$;
(vi) there exist $\tau > 0$ and $s \geq 1$ such that for $t \in [a,b]$, we have

$$\int_a^t \sqrt{\lambda(s)}ds \leq \sqrt{\frac{e^{-\tau}}{s}}.$$

Then the system of integral inclusions (3.1) has a solution.
**Proof .** Let $r \in \chi$ be such that $u \in \hat{S}r$ and $\xi(r(t), j(t)) \geq 0$ for all $t \in [a,b]$. Then $\Upsilon_r(t,s) \in \Gamma_r(t,s)$ for all $t,s \in [a,b]$ such that $u(t) = \int_a^t \Upsilon_r(t,s)ds + g(t) \in u(t) = \int_a^t \Gamma_r(t,s)ds + g(t)$, $t \in [a,b]$. But

$$H_b(\Gamma(t,s,r(s)), \Xi(t,s,j(s))) \leq Z(s) \max \left\{ \begin{array}{c} \check{d}_b(r(s), j(s)), D_b(r(s), \Gamma(t,s,r(s))), \\ D_b(j(s), \Xi(t,s,j(s))), \\ \frac{D_b(r(s),\Xi(t,s,j(s))) + D_b(j(s),\Gamma(t,s,r(s)))}{2s} \end{array} \right\}$$

for all $t \in [a,b]$, so there exists $j \in \chi$, $z(t,s) \in \Xi_j(t,s)$ for all $t,s \in [a,b]$ such that

$$|\Upsilon_r(t,s) - z(t,s)|^2 \leq \lambda(s) \max \left\{ \begin{array}{c} \check{d}_b(r(s), j(s)), D_b(r(s), \Gamma_r(t,s)), \\ \check{d}_b(j(s), z(t,s)), \\ \frac{\check{d}_b(r(s),z(t,s)) + D_b(j(s),\Gamma_r(t,s))}{2s} \end{array} \right\},$$

for all $t \in [a,b]$. Now, we can consider the multivalued operator $E : [a,b] \times [a,b] \longrightarrow CB(\mathbb{R})$ defined by

$$E(t,s) = \Xi_j(t,s) \cap \{L \in \mathbb{R} \,|\, |\Upsilon_r(t,s) - L| \leq \lambda(s) M_s(r,j)\},$$

for all $t,s \in [a,b]$. Taking into account the fact that the multivalued operator $E$ is lower semi-continuous, it follows that there exists a continuous operator $\Pi_y : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ such that $\Pi_j(t,s) \in E(t,s)$ for all $t,s \in [a,b]$. We have for $v \in \check{T}j$,

$$v(t) = \int_a^t \Pi_j(t,s)ds + g(t) \in \int_a^t \Xi_j(t,s)ds + g(t), \; t \in [a,b],$$

and

$$\begin{aligned} |u(t) - v(t)|^2 &\leq \left(\int_a^t |\Upsilon_r(t,s) - \Pi_j(t,s)|\, ds\right)^2 \\ &\leq \left(\int_a^t \sqrt{\lambda(s) \max \left\{ \begin{array}{c} \check{d}_b(r(s), j(s)), \check{d}_b(r(s), \Upsilon_r(t,s)), \\ \check{d}_b(j(s), \Pi_j(t,s)), \\ \frac{\check{d}_b(r(s),\Pi_j(t,s)) + \check{d}_b(j(s),\Upsilon_r(t,s))}{2s} \end{array} \right\}} ds\right)^2 \\ &\leq \left(\int_a^t \sqrt{\lambda(s)}ds\right)^2 \max \left\{ \begin{array}{c} \check{d}_b(r,j), D_b\left(r, \hat{S}r\right), D_b\left(j, \check{T}j\right) \\ , \frac{D_b\left(r,\check{T}j\right) + D_b\left(j,\hat{S}r\right)}{2s} \end{array} \right\}. \end{aligned}$$

Consequently, we have

$$d(u,v) \leq \frac{e^{-\tau}}{s} \max \left\{ \begin{array}{c} \check{d}_b(r,j), D_b\left(r, \hat{S}r\right), D_b\left(j, \check{T}j\right), \\ \frac{D_b\left(r,\check{T}j\right) + D_b\left(j,\hat{S}r\right)}{2s} \end{array} \right\}.$$

Now, by interchanging the role of $r$ and $j$, we reach to

$$sH_b\left(\hat{S}r,\check{T}j\right)\leq e^{-\tau}M_s(r,j),\ \ r,j\in\chi,$$

where

$$M_s(r,j)=\max\left\{\check{d}_b\left(r,j\right),D_b\left(r,\hat{S}r\right),D_b\left(j,\check{T}j\right),\frac{D_b\left(r,\check{T}j\right)+D_b\left(j,\hat{S}r\right)}{2s}\right\}.$$

As $\theta\left(t\right)=e^t\in\Theta_s$, applying it on above inequality and after some simplifications, we get

$$e^{\left(sH_b\left(\hat{S}r,\check{T}j\right)\right)}\leq\left[e^{\left(M_s(r,j)\right)}\right]^{e^{-\tau}},\ \ r,j\in\chi.$$

Define $\alpha,\eta:\chi\times\chi\longrightarrow[0,\infty)$ as

$$\alpha\left(r,j\right)=\begin{cases}1, & \text{if } \xi\left(r\left(t\right),j(t)\right)\geq 0, t\in[a,b]\\ 0, & \text{otherwise}\end{cases}$$

and

$$\eta\left(r,j\right)=\begin{cases}\frac{1}{3}, & \text{if } \xi\left(r\left(t\right),j(t)\right)\geq 0, t\in[a,b]\\ 1, & \text{otherwise.}\end{cases}$$

Let $r,j\in\chi$ be such that $\alpha\left(r,j\right)\geq\min\left\{\eta_*\left(r,\hat{S}r\right),\eta_*\left(j,\check{T}j\right)\right\}$. Then $\xi\left(r\left(t\right),j(t)\right)\geq 0$ for all $t\in[a,b]$. Thus

$$e^{\left(sH_b\left(\hat{S}r,\check{T}j\right)\right)}\leq\left[e^{\left(M_s(r,j)\right)}\right]^{e^{-\tau}}.$$

This implies that

$$\theta\left(sH_b\left(\hat{S}r,\check{T}j\right)\right)\leq\left[\theta\left(M_s(r,j)\right)\right]^k,\ \text{where } k=e^{-\tau}.$$

Hence, $\left(\hat{S},\check{T}\right)$ is a generalized multi-valued $\alpha_*$-$\eta_*$-$\theta$-contraction. By using (iv), for every $r\in\chi$ with $\alpha_*\left(r,\hat{S}r\right)\geq\eta_*\left(r,\hat{S}r\right)$ and $\alpha_*\left(r,\check{T}r\right)\geq\eta_*\left(r,\check{T}r\right)$, we get

$$\xi\left(\hat{S}r\left(t\right),\check{T}^2r(t)\right)\geq 0$$

and

$$\xi\left(\check{T}r\left(t\right),\hat{S}^2r(t)\right)\geq 0.$$

Therefore, $\alpha_*\left(\hat{S}r,\check{T}^2r\right)\geq\eta_*\left(\hat{S}r,\check{T}^2r\right)$ and $\alpha_*\left(\check{T}r,\hat{S}^2r\right)\geq\eta_*\left(\check{T}r,\hat{S}^2r\right)$. Let $r,j\in\chi$ be such that $\alpha\left(r,j\right)\geq\eta\left(r,j\right)$, $\alpha_*\left(j,\hat{S}j\right)\geq\eta_*\left(j,\hat{S}j\right)$ and $\alpha_*\left(j,\check{T}j\right)\geq\eta_*\left(j,\check{T}j\right)$. Then

$$\xi\left(r\left(t\right),j(t)\right)\geq 0,\ \ \xi\left(j\left(t\right),\hat{S}j(t)\right)\geq 0 \text{ and } \xi\left(j(t),\check{T}j\left(t\right)\right)\geq 0 \text{ for all } t\in[a,b].$$

By using (iii), we get that $\xi\left(r\left(t\right),\hat{S}j(t)\right)\geq 0$, $\xi\left(r\left(t\right),\check{T}j(t)\right)\geq 0$. So $\alpha_*\left(r,\hat{S}j\right)\geq\eta_*\left(r,\hat{S}j\right)$ and $\alpha_*\left(r,Tj\right)\geq\eta_*\left(r,\check{T}j\right)$. Then $\left(\hat{S},\check{T}\right)$ is triangular $\alpha_*$-$\eta_*$-orbital admissible pair. By, (ii), there exists $r_0\in\chi$ such that

$$\alpha_*\left(r_0,\hat{S}r_0\right)\geq\min\left\{\eta_*\left(r_0,\hat{S}r_0\right),\eta_*\left(\hat{S}r_0,\check{T}\hat{S}r_0\right)\right\}.$$

Let $\{r_n\}$ be a sequence in $\chi$ such that $r_n\longrightarrow r\in\chi$ as $n\longrightarrow\infty$. Then from (v), there exists a subsequence $\left\{r_{n(k)}\right\}$ of $\{r_n\}$ such that $\xi\left(r_{n(k)}\left(t\right),r\left(t\right)\right)\geq 0$, this implies that $\alpha\left(r_{n(k)},r\right)\geq\eta\left(r_{n(k)},r\right)$. Therefore, all hypothese of Theorem 2.4 are satisfied. Hence $\hat{S}$ and $\check{T}$ have a common fixed point, that is, the system of Volterra-type integral inclusions (3.1) has a solution. □

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