



Ćirić Type multi-valued α_* - η_* - θ -Contractions on b-metric spaces with Applications

Eskandar Ameer^{a,b}, Hassen Aydi^c, Muhammad Arshad^d, Aftab Hussain^{e,*}, Abdul Rahim Khan^f

^aDepartment of Mathematics, Taiz University, Taiz

^bDepartment of Mathematics, International Islamic University, H-10, Islamabad - 44000, Pakistan

^cDepartment of Mathematics, College of Education of Jubail, Imam Abdulrahman Bin Faisal University, P. O: 12020, Industrial Jubail 31961, Saudi Arabia

^dDepartment of Mathematics, International Islamic University, H-10, Islamabad - 44000, Pakistan

^eDepartment of Mathematics, Faculty of Natural Sciences, Centre for Mathematical Research, Khawaja Fareed University of Engineering and Information Technology Rahim Yar Khan, Pakistan

^fDepartment of Mathematics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

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Abstract

In this paper, we give sufficient conditions for the existence of solutions of a system of Volterra-type integral inclusion equations using new sort of multi-valued contractions, named as generalized multi-valued α_* - η_* - θ -contractions defined on α -complete b-metric spaces. We give its relevance to fixed point results. We set up an example to elucidate our main results.

Keywords: fixed point, α -complete b-metric space, α -continuous multi-valued mappings, triangular α -orbital admissible, generalized multi-valued α_* - η_* - θ -contractions.

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1. Introduction and Preliminaries

In 1989, Bakhtin [26] investigated the concept of b-metric spaces. However, Czerwik [29, 30] initiated study of fixed point of self-mappings in b-metric spaces and proved an analogue of Banach's fixed point theorem. Since then, numerous research articles have been published comprising fixed point theorems for various classes of single-valued and multivalued operators in b-metric spaces, (see e.g., [1, 11, 12, 13, 14, 15, 19, 20, 21, 22, 25, 27, 37, 38]) and related references therein.

*Corresponding author

Email addresses: eskandarameer@yahoo.com (Eskandar Ameer), hmaydi@uod.edu.sa (Hassen Aydi), marshad_zia@yahoo.com, (Muhammad Arshad), aftabshh@gmail.com (Aftab Hussain), arahim@kfupm.edu.sa (Abdul Rahim Khan)

Definition 1.1. [29] Let χ be a non-empty set and $s \geq 1$ ($s \in \mathbb{R}$). A function $\check{d}_b : \chi \times \chi \rightarrow [0, \infty)$ is said to be a b -metric, if for all $r, j, z \in \chi$,

- (i) $\check{d}_b(r, j) = 0 \Leftrightarrow r = j$;
- (ii) $\check{d}_b(r, j) = \check{d}_b(j, r)$;
- (ii) $\check{d}_b(r, j) \leq s [\check{d}_b(r, z) + \check{d}_b(z, j)]$.

The pair (χ, \check{d}_b) is called a b -metric space (with constant s).

Example 1.2. [21] Let $H^p = \{f \in W(U) : \|f\|_{H^p} < \infty\}$, $p \in (0, 1)$ be H^p space defined on the unit disk U , where $H(U)$ is the set of all holomorphic functions on U and

$$\|f\|_{H^p} = \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

Denote $\chi = H^p(U)$. Define a mapping $\check{d}_b : \chi \times \chi \rightarrow [0, \infty)$ by

$$\check{d}_b(f, g) = \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta}) - g(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}},$$

for all $f, g \in X$. Then (χ, \check{d}_b) is a b -metric space with coefficient $s = 2^{\frac{1}{p}-1}$.

Definition 1.3. [36] Let $\check{T} : \chi \rightarrow \chi$ be a self-map and $\alpha : \chi \times \chi \rightarrow [0, +\infty)$. Then \check{T} is said to be α -admissible, if $\alpha(r, j) \geq 1 \implies \alpha(\check{T}r, \check{T}j) \geq 1$.

Definition 1.4. [23] Let $\check{T} : \chi \rightarrow \chi$ be a self-map and $\alpha : \chi \times \chi \rightarrow [0, +\infty)$. Then \check{T} is said to be triangular α -admissible, if \check{T} satisfies:

- ($\check{T}1$) $\alpha(r, j) \geq 1 \implies \alpha(\check{T}r, \check{T}j) \geq 1$;
- ($\check{T}2$) $\alpha(r, u) \geq 1$ and $\alpha(u, j) \geq 1 \implies \alpha(r, j) \geq 1$.

Definition 1.5. [35] Let $\check{T} : \chi \rightarrow \chi$ be a self-map and $\alpha : \chi \times \chi \rightarrow [0, +\infty)$. Then \check{T} is said to be α -orbital admissible if

- ($\check{T}3$) $\alpha(r, \check{T}r) \geq 1 \implies \alpha(\check{T}r, \check{T}^2r) \geq 1$.

Definition 1.6. [35] Let $\check{T} : \chi \rightarrow \chi$ be a map and $\alpha : \chi \times \chi \rightarrow [0, +\infty)$. Then \check{T} is said to be triangular α -orbital admissible if \check{T} satisfies ($\check{T}3$),

- ($\check{T}4$) $\alpha(r, j) \geq 1$ and $\alpha(j, \check{T}j) \geq 1 \implies \alpha(r, \check{T}j) \geq 1$.

Definition 1.7. [6] Let $\check{T} : \chi \rightarrow \chi$ be a map and $\alpha, \eta : \chi \times \chi \rightarrow [0, +\infty)$. Then \check{T} is said to be α -orbital admissible with respect to η if,

- ($\check{T}5$) $r \in \chi$, $\alpha(r, \check{T}r) \geq \eta(r, \check{T}r) \implies \alpha(\check{T}r, \check{T}^2r) \geq \eta(\check{T}r, \check{T}^2r)$.

Definition 1.8. [6] Let $\check{T} : \chi \rightarrow \chi$ be a map and $\alpha, \eta : \chi \times \chi \rightarrow [0, +\infty)$. Then \check{T} is said to be triangular α -orbital admissible with respect to η if \check{T} satisfies ($\check{T}5$),

- ($\check{T}6$) $r, j \in \chi$, $\alpha(r, j) \geq \eta(r, j)$ and $\alpha(j, \check{T}j) \geq \eta(j, \check{T}j) \implies \alpha(r, \check{T}j) \geq \eta(r, \check{T}j)$.

Definition 1.9. [24] Let (χ, \check{d}_b) be a b -metric space and $\alpha, \eta : \chi \times \chi \rightarrow [0, \infty)$. Then χ is said to be α - η -complete, if every Cauchy sequence $\{r_n\}$ in χ with $\alpha(r_n, r_{n+1}) \geq \eta(r_n, r_{n+1})$ for all $n \in \mathbb{N}$ converges in χ .

Recently, Jleli and Samet [8, 9] presented the notion of a θ -contraction.

Definition 1.10. Let (χ, \check{d}) be a metric space. A map $\check{T} : \chi \rightarrow \chi$ is called θ -contraction, if there exists a constant $k \in (0, 1)$ and $\theta \in \Theta$ such that,

$$r, j \in \chi, \check{d}(\check{T}r, \check{T}j) \neq 0 \implies \theta(\check{d}(\check{T}r, \check{T}j)) \leq [\theta(\check{d}(r, j))]^k,$$

Where Θ is the set of functions $\theta : (0, \infty) \rightarrow (1, \infty)$ satisfying,

($\Theta 1$) θ is non-decreasing;

($\Theta 2$) for each sequence $\{\check{t}_n\} \subset (0, \infty)$,

$$\lim_{n \rightarrow \infty} \theta(\check{t}_n) = 1 \text{ if and only if } \lim_{n \rightarrow \infty} \check{t}_n = 0^+;$$

($\Theta 3$) there exists $q \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{\check{t} \rightarrow 0^+} \frac{\theta(\check{t}) - 1}{\check{t}^q} = \ell$.

Jleli and Samet [8] established the following fixed point theorem.

Theorem 1.11. [8] Let (χ, \check{d}) be a complete metric space and $\check{T} : \chi \rightarrow \chi$ be θ -contraction. Then \check{T} has a unique fixed point.

As in [10], we denote by μ the family of all functions $\theta : (0, \infty) \rightarrow (1, \infty)$ satisfying the assertions ($\Theta 1$), ($\Theta 2$) and ($\Theta' 3$), where

($\Theta' 3$) means θ is continuous on $(0, \infty)$.

Note that ($\Theta 3$) and ($\Theta' 3$) are independent of each other [10].

Example 1.12. [10] For all $t \in (0, \infty)$, consider

$$\begin{aligned} \phi_1(t) &= e^t, & \phi_4(t) &= \cosh t; \\ \phi_2(t) &= e^{\sqrt{te^t}}, & \phi_5(t) &= 1 + \ln(t + 1); \\ \phi_3(t) &= e^{\sqrt{t}}, & \phi_6(t) &= e^{te^t}. \end{aligned}$$

Then $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6 \in \mu$.

Very recently, Hussain et al. [5] defined a generalized (α, η) - Θ -contraction and extended the results of Jleli and Samet [8].

Let (χ, \check{d}_b) be a b -metric space. Let $CB_b(\chi)$ denote the set of all closed and bounded subsets of χ . For $r \in \chi$ and $A, B \in CB_b(\chi)$, define

$$D_b(r, A) = \inf_{a \in A} \check{d}_b(r, a) \text{ and } D_b(A, B) = \sup_{a \in A} D_b(a, B).$$

Define a mapping $H_b : CB_b(\chi) \times CB_b(\chi) \rightarrow [0, \infty)$ by

$$H_b(A, B) = \max \left\{ \sup_{r \in A} D_b(r, B), \sup_{j \in B} D_b(j, A) \right\},$$

for each $A, B \in CB_b(\chi)$. Hence the map H_b is called Hausdorff b -metric (induced by a b -metric space (χ, \check{d}_b)).

Lemma 1.13. [29] Let (χ, \check{d}_b) be a b -metric space. For any $A, B \in CB_b(\chi)$ and any $r, j \in \chi$, we have:

- (i) $D_b(r, B) \leq \check{d}_b(r, b)$ for any $b \in B$;
- (ii) $D_b(r, B) \leq H_b(A, B)$;
- (iii) $D_b(r, A) \leq s [d_b(r, j) + D_b(j, B)]$.

Lemma 1.14. [29] Let (χ, \check{d}_b) be a b -metric space, $A, B \in CB_b(\chi)$ and $M > 1$. Then for all $a \in A$, there exists $b \in B$ such that $\check{d}_b(a, b) \leq MH_b(A, B)$.

Definition 1.15. [33] Let $\check{T} : \chi \rightarrow CB_b(\chi)$ be a multi-valued mapping and $\alpha : \chi \times \chi \rightarrow [0, +\infty)$. Then \check{T} is said to be α_* -admissible if $\alpha(r, j) \geq 1 \implies \alpha_*(\check{T}r, \check{T}j) \geq 1$, where

$$\alpha_*(A, B) = \inf \{ \alpha(r, j) : r \in A, j \in B \}.$$

Now, we introduce the following definitions.

Definition 1.16. Let $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$ be two multi-valued maps and $\alpha, \eta : \chi \times \chi \rightarrow [0, +\infty)$ be two functions. We say that (\hat{S}, \check{T}) is triangular $\alpha_*\text{-}\eta_*$ -admissible pair, if:

(i) $\alpha(r, j) \geq \eta(r, j) \implies \alpha_*(\hat{S}r, \check{T}j) \geq \eta_*(\hat{S}r, \check{T}j)$ and $\alpha_*(\check{T}r, \hat{S}j) \geq \eta_*(\check{T}r, \hat{S}j)$, where

$$\begin{aligned} \alpha_*(A, B) &= \inf \{ \alpha(r, j) : r \in A, j \in B \}, \\ \eta_*(A, B) &= \inf \{ \eta(r, j) : r \in A, j \in B \}; \end{aligned}$$

(ii) $\alpha(r, u) \geq \eta(r, u)$ and $\alpha(u, j) \geq \eta(u, j) \implies \alpha(r, j) \geq \eta(r, j)$.

Definition 1.17. Let $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$ be two multi-valued maps and $\alpha, \eta : \chi \times \chi \rightarrow [0, +\infty)$ be functions. We say that (\hat{S}, \check{T}) is an $\alpha_*\text{-}\eta_*$ -orbital admissible pair, if,

(i) $\alpha_*(r, \hat{S}r) \geq \eta_*(r, \hat{S}r)$ and $\alpha_*(r, \check{T}r) \geq \eta_*(r, \check{T}r) \implies \alpha_*(\hat{S}r, \check{T}^2r) \geq \eta_*(\hat{S}r, \check{T}^2r)$ and $\alpha_*(\check{T}r, \hat{S}^2r) \geq \eta_*(\check{T}r, \hat{S}^2r)$.

Definition 1.18. Let $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$ be two multi-valued maps and $\alpha, \eta : \chi \times \chi \rightarrow [0, +\infty)$ be functions. Then (\hat{S}, \check{T}) is said to be triangular $\alpha_*\text{-}\eta_*$ -orbital admissible pair, if:

(i) (\hat{S}, \check{T}) is $\alpha_*\text{-}\eta_*$ -orbital admissible pair;

(ii) $\alpha(r, j) \geq \eta(r, j)$, $\alpha_*(j, \hat{S}j) \geq \eta_*(j, \hat{S}j)$ and $\alpha_*(j, \check{T}j) \geq \eta_*(j, \check{T}j) \implies \alpha_*(r, \hat{S}j) \geq \eta_*(r, \hat{S}j)$ and $\alpha_*(r, \check{T}j) \geq \eta_*(r, \check{T}j)$.

Lemma 1.19. Let $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$ such that (\hat{S}, \check{T}) is triangular $\alpha_*\text{-}\eta_*$ -orbital admissible pair. Assume that, there exists $r_0 \in \chi$ such that $\alpha_*(r_0, \hat{S}r_0) \geq \min \{ \eta_*(r_0, \hat{S}r_0), \eta_*(\hat{S}r_0, \check{T}\hat{S}r_0) \}$. Define the sequence $\{r_{\check{n}}\}$ in χ by $r_{2i+1} \in \hat{S}r_{2i}$ and $r_{2i+2} \in \check{T}r_{2i+1}$, where $i = 0, 1, 2, \dots$. Then for $\check{n}, m \in \mathbb{N} \cup \{0\}$ with $m > \check{n}$, we have $\alpha(r_{\check{n}}, r_m) \geq \eta(r_{\check{n}}, r_m)$.

Definition 1.20. Let (χ, \check{d}_b) be a b -metric space. Let $\hat{S} : \chi \rightarrow CB_b(\chi)$ and $\alpha, \eta : \chi \times \chi \rightarrow [0, +\infty)$. Then \hat{S} is said to be a multivalued $\alpha\text{-}\eta$ -continuous on $(CB_b(\chi), H_b)$ if whenever $\{r_{\check{n}}\}$ is a sequence in χ with $\alpha(r_{\check{n}}, r_{\check{n}+1}) \geq \eta(r_{\check{n}}, r_{\check{n}+1})$ for all $\check{n} \in \mathbb{N}$ and $r \in \chi$ such that $\lim_{\check{n} \rightarrow \infty} \check{d}_b(r_{\check{n}}, r) = 0$, then

$$\lim_{\check{n} \rightarrow \infty} H_b(\hat{S}r_{\check{n}}, \hat{S}r) = 0.$$

2. Fixed point results

First, inspired by Jleli and Samet [8, 9], we give the following definition.

Definition 2.1. Let $s \geq 1$. We denote by Θ_s the set of all functions $\theta : (0, \infty) \rightarrow (1, \infty)$, with the following properties:

- (Θ_s 1) θ is non-decreasing;
- (Θ_s 2) for each sequence $\{\check{t}_n\} \subset (0, \infty)$,

$$\lim_{n \rightarrow \infty} \theta(\check{t}_n) = 1 \Leftrightarrow \lim_{n \rightarrow \infty} \check{t}_n = 0^+;$$

(Θ_s 3) there exists $q \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{\check{t} \rightarrow 0^+} \frac{\theta(\check{t}) - 1}{\check{t}^q} = \ell$.

(Θ_s 4) for each sequence $\{\check{t}_n\} \subset (0, \infty)$ such that $\theta(s\check{t}_n) \leq [\theta(\check{t}_{n-1})]^k$, for all $\check{n} \in \mathbb{N}$ then $\theta(s^n\check{t}_n) \leq [\theta(s^{n-1}\check{t}_{n-1})]^k$, for some $k \in (0, 1)$ and for all $\check{n} \in \mathbb{N}$.

Example 2.2. Let $\theta : (0, \infty) \rightarrow (1, \infty)$ defined by $\theta(t) = e^{\sqrt{t}}$. Then clearly, θ satisfies (Θ_s 1)-(Θ_s 4). Now we show only, (Θ_s 4). suppose that, for some $k \in (0, 1)$ and for all $n \in \mathbb{N}$, we $e^{\sqrt{s\check{t}_n}} \leq [\theta(e^{\sqrt{\check{t}_{n-1}}})]^k$. Thus,

$$\begin{aligned} e^{\sqrt{s^n\check{t}_n}} &= e^{\sqrt{s^{n-1}s\check{t}_n}} = \left[e^{\sqrt{s\check{t}_n}} \right]^{\sqrt{s^{n-1}}} \\ &\leq \left[\left(e^{\sqrt{\check{t}_{n-1}}} \right)^k \right]^{\sqrt{s^{n-1}}} = \left[e^{\sqrt{s^{n-1}\check{t}_{n-1}}} \right]^k, \end{aligned}$$

hence (Θ_s 4) holds true. Note that also, $\theta(t) = e^{\sqrt{t}} \in \Theta_s$.

Now, we introduce the concept of generalized multi-valued α_* - η_* - θ -contractions as follows:

Definition 2.3. Let (χ, \check{d}_b) be a b -metric space, and $\alpha, \eta : \chi \times \chi \rightarrow [0, \infty)$ be two functions. Let $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$ be two multi-valued maps. Then (\hat{S}, \check{T}) is called a generalized multi-valued α_* - η_* - θ -contraction if for $r, j \in \chi$, with $\alpha(r, j) \geq \min \{ \eta_*(r, \hat{S}r), \eta_*(j, \check{T}j) \}$ and $H_b(\hat{S}r, \check{T}j) > 0$, we have

$$\theta \left(sH_b(\hat{S}r, \check{T}j) \right) \leq [\theta(M_s(r, j))]^k, \tag{2.1}$$

where $\theta \in \Theta_s$, $k \in (0, 1)$ and

$$M_s(r, j) = \max \left\{ \check{d}_b(r, j), D_b(r, \hat{S}r), D_b(j, \check{T}j), \frac{D_b(r, \check{T}j) + D_b(j, \hat{S}r)}{2s} \right\}. \tag{2.2}$$

The following theorem is our main result.

Theorem 2.4. Let (χ, \check{d}_b) be a b -metric space and $\alpha, \eta : \chi \times \chi \rightarrow [0, \infty)$ be two functions. Let $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$ be such that (\hat{S}, \check{T}) is a generalized multi-valued α_* - η_* - θ -contraction. Suppose that,

- (i) (X, \check{d}_b) is an α - η -complete b -metric space;

- (ii) (\hat{S}, \check{T}) is triangular α_* - η_* -orbital admissible pair;
 - (iii) there exists $r_0 \in \chi$ such that $\alpha_*(r_0, \hat{S}r_0) \geq \min \left\{ \eta_*(r_0, \hat{S}r_0), \eta_*(\hat{S}r_0, \check{T}\hat{S}r_0) \right\}$;
 - (iv) \hat{S} and \check{T} are multi-valued α - η -continuous.
- Then \hat{S} and \check{T} have a common fixed point $r^* \in \chi$.

Proof . Let $r_0 \in \chi$ be such that $\alpha_*(r_0, \hat{S}r_0) \geq \min \left\{ \eta_*(r_0, \hat{S}r_0), \eta_*(\hat{S}r_0, \check{T}\hat{S}r_0) \right\}$. Choose $r_1 \in \hat{S}r_0$ such that

$$\alpha(r_0, r_1) \geq \min \left\{ \eta_*(r_0, \hat{S}r_0), \eta_*(r_1, \check{T}r_1) \right\}$$

and $r_1 \neq r_0$. By (2.1) and Lemma 1.12, we have

$$0 < \theta(sD_b(r_1, \check{T}r_1)) \leq \theta(sH_b(\hat{S}r_0, \check{T}r_1)). \tag{2.3}$$

There exists $x_2 \in \check{T}r_1$ such that

$$\begin{aligned} 0 &\leq \theta(s\check{d}_b(r_1, r_2)) \leq \theta(sH_b(\hat{S}r_0, \check{T}r_1)) \\ &\leq [\theta(M_s(r_0, r_1))]^k, \end{aligned}$$

which implies that

$$0 < \theta(s\check{d}_b(r_1, r_2)) \leq [\theta(M_b(r_0, r_1))]^k, \tag{2.4}$$

where

$$\begin{aligned} M_s(r_0, r_1) &= \max \left\{ \check{d}_b(r_0, r_1), D_b(r_0, \hat{S}r_0), D_b(r_1, \check{T}r_1), \frac{D_b(r_0, \check{T}r_1) + D_b(r_1, \hat{S}r_0)}{2s} \right\} \\ &\leq \max \left\{ \check{d}_b(r_0, r_1), \check{d}_b(r_0, r_1), \check{d}_b(r_1, r_2), \frac{D_b(r_0, \check{T}r_1) + \check{d}_b(r_1, r_1)}{2s} \right\} \\ &\leq \max \left\{ \check{d}_b(r_0, r_1), \check{d}_b(r_1, r_2), \frac{D_b(r_0, \check{T}r_1)}{2s} \right\}. \end{aligned}$$

Since

$$\begin{aligned} \frac{D_b(r_0, \check{T}r_1)}{2s} &\leq \frac{s[\check{d}_b(r_0, r_1) + D_b(r_1, \check{T}r_1)]}{2s} \\ &\leq \frac{[\check{d}_b(r_0, r_1) + D_b(r_1, \check{T}r_1)]}{2} \leq \max \{ \check{d}_b(r_0, r_1), D_b(r_1, \check{T}r_1) \}, \end{aligned}$$

then we get

$$M_s(r_0, r_1) \leq \max \{ \check{d}_b(r_0, r_1), D_b(r_1, \check{T}r_1) \}.$$

If $\max \{ \check{d}_b(r_0, r_1), D_b(r_1, \check{T}r_1) \} = D_b(r_1, \check{T}r_1)$, then from (2.4), we have

$$\theta(sD_b(r_1, \check{T}r_1)) \leq [\theta(D_b(r_1, \check{T}r_1))]^k < \theta(D_b(r_1, \check{T}r_1)),$$

which is a contradiction. Therefore,

$$\max \{ \check{d}_b(r_0, r_1), D_b(r_1, \check{T}r_1) \} = \check{d}_b(r_0, r_1).$$

By (2.4), we get that $\theta (s\check{d}_b (r_1, r_2)) < \theta (\check{d}_b (r_0, r_1))$. Similarly, for $r_2 \in \check{T}r_1$ and $r_3 \in \hat{S}r_2$,

$$\begin{aligned} \theta (s\check{d}_b (r_2, r_3)) &\leq \theta \left(sD_b \left(r_2, \hat{S}r_2 \right) \right) \\ &\leq \theta \left(sH_b \left(\check{T}r_1, \hat{S}r_2 \right) \right) \\ &\leq \theta (\check{d}_b (r_1, r_2)), \end{aligned}$$

which implies that

$$\theta (s\check{d}_b (r_2, r_3)) \leq \theta (\check{d}_b (r_1, r_2)). \tag{2.5}$$

Continuing in this way, we define a sequence $\{r_n\}$ in χ such that $r_{2i+1} \in \hat{S}r_{2i}$ and $r_{2i+2} \in \check{T}r_{2i+1}$, $i = 0, 1, 2, \dots$

Since $\alpha_* (r_0, \hat{S}r_0) \geq \min \left\{ \eta_* (r_0, \hat{S}r_0), \eta_* (\hat{S}r_0, \check{T}\hat{S}r_0) \right\}$ and (\hat{S}, \check{T}) is triangular α_* - η_* -orbital admissible pair, so by using Lemma 1.19, we get

$$\alpha (r_n, r_{n+1}) \geq \eta (r_n, r_{n+1}), \text{ for all } n \in \mathbb{N}.$$

Then

$$\begin{aligned} 0 < \theta (s\check{d}_b (r_{2i+1}, r_{2i+2})) &\leq \theta \left(sH_b \left(\hat{S}r_{2i}, \check{T}r_{2i+1} \right) \right) \\ &\leq [\theta (M_s (r_{2i}, r_{2i+1}))]^k, \end{aligned} \tag{2.6}$$

for all $i \in \mathbb{N}$, where

$$\begin{aligned} M_s (r_{2i}, r_{2i+1}) &= \max \left\{ \check{d}_b (r_{2i}, r_{2i+1}), D_b \left(r_{2i}, \hat{S}r_{2i} \right), D_b \left(r_{2i+1}, \check{T}r_{2i+1} \right), \right. \\ &\quad \left. \frac{D_b (r_{2i}, \check{T}r_{2i+1}) + D_b (r_{2i+1}, \hat{S}r_{2i})}{2s} \right\} \\ &\leq \max \left\{ \check{d}_b (r_{2i}, r_{2i+1}), \check{d}_b (r_{2i}, r_{2i+1}), \check{d}_b (r_{2i+1}, r_{2i+2}), \right. \\ &\quad \left. \frac{D_b (r_{2i}, \check{T}r_{2i+1})}{2s} \right\} \\ &\leq \max \left\{ \check{d}_b (r_{2i}, r_{2i+1}), \check{d}_b (r_{2i+1}, r_{2i+2}), \frac{D_b (r_{2i}, \check{T}r_{2i+1})}{2s} \right\}. \end{aligned}$$

Since

$$\begin{aligned} \frac{D_b (r_{2i}, \check{T}r_{2i+1})}{2s} &\leq \frac{s [\check{d}_b (r_{2i}, r_{2i+1}) + \check{d}_b (r_{2i+1}, r_{2i+2})]}{2s} \\ &\leq \frac{[\check{d}_b (r_{2i}, r_{2i+1}) + \check{d}_b (r_{2i+1}, r_{2i+2})]}{2} \\ &\leq \max \{ \check{d}_b (r_{2i}, r_{2i+1}), \check{d}_b (r_{2i+1}, r_{2i+2}) \}, \end{aligned}$$

then we get

$$M_s (r_{2i}, r_{2i+1}) \leq \max \{ \check{d}_b (r_{2i}, r_{2i+1}), \check{d}_b (r_{2i+1}, r_{2i+2}) \}, \quad \forall i \geq 0.$$

If for some i , $\max \{ \check{d}_b (r_{2i}, r_{2i+1}), \check{d}_b (r_{2i+1}, r_{2i+2}) \} = \check{d}_b (r_{2i+1}, r_{2i+2})$, then by (2.6) we have

$$\begin{aligned} 1 &< \theta (\check{d}_b (r_{2i+1}, r_{2i+2})) \leq [\theta (\check{d}_b (r_{2i+1}, r_{2i+2}))]^k \\ &< \theta (\check{d}_b (r_{2i+1}, r_{2i+2})), \end{aligned}$$

which is a contradiction. Thus

$$\max \{ \check{d}_b(r_{2i}, r_{2i+1}), \check{d}_b(r_{2i+1}, r_{2i+2}) \} = \check{d}_b(r_{2i}, r_{2i+1}) \quad \forall i \geq 0.$$

By (2.6), we get that

$$1 < \theta(s\check{d}_b(r_{2i+1}, r_{2i+2})) \leq [\theta(\check{d}_b(r_{2i}, r_{2i+1}))]^k < \theta(\check{d}_b(r_{2i}, r_{2i+1})) \quad \forall i \geq 0.$$

This implies that

$$1 < \theta(s\check{d}_b(r_{n+1}, r_{n+2})) \leq [\theta(\check{d}_b(r_n, r_{n+1}))]^k < \theta(\check{d}_b(r_n, r_{n+1})) \quad \forall n \geq 0. \tag{2.7}$$

From (2.7) and axiom (Θ_s4) , we have

$$1 < \theta(s^n\check{d}_b(r_{n+1}, r_{n+2})) \leq [\theta(s^{n-1}\check{d}_b(r_{n-1}, r_n))]^k < \theta(s^{n-1}\check{d}_b(r_{n-1}, r_n)) \quad \forall n \geq 0. \tag{2.8}$$

Further,

$$\begin{aligned} 1 < \theta(s^n\check{d}_b(r_{n+1}, r_{n+2})) &= \theta\left(s^n\check{d}_b\left(\hat{S}r_n, \check{T}r_{n+1}\right)\right) \leq [\theta(s^{n-1}\check{d}_b(r_{n-1}, r_n))]^k \\ &= \left[\theta\left(s^{n-1}\check{d}_b\left(\hat{S}r_{n-2}, \check{T}r_{n-1}\right)\right)\right]^k \leq [\theta(s^{n-2}\check{d}_b(r_{n-1}, r_{n-2}))]^{k^2} \\ &\leq \dots \leq [\theta(\check{d}_b(r_0, r_1))]^{k^n}, \end{aligned}$$

Which implies,

$$1 < \theta(s^n\check{d}_b(r_{n+1}, r_{n+2})) \leq [\theta(\check{d}_b(r_0, r_1))]^{k^n}, \tag{2.9}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \rightarrow \infty$ in (2.9), since $\theta \in \Theta_s$, we have

$$\lim_{n \rightarrow \infty} \theta(s^n\check{d}_b(r_{n+1}, r_{n+2})) = 1,$$

By (Θ_s2) , we get

$$\lim_{n \rightarrow \infty} s^n\check{d}_b(r_{n+1}, r_{n+2}) = 0. \tag{2.10}$$

From condition (Θ_s3) , there exist $q \in (0, 1)$ and $\ell \in (0, \infty]$ such that

$$\lim_{n \rightarrow \infty} \frac{\theta(s^n\check{d}_b(r_{n+1}, r_{n+2})) - 1}{[s^n\check{d}_b(r_{n+1}, r_{n+2})]^q} = \ell.$$

Suppose that $\ell < \infty$. Let $W = \frac{\ell}{2} > 0$. From the definition of the limit, there exists $n_0 \geq 1$ such that

$$\left| \frac{\theta(s^n\check{d}_b(r_{n+1}, r_{n+2})) - 1}{[s^n\check{d}_b(r_{n+1}, r_{n+2})]^q} - \ell \right| \leq W \quad \text{for all } n \geq n_0.$$

This implies

$$\frac{\theta(s^n\check{d}_b(r_{n+1}, r_{n+2})) - 1}{[s^n\check{d}_b(r_{n+1}, r_{n+2})]^q} \geq \ell - W = W \quad \text{for all } n \geq n_0.$$

Then

$$n [s^n\check{d}_b(r_{n+1}, r_{n+2})]^q \leq An [\theta(s^n\check{d}_b(r_{n+1}, r_{n+2})) - 1] \quad \text{for all } n \geq n_0,$$

where $P = \frac{1}{W}$. Suppose now that $\ell = \infty$. Let $W > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \geq 1$ such that

$$\frac{\theta (s^n \check{d}_b (r_{n+1}, r_{n+2})) - 1}{[s^n \check{d}_b (r_{n+1}, r_{n+2})]^q} \geq W \text{ for all } n \geq n_0.$$

Which implies

$$n [s^n \check{d}_b (r_{n+1}, r_{n+2})]^q \leq Pn [\theta (s^n \check{d}_b (r_{n+1}, r_{n+2})) - 1] \text{ for all } n \geq n_0,$$

where $P = \frac{1}{W}$. Thus, in all cases, there exist $P > 0$ and $n_0 \geq 1$ such that

$$n [s^n \check{d}_b (r_{n+1}, r_{n+2})]^q \leq Pn [\theta (s^n \check{d}_b (r_{n+1}, r_{n+2})) - 1] \text{ for all } n \geq n_0.$$

By using (2.9), we get

$$n [s^n \check{d}_b (r_{n+1}, r_{n+2})]^q \leq Pn \left([\theta (d(r_0, r_1))]^{k^n} - 1 \right) \text{ for all } n \geq n_0. \tag{2.11}$$

Setting $n \rightarrow \infty$ in the inequality (2.11), we get

$$\lim_{n \rightarrow \infty} n [s^n \check{d}_b (r_{n+1}, r_{n+2})]^q = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$s^n \check{d}_b (r_{n+1}, r_{n+2}) \leq \frac{1}{n^{\frac{1}{q}}} \text{ for all } n \geq n_1. \tag{2.12}$$

To prove $\{r_n\}$ is a Cauchy sequence, we use (2.12) and for $m > n \geq n_1$,

$$\begin{aligned} \check{d}_b (r_n, r_m) &\leq \sum_{i=n}^{m-1} s^i \check{d}_b (r_i, r_{i+1}) \leq \sum_{i=n}^{\infty} s^i \check{d}_b (r_i, r_{i+1}) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{q}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{q}}}$ entails $\lim_{n \rightarrow \infty} \check{d}_b (r_n, r_m) = 0$. Thus $\{r_n\}$ is a Cauchy sequence. Since χ is an α - η -complete b-metric space and $\alpha (r_n, r_{n+1}) \geq \eta (r_n, r_{n+1})$, for all $n \in \mathbb{N}$, there exists $r^* \in \chi$ such that $\lim_{n \rightarrow \infty} d (r_n, r^*) = 0$. This implies that $\lim_{i \rightarrow \infty} \check{d}_b (r_{2i+1}, r^*) = 0$ and $\lim_{i \rightarrow \infty} \check{d}_b (r_{2i+2}, r^*) = 0$. As \check{T} is an α - η -continuous multivalued mapping, so $\lim_{i \rightarrow \infty} H_b (r_{2i+1}, r^*) = 0$. Thus

$$D_b (r^*, \check{T}r^*) = \lim_{i \rightarrow \infty} D_b (r_{2i+2}, \check{T}r^*) \leq \lim_{i \rightarrow \infty} H_b (\check{T}r_{2i+1}, \check{T}r^*) = 0.$$

Consequently, $r^* \in \check{T}r^*$. Similarly, $r^* \in \hat{S}r^*$. Therefore, $r^* \in \chi$ is a common fixed point of \hat{S} and \check{T} . \square

Theorem 2.5. Let (χ, \check{d}_b) be a b-metric space, and $\alpha, \eta : \chi \times \chi \rightarrow [0, \infty)$. Let $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$ be such that (\hat{S}, \check{T}) is a generalized multi-valued α_* - η_* - θ -contraction. Suppose that,

- (i) (χ, \check{d}_b) is an α - η -complete b-metric space;
- (ii) (\hat{S}, \check{T}) is triangular α_* - η_* -orbital admissible pair;
- (iii) there exists $r_0 \in \chi$ such that $\alpha_* (r_0, \hat{S}r_0) \geq \min \left\{ \eta_* (r_0, \hat{S}r_0), \eta_* (\hat{S}r_0, \check{T}\hat{S}r_0) \right\}$;
- (iv) if $\{r_n\}$ is a sequence in χ such that $\alpha (r_n, r_{n+1}) \geq \eta (r_n, r_{n+1})$ for all $n \in \mathbb{N}$ and $r_n \rightarrow r^* \in \chi$ as $n \rightarrow \infty$, then either $\alpha_* (\hat{S}r_n, r^*) \geq \eta_* (\hat{S}r_n, r^*)$ or $\alpha_* (\check{T}r_{n+1}, r^*) \geq \eta_* (\check{T}r_{n+1}, r^*)$ holds for all $n \in \mathbb{N}$.

Then \hat{S} and \check{T} have a common fixed point $r^* \in \chi$.

Proof . Let $r_0 \in \chi$ be such that $\alpha_* \left(r_0, \hat{S}r_0 \right) \geq \min \left\{ \eta_* \left(r_0, \hat{S}r_0 \right), \eta_* \left(\hat{S}r_0, \check{T}\hat{S}r_0 \right) \right\}$. As in proof of Theorem 2.4, we construct a sequence $\{r_n\}$ in χ defined by $r_{2i+1} \in \hat{S}r_{2i}$ and $r_{2i+2} \in \check{T}r_{2i+1}$, where $i \geq 0$, $\alpha \left(r_n, r_{n+1} \right) \geq \eta \left(r_n, r_{n+1} \right)$, for all $n \in \mathbb{N}$ and $\{r_n\}$ converges to $r^* \in \chi$. Since $\alpha \left(r_n, x_{rn+1} \right) \geq \eta \left(r_n, r_{n+1} \right)$ for all $n \in \mathbb{N}$ and $r_n \rightarrow r^* \in \chi$ as $n \rightarrow \infty$, by condition (iv), either $\alpha_* \left(\hat{S}r_n, r^* \right) \geq \eta_* \left(\hat{S}r_n, r^* \right)$ or $\alpha_* \left(\check{T}r_{n+1}, r^* \right) \geq \eta_* \left(\check{T}r_{n+1}, r^* \right)$ holds all $n \in \mathbb{N}$. Thus,

$$\alpha \left(r_{n+1}, r^* \right) \geq \eta \left(r_{n+1}, r^* \right) \text{ or } \alpha \left(r_{n+2}, r^* \right) \geq \eta \left(r_{n+2}, r^* \right), \text{ holds for all } n \in \mathbb{N}.$$

Equivalently, there exists a subsequence $\{r_{n(k)}\}$ of $\{r_n\}$ such that

$$\alpha \left(r_{n(k)}, r^* \right) \geq \eta \left(r_{n(k)}, r^* \right) \text{ for all } k \in \mathbb{N} \tag{2.13}$$

From (2.13), we deduce that

$$\begin{aligned} \theta \left(D_b \left(r_{2n(k)+1}, \check{T}r^* \right) \right) &\leq \theta \left(D_b \left(\hat{S}r_{2n(k)}, \check{T}r^* \right) \right) \leq \theta \left(sH_b \left(\hat{S}r_{2n(k)}, \check{T}r^* \right) \right) \\ &\leq \left[\theta \left(M_s \left(r_{2n(k)}, r^* \right) \right) \right]^k. \end{aligned}$$

This implies that

$$\theta \left(D_b \left(r_{2n(k)+1}, \check{T}r^* \right) \right) \leq \left[\theta \left(M_s \left(r_{2n(k)}, r^* \right) \right) \right]^k < \theta \left(M_s \left(r_{2n(k)}, r^* \right) \right), \tag{2.14}$$

where

$$\begin{aligned} M_s \left(r_{2n(k)}, r^* \right) &= \max \left\{ \begin{aligned} &\check{d}_b \left(r_{2n(k)}, r^* \right), D_b \left(r_{2n(k)}, \hat{S}r_{2n(k)} \right), D_b \left(r^*, \check{T}r^* \right), \\ &\frac{D_b \left(r_{2n(k)}, \hat{S}r^* \right) + D_b \left(r^*, \check{T}r_{2n(k)} \right)}{2s} \end{aligned} \right\} \\ &\leq \max \left\{ \begin{aligned} &\check{d}_b \left(r_{2n(k)}, r^* \right), \check{d}_b \left(r_{2n(k)}, r_{2n(k)+1} \right), D_b \left(r^*, \check{T}r^* \right), \\ &\frac{D_b \left(r_{2n(k)}, \check{T}r^* \right) + D_b \left(r^*, \hat{S}r_{2n(k)} \right)}{2s} \end{aligned} \right\}. \end{aligned}$$

Suppose that $r^* \notin \check{T}r^*$, then $D_b \left(r^*, \check{T}r^* \right) > 0$. Taking the limit as $k \rightarrow \infty$ in (2.14) and using the condition $(\Theta'3)$, we have

$$\theta \left(D_b \left(r^*, \check{T}r^* \right) \right) < \theta \left(D_b \left(r^*, \check{T}r^* \right) \right).$$

It is a contradiction. Hence $D_b \left(r^*, \check{T}r^* \right) = 0$, and so $r^* \in \check{T}r^*$. Similarly, we can show that $r^* \in \hat{S}r^*$. Thus $r^* \in \chi$ is a common fixed point of \hat{S} and \check{T} . \square

Example 2.6. Let $\chi = [-1, 1]$ and define the function $\check{d}_b : \chi \times \chi \rightarrow [0, +\infty)$ by $\check{d}_b(r, j) = |r - j|^2$. Clearly, (χ, \check{d}_b) is a complete b -metric space with $s = 2$. Let $\theta(t) = e^t, t > 0$, then $\theta \in \Theta_s$. Define the mappings $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$ by

$$\check{T}r = \begin{cases} \left[0, \frac{2r}{245} \right], & \text{if } r \in [-1, 0] \\ \{1\}, & \text{if } r \in (0, 1] \end{cases},$$

and

$$\hat{S}r = \begin{cases} \left[0, \frac{r}{300} \right], & \text{if } r \in [-1, 0] \\ \{1\}, & \text{if } r \in (0, 1] \end{cases}.$$

Moreover, define the functions $\alpha, \eta : \chi \times \chi \rightarrow [0, \infty)$ by

$$\alpha(r, j) = \begin{cases} 1, & \text{if } r, j \in [-1, 0] \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\eta(r, j) = \begin{cases} \frac{1}{5}, & \text{if } r, j \in [-1, 0] \\ 3, & \text{otherwise.} \end{cases}$$

If $\{r_n\}$ is a Cauchy sequence such that $\alpha(r_n, r_{n+1}) \geq \eta(r_n, r_{n+1})$ for all $n \in \mathbb{N}$, then $\{r_n\} \subseteq [-1, 0]$. Since $([-1, 0], \check{d}_b)$ is a complete b -metric space, then the sequence $\{r_n\}$ converges in $[-1, 0] \subseteq \chi$. Thus (χ, \check{d}_b) is an α - η -complete b -metric space. Let $\alpha_*(r, \hat{S}r) \geq \eta_*(r, \hat{S}r)$ and $\alpha_*(r, \check{T}r) \geq \eta_*(r, \check{T}r)$. So, $r \in [-1, 0]$ and $\hat{S}r, \check{T}r \in [-1, 0]$. Hence $\hat{S}^2r = \hat{S}(\hat{S}r), \check{T}^2r = \check{T}(\check{T}r) \in [-1, 0]$. Then $\alpha_*(\hat{S}r, \check{T}^2r) \geq \eta_*(\hat{S}r, \check{T}^2r)$ and $\alpha_*(\check{T}r, \hat{S}^2r) \geq \eta_*(\check{T}r, \hat{S}^2r)$. Thus, (\hat{S}, \check{T}) is α_* - η_* -orbital admissible. Let $r, j \in \chi$ be such that $\alpha(r, j) \geq \eta(r, j)$, $\alpha_*(j, \hat{S}j) \geq \eta_*(j, \hat{S}j)$ and $\alpha_*(j, \check{T}j) \geq \eta_*(j, \check{T}j)$. Then we have $r, j, \hat{S}j, \check{T}j \in [-1, 0]$, which implies that $\alpha_*(r, \hat{S}j) \geq \eta_*(r, \hat{S}j)$ and $\alpha_*(r, \check{T}j) \geq \eta_*(r, \check{T}j)$. Hence, (\hat{S}, \check{T}) is triangular α_* - η_* -orbital admissible pair. Let $\{r_n\}$ be a sequence such that $r_n \rightarrow r$ as $n \rightarrow \infty$ and $\alpha(r_n, r_{n+1}) \geq \eta(r_n, r_{n+1})$ for all $n \in \mathbb{N}$. Then $\{r_n\} \subseteq [-1, 0]$ for all $n \in \mathbb{N}$. So $\lim_{n \rightarrow \infty} \check{T}r_n = \lim_{n \rightarrow \infty} [0, \frac{2}{245}r_n] = [0, \frac{2}{245}r] = \check{T}r$. Hence \check{T} is a multi-valued α - η -continuous. Similarly, we can check that S is a multi-valued α - η -continuous. Let $r_0 = -\frac{1}{2}$. Then

$$\begin{aligned} \alpha_*\left(-\frac{1}{2}, \hat{S}\left(-\frac{1}{2}\right)\right) &= \alpha_*\left(-\frac{1}{2}, 0\right) = 1 \\ &\geq \min \left\{ \begin{array}{l} \eta_*\left(-\frac{1}{2}, \hat{S}\left(-\frac{1}{2}\right)\right), \\ \eta_*\left(\hat{S}\left(-\frac{1}{2}\right), \check{T}\left(\hat{S}\left(-\frac{1}{2}\right)\right)\right) \end{array} \right\} = \frac{1}{5}. \end{aligned}$$

Let $r, j \in \chi$ be such that $\alpha(r, j) \geq \min\{\eta_*(r, \hat{S}r), \eta_*(j, \check{T}j)\}$. Then $r, j \in [-1, 0]$ and $H_b(\hat{S}r, \check{T}j) > 0$. So

$$\theta(sH_b(\hat{S}r, \check{T}j)) \leq [\theta(M_s(r, j))]^k,$$

where $k \in (\frac{4}{5}, 1)$. Hence all hypotheses of Theorem 2.4 are satisfied. Thus, \hat{S} and \check{T} have a common fixed point.

Corollary 2.7. Let (χ, \check{d}_b) be a complete b -metric space, and $\alpha, \eta : \chi \times \chi \rightarrow [0, \infty)$. Let $\hat{S} : \chi \rightarrow CB_b(\chi)$ be such that \hat{S} is a generalized multi-valued α_* - η_* - θ -contraction. Suppose that

- (i) (χ, \check{d}_b) is an α - η -complete b -metric space;
- (ii) \hat{S} is triangular α_* - η_* -orbital admissible;
- (iii) there exists $r_0 \in \chi$ such that $\alpha_*(r_0, \hat{S}r_0) \geq \min\{\eta_*(r_0, \hat{S}r_0), \eta_*(\hat{S}r_0, \check{T}\hat{S}r_0)\}$;
- (iv) either \hat{S} is a multi-valued α - η -continuous or if $\{r_n\}$ is a sequence in χ such that $\alpha(r_n, r_{n+1}) \geq \eta(r_n, r_{n+1})$ for all $n \in \mathbb{N}$ and $r_n \rightarrow r^* \in \chi$ as $n \rightarrow \infty$, then either $\alpha_*(\hat{S}r_n, r^*) \geq \eta_*(\hat{S}r_n, r^*)$ or $\alpha_*(\hat{S}r_{n+1}, r^*) \geq \eta_*(\hat{S}r_{n+1}, r^*)$ holds for all $n \in \mathbb{N}$.

Then \hat{S} has a fixed point $r^* \in \chi$.

Definition 2.8. Let (χ, \check{d}_b) be a b -metric space. Let $\alpha, \eta : \chi \times \chi \rightarrow [0, \infty)$ and $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$ be two multi-valued mappings. Then (\hat{S}, \check{T}) is said to be a multi-valued $\alpha_*\eta_*$ - θ -contraction mapping, if there exists $\theta \in \Theta_s$ such that for all $r, j \in \chi$ with $\alpha(r, j) \geq \min \left\{ \eta_*(r, \hat{S}r), \eta_*(j, \check{T}j) \right\}$, (\hat{S}, \check{T}) satisfies:

$$\theta \left(sH_b \left(\hat{S}r, \check{T}j \right) \right) \leq [\theta \left(\check{d}_b(r, j) \right)]^k, \quad k \in (0, 1).$$

Theorem 2.9. Let (χ, \check{d}_b) be a b -metric space, and $\alpha, \eta : \chi \times \chi \rightarrow [0, \infty)$. Let $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$ be such that (\hat{S}, \check{T}) is a multi-valued $\alpha_*\eta_*$ - θ -contraction. Suppose that,

- (i) (χ, \check{d}_b) is an α - η -complete b -metric space;
- (ii) (\hat{S}, \check{T}) is triangular $\alpha_*\eta_*$ -orbital admissible pair;
- (iii) there exists $r_0 \in \chi$ such that $\alpha_*(r_0, \hat{S}r_0) \geq \min \left\{ \eta_*(r_0, \hat{S}r_0), \eta_*(\hat{S}r_0, \check{T}\hat{S}r_0) \right\}$;
- (iv) either \hat{S} and \check{T} are multi-valued α - η -continuous or if $\{r_n\}$ is a sequence in χ such that $\alpha(r_n, r_{n+1}) \geq \eta(r_n, r_{n+1})$ for all $n \in \mathbb{N}$ and $r_n \rightarrow r^* \in \chi$ as $n \rightarrow \infty$, then either $\alpha_*(\hat{S}r_n, r^*) \geq \eta_*(\hat{S}r_n, r^*)$ or $\alpha_*(\check{T}r_{n+1}, r^*) \geq \eta_*(\check{T}r_{n+1}, r^*)$ holds for all $n \in \mathbb{N}$.

Then \hat{S} and \check{T} have a common fixed point $r^* \in \chi$.

Corollary 2.10. Let (χ, \preceq) be a partially ordered set and $\hat{S}, \check{T} : \chi \rightarrow \chi$. Suppose that there exists a b -metric \check{d}_b on χ such that (χ, \check{d}_b) is a complete b -metric space. Assume that,

- (i) there exists $\theta \in \Theta_s$ such that

$$\theta \left(sd \left(\hat{S}r, \check{T}j \right) \right) \leq [\theta \left(M_s(r, j) \right)]^k,$$

where $k \in (0, 1)$ and

$$M_s(r, j) = \max \left\{ \check{d}_b(r, j), \check{d}_b(r, \hat{S}r), \check{d}_b(j, \check{T}j), \frac{\check{d}_b(r, \check{T}j) + \check{d}_b(j, \hat{S}r)}{2s} \right\}$$

for all $r, j \in \chi$ with $r \preceq j$ and $\check{d}_b(\hat{S}r, \check{T}j) > 0$;

- (ii) \hat{S} and \check{T} are nondecreasing (that is, if for all $r, j \in \chi$, $r \preceq j$ implies $\hat{S}r \preceq \hat{S}j$);
- (iii) there exists $r_0 \in \chi$ such that $r_0 \preceq \hat{S}r_0$;
- (iv) either \hat{S} and \check{T} are continuous or if $\{r_n\}$ is a sequence in χ such that $r_n \preceq r_{n+1}$ for all $n \in \mathbb{N}$ and $r_n \rightarrow r^* \in \chi$ as $n \rightarrow \infty$, then either $\hat{S}r_n \preceq r^*$ or $\check{T}r_{n+1} \preceq r^*$ holds for all $n \in \mathbb{N}$.

Then \hat{S} and \check{T} have a common fixed point $r^* \in \chi$.

Now, we deduce certain Suzuki-Samet type fixed point results.

Theorem 2.11. Let (χ, \check{d}_b) be a complete b -metric space. Let $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$ be two continuous multi-valued mappings. If for all $r, j \in \chi$ with

$$\frac{1}{2} \min \left\{ D_b(r, \hat{S}r), D_b(j, \check{T}j) \right\} \leq \check{d}_b(r, j),$$

and $H_b(\hat{S}r, \check{T}j) > 0$, we have

$$\theta \left(sH_b \left(\hat{S}r, \check{T}j \right) \right) \leq [\theta \left(M_s(r, j) \right)]^k,$$

where $\theta \in \Theta_s$. Then \hat{S} and \check{T} have a common fixed point.

Proof . Define $\alpha, \eta : \chi \times \chi \longrightarrow [0, \infty)$ by

$$\alpha(r, j) = \check{d}_b(r, j) \text{ and } \eta(r, j) = \frac{1}{2}\check{d}_b(r, j),$$

for all $r, j \in \chi$. Since $\frac{1}{2}\check{d}_b(r, j) \leq \check{d}_b(r, j)$ for all $r, j \in \chi$, so $\eta(r, j) \leq \alpha(r, j)$ for all $r, j \in \chi$. Hence the conditions (i), (iii) and (iv) of Theorem 2.2 hold. Since \hat{S} and \check{T} are continuous, \hat{S} and \check{T} are α - η -continuous multi-valued mappings. Let $\min \left\{ \eta_* \left(r, \hat{S}r \right), \eta_* \left(r, \check{T}r \right) \right\} \leq \alpha(r, j)$ with $H_b \left(\hat{S}r, \check{T}j \right) > 0$. Equivalently, if $\frac{1}{2} \min \left\{ D_b \left(r, \hat{S}r \right), D_b \left(j, \check{T}j \right) \right\} \leq \check{d}_b(r, j)$ with $H_b \left(\hat{S}r, \check{T}j \right) > 0$, then we have

$$\theta \left(sH_b \left(\hat{S}r, \check{T}j \right) \right) \leq [\theta \left(M_s(r, j) \right)]^k.$$

That is, (\hat{S}, \check{T}) is a generalized multi-valued α_* - η_* - θ -contraction. Hence, all conditions of Theorem 2.2 hold. Thus \hat{S} and \check{T} have a common fixed point. \square

Theorem 2.12. Let (χ, \check{d}_b) be a complete b -metric space. Let $\hat{S}, \check{T} : \chi \longrightarrow CB_b(\chi)$. If for all $r, j \in \chi$ with

$$\frac{1}{2(1 + \pi)} \min \left\{ D_b \left(r, \hat{S}r \right), D_b \left(j, \check{T}j \right) \right\} \leq \check{d}_b(r, j),$$

$\pi > 0$ and $H_b \left(\hat{S}r, \check{T}j \right) > 0$, we have

$$\theta \left(sH_b \left(\hat{S}r, \check{T}j \right) \right) \leq [\theta \left(M_s(r, j) \right)]^k,$$

where $\theta \in \Theta_s$. Then \hat{S} and \check{T} have a common fixed point.

Proof . The result follows from Theorem 2.3 by taking $\alpha, \eta : \chi \times \chi \longrightarrow [0, \infty)$ as

$$\alpha(r, j) = \check{d}_b(r, j) \text{ and } \eta(r, j) = \frac{1}{2(1 + \pi)}\check{d}_b(r, j).$$

\square

3. Application

we apply the result given by Theorem 2.4 to study the existence of a solution for a system of Volterra-type integral inclusions. For instance,

Consider the following system of Volterra-type integral inclusions:

$$r(t) \in \int_a^t \Gamma(t, s, r(s))ds + f(t) \text{ and } j(t) \in \int_a^t \Xi(t, s, j(s))ds + g(t) \tag{3.1}$$

where $\Gamma, \Xi : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow CVB(\mathbb{R})$, and $CVB(\mathbb{R})$ denotes the family of nonempty closed, convex and bounded subsets of \mathbb{R} (set of all real numbers). let $\chi = C([a, b], \mathbb{R})$ be the space of all continuous real valued functions on $[a, b]$. Note that χ is a complete b -metric space by considering $\check{d}_b(r, j) = \sup_{t \in [a, b]} |r(t) - j(t)|^2$ with $s = 2$. For each $r, j \in C([a, b], \mathbb{R})$, the operators $\Gamma(\cdot, \cdot, x)$ and $\Xi(\cdot, \cdot, y)$ are lower semi-continuous. Further, the functions $f, g : [a, b] \longrightarrow \mathbb{R}$ are continuous.

For the system of integrals inclusion given above, we can define multivalued operators $\hat{S}, \check{T} : C([a, b], \mathbb{R}) \rightarrow CB(C([a, b], \mathbb{R}))$ as follows:

$$\hat{S}r(t) = \left\{ u \in C([a, b], \mathbb{R}) : u \in \int_a^t \Gamma(t, s, r(s))ds + f(t), t \in [a, b] \right\},$$

and

$$\check{T}j(t) = \left\{ v \in C([a, b], \mathbb{R}) : v \in \int_a^t \Xi(t, s, j(s))ds + g(t), t \in [a, b] \right\}.$$

Let $r, j \in C([a, b], \mathbb{R})$ and denote $\Gamma_r := \Gamma(t, s, r(s))$ and $\Xi_j := \Xi(t, s, j(s))$, $t, s \in [a, b]$. Now for $\Gamma_r, \Xi_j : [a, b] \times [a, b] \rightarrow CVB(\mathbb{R})$, by Michael's selection theorem, there exist continuous operators $\Upsilon_r, \Pi_j : [a, b] \times [a, b] \rightarrow \mathbb{R}$ with $\Upsilon_r(t, s) \in \Gamma_r(t, s)$ and $\Pi_j(t, s) \in \Xi_j(t, s)$ for all $t, s \in [a, b]$. This shows that $\int_a^t \Upsilon_r(t, s)ds + f(t) \in \hat{S}r(t)$ and $\int_a^t \Pi_j(t, s)ds + g(t) \in \check{T}j(t)$. Thus, the operators $\hat{S}r$ and $\check{T}j$ are nonempty. Since g, Υ_r and Π_j are continuous on $[a, b]$ (resp. $[a, b] \times [a, b]$), their ranges are bounded and hence $\hat{S}r$ and $\check{T}j$ are bounded (i.e., $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$).

Theorem 3.1. Take $\chi = C([a, b], \mathbb{R})$. Consider the multivalued operators $\hat{S}, \check{T} : \chi \rightarrow CB_b(\chi)$,

$$\hat{S}r(t) = \left\{ u \in C([a, b], \mathbb{R}) : u \in \int_a^t \Gamma(t, s, r(s))ds + f(t), t \in [a, b] \right\},$$

and

$$\check{T}j(t) = \left\{ v \in C([a, b], \mathbb{R}) : v \in \int_a^t \Xi(t, s, j(s))ds + g(t), t \in [a, b] \right\},$$

where $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous and $\Gamma, \Xi : [a, b] \times [a, b] \times \mathbb{R} \rightarrow CVB(\mathbb{R})$ is such that for each $r \in C([a, b], \mathbb{R})$, the operators $\Gamma(\cdot, \cdot, r)$ and $\Xi(\cdot, \cdot, j)$ are lower semi-continuous.

Assume that the following conditions hold:

(i) there exist a function $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}$ and a continuous mapping $\lambda : \chi \rightarrow [0, \infty)$ such that for all $r, j \in \chi$, we have

$$H_b(\Gamma(t, s, r(s)), \Xi(t, s, j(s))) \leq \lambda(s) M_s(r, j) \text{ for all } t \in [a, b],$$

where

$$M_s(r, j) = \max \left\{ \begin{array}{l} \check{d}_b(r(s), j(s)), D_b(r(s), \Gamma(t, s, r(s))), \\ D_b(j(s), \Xi(t, s, j(s))), \frac{D_b(r(s), \Xi(t, s, j(s))) + D_b(j(s), \Gamma(t, s, r(s)))}{2s} \end{array} \right\};$$

(ii) there exists $r_0 \in C([a, b], \mathbb{R})$ such that for all $t \in [a, b]$,

$$\xi \left(r_0(t), \int_a^t \Gamma(t, s, r_0(s))ds + f(t) \right) \geq 0;$$

(iii) for all $t \in [a, b]$ and for all $r, j, z \in C([a, b], \mathbb{R})$,

$$\xi(r(t), j(t)) \geq 0 \text{ and } \xi(j(t), z(t)) \geq 0 \implies \xi(r(t), z(t)) \geq 0;$$

(iv) for all $t \in [a, b]$ and for all $r, j \in C([a, b], \mathbb{R})$,

$$\xi(r(t), j(t)) \geq 0 \text{ implies } \xi \left(\int_a^t \Gamma(t, s, r(s))ds + f(t), \int_a^t \Xi(t, s, j(s))ds + g(t) \right) \geq 0;$$

- (v) if a sequence $\{r_n\}$ in $C([a, b], \mathbb{R})$ with $\xi(r_n(t), r_{n+1}(t)) \geq 0$ for all $n \in \mathbb{N}$ and for all $t \in [a, b]$ such that $r_n \rightarrow r \in C([a, b])$ as $n \rightarrow \infty$, then there exists a subsequence $\{r_{n(k)}\}$ of $\{r_n\}$ such that $\xi(r_{n(k)}(t), r(t)) \geq 0$ for all $k \in \mathbb{N}$ and for all $t \in [a, b]$;
- (vi) there exist $\tau > 0$ and $s \geq 1$ such that for $t \in [a, b]$, we have

$$\int_a^t \sqrt{\lambda(s)} ds \leq \sqrt{\frac{e^{-\tau}}{s}}.$$

Then the system of integral inclusions (3.1) has a solution.

Proof. Let $r \in \chi$ be such that $u \in \hat{S}r$ and $\xi(r(t), j(t)) \geq 0$ for all $t \in [a, b]$. Then $\Upsilon_r(t, s) \in \Gamma_r(t, s)$ for all $t, s \in [a, b]$ such that $u(t) = \int_a^t \Upsilon_r(t, s) ds + g(t) \in u(t) = \int_a^t \Gamma_r(t, s) ds + g(t)$, $t \in [a, b]$. But

$$H_b(\Gamma(t, s, r(s)), \Xi(t, s, j(s))) \leq Z(s) \max \left\{ \begin{array}{l} \check{d}_b(r(s), j(s)), D_b(r(s), \Gamma(t, s, r(s))), \\ D_b(j(s), \Xi(t, s, j(s))), \\ \frac{D_b(r(s), \Xi(t, s, j(s))) + D_b(j(s), \Gamma(t, s, r(s)))}{2s} \end{array} \right\}$$

for all $t \in [a, b]$, so there exists $j \in \chi$, $z(t, s) \in \Xi_j(t, s)$ for all $t, s \in [a, b]$ such that

$$|\Upsilon_r(t, s) - z(t, s)|^2 \leq \lambda(s) \max \left\{ \begin{array}{l} \check{d}_b(r(s), j(s)), D_b(r(s), \Gamma_r(t, s)), \\ \check{d}_b(j(s), z(t, s)), \\ \frac{\check{d}_b(r(s), z(t, s)) + D_b(j(s), \Gamma_r(t, s))}{2s} \end{array} \right\},$$

for all $t \in [a, b]$. Now, we can consider the multivalued operator $E : [a, b] \times [a, b] \rightarrow CB(\mathbb{R})$ defined by

$$E(t, s) = \Xi_j(t, s) \cap \{L \in \mathbb{R} \mid |\Upsilon_r(t, s) - L| \leq \lambda(s) M_s(r, j)\},$$

for all $t, s \in [a, b]$. Taking into account the fact that the multivalued operator E is lower semi-continuous, it follows that there exists a continuous operator $\Pi_j : [a, b] \times [a, b] \rightarrow \mathbb{R}$ such that $\Pi_j(t, s) \in E(t, s)$ for all $t, s \in [a, b]$. We have for $v \in \check{T}j$,

$$v(t) = \int_a^t \Pi_j(t, s) ds + g(t) \in \int_a^t \Xi_j(t, s) ds + g(t), \quad t \in [a, b],$$

and

$$\begin{aligned} |u(t) - v(t)|^2 &\leq \left(\int_a^t |\Upsilon_r(t, s) - \Pi_j(t, s)| ds \right)^2 \\ &\leq \left(\int_a^t \sqrt{\lambda(s) \max \left\{ \begin{array}{l} \check{d}_b(r(s), j(s)), \check{d}_b(r(s), \Upsilon_r(t, s)), \\ \check{d}_b(j(s), \Pi_j(t, s)), \\ \frac{\check{d}_b(r(s), \Pi_j(t, s)) + \check{d}_b(j(s), \Upsilon_r(t, s))}{2s} \end{array} \right\}} ds \right)^2 \\ &\leq \left(\int_a^t \sqrt{\lambda(s)} ds \right)^2 \max \left\{ \begin{array}{l} \check{d}_b(r, j), D_b(r, \hat{S}r), D_b(j, \check{T}j) \\ \frac{D_b(r, \check{T}j) + D_b(j, \hat{S}r)}{2s} \end{array} \right\}. \end{aligned}$$

Consequently, we have

$$d(u, v) \leq \frac{e^{-\tau}}{s} \max \left\{ \begin{array}{l} \check{d}_b(r, j), D_b(r, \hat{S}r), D_b(j, \check{T}j), \\ \frac{D_b(r, \check{T}j) + D_b(j, \hat{S}r)}{2s} \end{array} \right\}.$$

Now, by interchanging the role of r and j , we reach to

$$sH_b \left(\hat{S}r, \check{T}j \right) \leq e^{-\tau} M_s(r, j), \quad r, j \in \chi,$$

where

$$M_s(r, j) = \max \left\{ \check{d}_b(r, j), D_b(r, \hat{S}r), D_b(j, \check{T}j), \frac{D_b(r, \check{T}j) + D_b(j, \hat{S}r)}{2s} \right\}.$$

As $\theta(t) = e^t \in \Theta_s$, applying it on above inequality and after some simplifications, we get

$$e^{(sH_b(\hat{S}r, \check{T}j))} \leq [e^{(M_s(r, j))}]^{e^{-\tau}}, \quad r, j \in \chi.$$

Define $\alpha, \eta : \chi \times \chi \rightarrow [0, \infty)$ as

$$\alpha(r, j) = \begin{cases} 1, & \text{if } \xi(r(t), j(t)) \geq 0, t \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$

and

$$\eta(r, j) = \begin{cases} \frac{1}{3}, & \text{if } \xi(r(t), j(t)) \geq 0, t \in [a, b] \\ 1, & \text{otherwise.} \end{cases}$$

Let $r, j \in \chi$ be such that $\alpha(r, j) \geq \min \{ \eta_*(r, \hat{S}r), \eta_*(j, \check{T}j) \}$. Then $\xi(r(t), j(t)) \geq 0$ for all $t \in [a, b]$. Thus

$$e^{(sH_b(\hat{S}r, \check{T}j))} \leq [e^{(M_s(r, j))}]^{e^{-\tau}}.$$

This implies that

$$\theta \left(sH_b \left(\hat{S}r, \check{T}j \right) \right) \leq [\theta(M_s(r, j))]^k, \quad \text{where } k = e^{-\tau}.$$

Hence, (\hat{S}, \check{T}) is a generalized multi-valued α_* - η_* - θ -contraction. By using (iv), for every $r \in \chi$ with $\alpha_*(r, \hat{S}r) \geq \eta_*(r, \hat{S}r)$ and $\alpha_*(r, \check{T}r) \geq \eta_*(r, \check{T}r)$, we get

$$\xi(\hat{S}r(t), \check{T}^2r(t)) \geq 0$$

and

$$\xi(\check{T}r(t), \hat{S}^2r(t)) \geq 0.$$

Therefore, $\alpha_*(\hat{S}r, \check{T}^2r) \geq \eta_*(\hat{S}r, \check{T}^2r)$ and $\alpha_*(\check{T}r, \hat{S}^2r) \geq \eta_*(\check{T}r, \hat{S}^2r)$. Let $r, j \in \chi$ be such that $\alpha(r, j) \geq \eta(r, j)$, $\alpha_*(j, \hat{S}j) \geq \eta_*(j, \hat{S}j)$ and $\alpha_*(j, \check{T}j) \geq \eta_*(j, \check{T}j)$. Then

$$\xi(r(t), j(t)) \geq 0, \quad \xi(j(t), \hat{S}j(t)) \geq 0 \text{ and } \xi(j(t), \check{T}j(t)) \geq 0 \text{ for all } t \in [a, b].$$

By using (iii), we get that $\xi(r(t), \hat{S}j(t)) \geq 0$, $\xi(r(t), \check{T}j(t)) \geq 0$. So $\alpha_*(r, \hat{S}j) \geq \eta_*(r, \hat{S}j)$ and $\alpha_*(r, \check{T}j) \geq \eta_*(r, \check{T}j)$. Then (\hat{S}, \check{T}) is triangular α_* - η_* -orbital admissible pair. By, (ii), there exists $r_0 \in \chi$ such that

$$\alpha_*(r_0, \hat{S}r_0) \geq \min \{ \eta_*(r_0, \hat{S}r_0), \eta_*(\hat{S}r_0, \check{T}\hat{S}r_0) \}.$$

Let $\{r_n\}$ be a sequence in χ such that $r_n \rightarrow r \in \chi$ as $n \rightarrow \infty$. Then from (v), there exists a subsequence $\{r_{n(k)}\}$ of $\{r_n\}$ such that $\xi(r_{n(k)}(t), r(t)) \geq 0$, this implies that $\alpha(r_{n(k)}, r) \geq \eta(r_{n(k)}, r)$. Therefore, all hypotheses of Theorem 2.4 are satisfied. Hence \hat{S} and \check{T} have a common fixed point, that is, the system of Volterra-type integral inclusions (3.1) has a solution. \square

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