Int. J. Nonlinear Anal. Appl. 12 (2021) No. 1, 649-668
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

Construction of Generating Functions of the Products of Vieta Polynomials with Gaussian Numbers and Polynomials

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(Communicated by Tiantian Zhang)

Abstract

In the present paper, we introduce the recurrence relations of Vieta Fibonacci, Vieta Lucas, Vieta Pell and Vieta Pell Lucas polynomials. We obtain the generating functions of these polynomials, then we give the new generating functions of the products of these polynomials and the products of these polynomials with Gaussian numbers and polynomials. These results are based on the relation between Vieta polynomials and Chebyshev polynomials of first and second kinds.

Keywords: Generating functions, Vieta Fibonacci polynomials, Vieta Lucas polynomials, Vieta Pell polynomials, Gaussian numbers.
2010 MSC: 05E05, 11B39.

1. Introduction

In [5], Horadam consider the Vieta Fibonacci polynomials and Vieta Lucas polynomials which are defined by the following recurrence relations:

$$\begin{cases} V_n(x) = xV_{n-1}(x) - V_{n-2}(x), \ \forall n \geq 2 \\ V_0(x) = 0, V_1(x) = 1 \end{cases},$$

and

$$\begin{cases} v_n(x) = xv_{n-1}(x) - v_{n-2}(x), \ \forall n \geq 2 \\ v_0(x) = 2, v_1(x) = x \end{cases},$$

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Received: 6 April 2020 *Accepted:* 28 September 2020

respectively.

Vieta Pell polynomials and Vieta Pell Lucas polynomials are defined in [6] by the following recurrence relations:

$$\begin{cases} t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x), \ \forall n \geq 2 \\ t_0(x) = 0, t_1(x) = 1 \end{cases},$$

and

$$\begin{cases} s_n(x) = 2xs_{n-1}(x) - s_{n-2}(x), \ \forall n \geq 2 \\ s_0(x) = 2, s_1(x) = 2x. \end{cases},$$

respectively.

Chebyshev polynomials are a sequence of orthogonal polynomials which can be defined recursively. Recall that the n th Chebyshev polynomials of the first and second kinds are denoted by $(T_n(x))_{n\in\mathbb{N}}$ and $(U_n(x))_{n\in\mathbb{N}}$, respectively.

The Chebyshev polynomials of the first kind $(T_n(x))_{n\in\mathbb{N}}$ is given either by the generating function (see [4]):

$$\sum_{n=0}^{+\infty} T_n(x)z^n = \frac{1-xz}{1-2xz+z^2},$$

or by the recurrence relation

$$\begin{cases} T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \ \forall n \geq 2 \\ T_0(x) = 1, \ T_1(x) = x \end{cases}.$$

The Chebyshev polynomials of the second kind $(U_n(x))_{n\in\mathbb{N}}$ are given either by the generating function (see [4]):

$$\sum_{n=0}^{+\infty} U_n(x)z^n = \frac{1}{1-2xz+z^2},$$

or by the recurrence relation

$$\begin{cases} U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \ \forall n \geq 2 \\ U_0(x) = 1, \ U_1(x) = 2x \end{cases}.$$

It is well known that the Chebyshev polynomials of the first and second kinds are closely related to Vieta Fibonacci and Vieta Lucas polynomials, Vieta Pell and Vieta Pell Lucas polynomials. So, in [19] Vitula and Slota redefined Vieta polynomials as modified Chebyshev polynomials.

Corollary 1.1. *[8, 7, 6] For $n \in \mathbb{N}$, the related features of Vieta and Chebyshev polynomials are given as*

$$v_n(x) = 2T_n(\frac{1}{2}x). \tag{1.1}$$

$$s_n(x) = 2T_n(x). \tag{1.2}$$

$$t_{n+1}(x) = U_n(x). \tag{1.3}$$

$$V_{n+1}(x) = U_n(\frac{1}{2}x). \tag{1.4}$$

The characteristic equation of the Vieta Pell and Vieta Pell Lucas polynomials is

$$\lambda^2 - 2x\lambda + 1 = 0. \tag{1.5}$$

Let α and β be the roots of the characteristic equation (1.5). α and β satisfy the following equations

$$\alpha + \beta = 2x,\ \alpha\beta = 1,\ \alpha - \beta = 2\sqrt{x^2 - 1}.$$

Using the standart tecniques, we have the Binet's formulas of $(t_n(x))_{n\in\mathbb{N}}$ and $(s_n(x))_{n\in\mathbb{N}}$ polynomials as

$$t_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

and

$$s_n(x) = \alpha^n + \beta^n,$$

respectively.

The authors in [13] are calculated the generating functions of the products of Gaussian numbers with Chebyshev polynomials of first and second kinds.

In [9], the authors are calculated the generating functions of the products of Gaussian polynomials with Chebyshev polynomials of first and second kinds.

In the papers [1, 2], second-order linear recurrence sequence $(U_n(a, b; p, q))_{n\geq 0}$ or briefly $(U_n)_{n\geq 0}$ is considered by the recurrence relation:

$$U_{n+2} = pU_{n+1} + qU_n,\ \forall n \geq 0,$$

with the initial conditions $U_0 = a$ and $U_1 = b$, special cases are listed in the Table below:

a	b	p	q	$(U_n)_{n\geq 0}$	**Gaussian numbers**
i	1	1	1	$(GF_n)_{n\geq 0}$	Gaussian Fibonacci numbers
$2-i$	$1+2i$	1	1	$(GL_n)_{n\geq 0}$	Gaussian Lucas numbers
$\frac{i}{2}$	1	1	2	$(GJ_n)_{n\geq 0}$	Gaussian Jacobsthal numbers
$2-\frac{i}{2}$	$1+2i$	1	2	$(Gj_n)_{n\geq 0}$	Gaussian Jacobsthal Lucas numbers
i	1	2	1	$(GP_n)_{n\geq 0}$	Gaussian Pell numbers
$2-2i$	$2+2i$	2	1	$(GQ_n)_{n\geq 0}$	Gaussian Pell Lucas numbers

Table 1. The Gaussian numbers.

The Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials $(GJ_n(x))_{n\in\mathbb{N}}$ and $(Gj_n(x))_{n\in\mathbb{N}}$ are defined and studied by authors [15]. They give generating functions, Binet's formulas, explicit formulas, Q matrix, determinantal representations and partial derivation of these polynomials.

The Gaussian Pell polynomials, say $(GP_n(x))_{n\in\mathbb{N}}$, is defined in [18] recurrently by

$$\begin{cases} GP_0(x) = i, GP_1(x) = 1 \\ GP_n(x) = 2xGP_{n-1}(x) + GP_{n-2}(x),\ \forall n \geq 2 \end{cases}.$$

It is note that we have an important relation between Gaussian Pell polynomials and usual Pell polynomials as follows.

$$GP_n(x) = P_{n-1}(x) + iP_{n-2}(x),\quad \forall\ n \geq 2.$$

2. Generating Functions of Vieta Polynomials

In this section, we calculate the generating functions of Vieta polynomials.

Theorem 2.1. *For $n \in \mathbb{N}$, the generating function of Vieta Fibonacci polynomials is given by*

$$\sum_{n=0}^{+\infty} V_n(x)z^n = \frac{z}{1 - xz + z^2}.$$

Proof . We have

$$\sum_{n=0}^{+\infty} U_n(x)z^n = \frac{1}{1 - 2xz + z^2}.$$

From which it follows

$$\sum_{n=0}^{+\infty} U_{n-1}(x)z^n = \frac{z}{1 - 2xz + z^2}.$$

According to the relationship (1.4), we obtain

$$\sum_{n=0}^{+\infty} V_n(x)z^n = \sum_{n=0}^{+\infty} U_{n-1}(\frac{1}{2}x)z^n = \frac{z}{1 - 2(\frac{x}{2})z + z^2} = \frac{z}{1 - xz + z^2}.$$

The proof is completed. □

Theorem 2.2. *For $n \in \mathbb{N}$, the generating function of Vieta Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} v_n(x)z^n = \frac{2 - xz}{1 - xz + z^2}.$$

Proof . We have

$$\sum_{n=0}^{+\infty} T_n(x)z^n = \frac{1 - xz}{1 - 2xz + z^2}.$$

According to the relationship (1.1), we obtain

$$\sum_{n=0}^{+\infty} v_n(x)z^n = 2\sum_{n=0}^{+\infty} T_n(\frac{1}{2}x)z^n = 2(\frac{1 - \frac{x}{2}z}{1 - 2(\frac{x}{2})z + z^2}) = \frac{2 - xz}{1 - xz + z^2}.$$

The proof is completed. □

Theorem 2.3. *For $n \in \mathbb{N}$, the generating function of Vieta Pell polynomials is given by*

$$\sum_{n=0}^{+\infty} t_n(x)z^n = \frac{z}{1 - 2xz + z^2}.$$

Proof . We have

$$\sum_{n=0}^{+\infty} U_{n-1}(x)z^n = \frac{z}{1-2xz+z^2}.$$

Then, according to the relationship (1.3), we obtain

$$\sum_{n=0}^{+\infty} t_n(x)z^n = \sum_{n=0}^{+\infty} U_{n-1}(x)z^n = \frac{z}{1-2xz+z^2}.$$

This completes the proof. □

Theorem 2.4. *For $n \in \mathbb{N}$, the generating function of Vieta Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} s_n(x)z^n = \frac{2-2xz}{1-2xz+z^2}.$$

Proof . We have

$$s_n(x) = 2T_n(x).$$

From which it follows

$$\sum_{n=0}^{+\infty} s_n(x)z^n = 2\sum_{n=0}^{+\infty} T_n(x)z^n = \frac{2-2xz}{1-2xz+z^2}.$$

The proof is completed. □

3. Generating Functions of the Products of Vieta Polynomials

In this section, we calculate the new generating functions of the products of Vieta polynomials by using the related features of Vieta and Chebyshev polynomials, we have the following results.

Theorem 3.1. *For $n \in \mathbb{N}$, the new generating function of the product of Vieta Fibonacci polynomials is given by*

$$\sum_{n=0}^{+\infty} V_n(x)V_n(y)z^n = \frac{z-z^3}{1-xyz+(x^2+y^2-2)z^2-xyz^3+z^4}.$$

Proof . We have

$$\sum_{n=0}^{+\infty} U_n(x)U_n(y)z^n = \frac{1-z^2}{1-4xyz+(4x^2+4y^2-2)z^2-4xyz^3+z^4}, \text{ (see [3]).}$$

From which it follows

$$\sum_{n=0}^{+\infty} U_{n-1}(x)U_{n-1}(y)z^n = \frac{z-z^3}{1-4xyz+(4x^2+4y^2-2)z^2-4xyz^3+z^4}.$$

Then, according to the relationship (1.4), we obtain

$$\begin{aligned}
\sum_{n=0}^{+\infty} V_n(x)V_n(y)z^n &= \sum_{n=0}^{+\infty} U_{n-1}(\frac{x}{2})U_{n-1}(\frac{y}{2})z^n \\
&= \frac{z-z^3}{1-4(\frac{x}{2})(\frac{y}{2})z+(x^2+y^2-2)z^2-4(\frac{x}{2})(\frac{y}{2})z^3+z^4} \\
&= \frac{z-z^3}{1-xyz+(x^2+y^2-2)z^2-xyz^3+z^4}.
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *For $n \in \mathbb{N}$, the new generating function of the product of Vieta Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} v_n(x)v_n(y)z^n = \frac{4-3xyz+(2x^2+2y^2-4)z^2-xyz^3}{1-xyz+(x^2+y^2-2)z^2-xyz^3+z^4}.$$

Proof . We have

$$\sum_{n=0}^{+\infty} T_n(x)T_n(y)z^n = \frac{1-3xyz+(2x^2+2y^2-1)z^2-xyz^3}{1-4xyz+(4x^2+4y^2-2)z^2-4xyz^3+z^4}, \text{ (see [3])}.$$

Then

$$\begin{aligned}
\sum_{n=0}^{+\infty} v_n(x)v_n(y)z^n &= \sum_{n=0}^{+\infty} \left(2T_n(\frac{1}{2}x)\right)\left(2T_n(\frac{1}{2}y)\right)z^n \\
&= 4\sum_{n=0}^{+\infty} T_n(\frac{1}{2}x)T_n(\frac{1}{2}y)z^n \\
&= \frac{4\left[1-3\left(\frac{x}{2}\right)\left(\frac{y}{2}\right)z+(2\left(\frac{x}{2}\right)^2+2\left(\frac{y}{2}\right)^2-1)z^2-\left(\frac{x}{2}\right)\left(\frac{y}{2}\right)z^3\right]}{1-4\left(\frac{x}{2}\right)\left(\frac{y}{2}\right)z+(4\left(\frac{x}{2}\right)^2+4\left(\frac{y}{2}\right)^2-2)z^2-4\left(\frac{x}{2}\right)\left(\frac{y}{2}\right)z^3+z^4} \\
&= \frac{4-3xyz+(2x^2+2y^2-4)z^2-xyz^3}{1-xyz+(x^2+y^2-2)z^2-xyz^3+z^4}.
\end{aligned}$$

This completes the proof. □

Proposition 3.3. *For $n \in \mathbb{N}$, the new generating function of the product of Vieta Pell polynomials is given by*

$$\sum_{n=0}^{+\infty} t_n(x)t_n(y)z^n = \frac{z-z^3}{1-4xyz+(4x^2+4y^2-2)z^2-4xyz^3+z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} U_{n-1}(x)U_{n-1}(y)z^n = \frac{z-z^3}{1-4xyz+(4x^2+4y^2-2)z^2-4xyz^3+z^4},$$

and according the relationship (1.3), we get

$$\sum_{n=0}^{+\infty} t_n(x)t_n(y)z^n = \frac{z-z^3}{1-4xyz+(4x^2+4y^2-2)z^2-4xyz^3+z^4}.$$

This completes the proof. □

Theorem 3.4. *For $n \in \mathbb{N}$, the new generating function of the product of Vieta Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} s_n(x)s_n(y)z^n = \frac{4 - 12xyz + (8x^2 + 8y^2 - 4)z^2 - 4xyz^3}{1 - 4xyz + (4x^2 + 4y^2 - 2)z^2 - 4xyz^3 + z^4}.$$

Proof . We have $s_n(x) = 2T_n(x)$, then

$$\sum_{n=0}^{+\infty} s_n(x)s_n(y)z^n = \sum_{n=0}^{+\infty} (2T_n(x))(2T_n(y))z^n = 4\sum_{n=0}^{+\infty} T_n(x)T_n(y)z^n,$$

since

$$\sum_{n=0}^{+\infty} T_n(x)T_n(y)z^n = \frac{1 - 3xyz + (2x^2 + 2y^2 - 1)z^2 - xyz^3}{1 - 4xyz + (4x^2 + 4y^2 - 2)z^2 - 4xyz^3 + z^4}, \text{ (see [3])}.$$

Therefore

$$\sum_{n=0}^{+\infty} s_n(x)s_n(y)z^n = \frac{4 - 12xyz + (8x^2 + 8y^2 - 4)z^2 - 4xyz^3}{1 - 4xyz + (4x^2 + 4y^2 - 2)z^2 - 4xyz^3 + z^4}.$$

This completes the proof. □

4. Generating Functions of the Products of Vieta Polynomials with Gaussian Numbers and Polynomials

In this part, we now derive the new generating functions of the products of Vieta polynomials with Gaussian numbers and polynomials.

Theorem 4.1. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Fibonacci numbers with Vieta Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GF_n v_n(x)z^n = \frac{2i + (1 - 2i)xz + (i(4 - x^2) - 2)z^2 - (1 - i)xz^3}{1 - xz + (3 - x^2)z^2 + xz^3 + z^4}.$$

Proof . We know that

$$\sum_{n=0}^{+\infty} GF_n T_n(x)z^n = \frac{i + (1 - 2i)xz + (2i(1 - x^2) - 1)z^2 - (1 - i)xz^3}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4}, \text{ (see [13])}.$$

From which it follows

$$\begin{aligned}
\sum_{n=0}^{+\infty} GF_n v_n(x)z^n &= 2\sum_{n=0}^{+\infty} GF_n T_n(\frac{x}{2})z^n \\
&= \frac{2\left(i + (1 - 2i)\left(\frac{x}{2}\right)z + (2i(1 - \left(\frac{x}{2}\right)^2) - 1)z^2 - (1 - i)\left(\frac{x}{2}\right)z^3\right)}{1 - xz + (3 - x^2)z^2 + xz^3 + z^4} \\
&= \frac{2i + (1 - 2i)xz + (i(4 - x^2) - 2)z^2 - (1 - i)xz^3}{1 - xz + (3 - x^2)z^2 + xz^3 + z^4}.
\end{aligned}$$

The proof is completed. □

Proposition 4.2. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Fibonacci numbers with Vieta Pell polynomials $t_{n+1}(x)$ is given by*

$$\sum_{n=0}^{+\infty} GF_n t_{n+1}(x) z^n = \frac{i + 2(1-i)xz + (2i-1)z^2}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} GF_n U_n(x) z^n = \frac{i + 2(1-i)xz + (2i-1)z^2}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4} \text{ (see [13])},$$

and according the relationship (1.3), we get

$$\sum_{n=0}^{+\infty} GF_n t_{n+1}(x) z^n = \frac{i + 2(1-i)xz + (2i-1)z^2}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4}.$$

The proof is completed. □

Theorem 4.3. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Fibonacci numbers with Vieta Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GF_n s_n(x) z^n = \frac{2i + 2(1-2i)xz + (4i(1-x^2) - 2)z^2 - 2(1-i)xz^3}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4}.$$

Proof . By using the relationship (1.2) , we obtain

$$\begin{aligned} \sum_{n=0}^{+\infty} GF_n s_n(x) z^n &= 2\sum_{n=0}^{+\infty} GF_n T_n(x) z^n \\ &= \frac{2\left(i + (1-2i)xz + (2i(1-x^2) - 1)z^2 - (1-i)xz^3\right)}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4} \\ &= \frac{2i + 2(1-2i)xz + (4i(1-x^2) - 2)z^2 - 2(1-i)xz^3}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4}. \end{aligned}$$

The proof is completed. □

Theorem 4.4. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Lucas numbers with Vieta Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GL_n v_n(x) z^n = \frac{4 - 2i + (4i-3)xz + (6 - 2x^2 + i(x^2 - 8))z^2 + (1-3i)xz^3}{1 - xz + (3 - x^2)z^2 + xz^3 + z^4}.$$

Proof . We have

$$\sum_{n=0}^{+\infty} GL_n T_n(x) z^n = \frac{2 - i + (4i-3)xz + (3 - 4x^2 + 2i(x^2 - 2))z^2 + (1-3i)xz^3}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4}, \text{ (see [13])}.$$

We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} GL_n v_n(x) z^n &= 2\sum_{n=0}^{+\infty} GL_n T_n(\frac{x}{2}) z^n \\ &= \frac{2\left[2-i+(4i-3)\left(\frac{x}{2}\right)z+(3-x^2+2i(\frac{x^2}{4}-2))z^2+(1-3i)\left(\frac{x}{2}\right)z^3\right]}{1-xz+(3-x^2)z^2+xz^3+z^4} \\ &= \frac{4-2i+(4i-3)xz+(6-2x^2+i(x^2-8))z^2+(1-3i)xz^3}{1-xz+(3-x^2)z^2+xz^3+z^4}.\end{aligned}$$

The proof is completed. □

Proposition 4.5. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Lucas numbers with Vieta Pell polynomials $t_{n+1}(x)$ is given by*

$$\sum_{n=0}^{+\infty} GL_n t_{n+1}(x) z^n = \frac{2-i+2(3i-1)xz+(3-4i)z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} GL_n U_n(x) z^n = \frac{2-i+2(3i-1)xz+(3-4i)z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} \quad \text{(see [13])},$$

and according to the relationship (1.3), we get

$$\sum_{n=0}^{+\infty} GL_n t_{n+1}(x) z^n = \frac{2-i+2(3i-1)xz+(3-4i)z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}.$$

This completes the proof. □

Proposition 4.6. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Lucas numbers with Vieta Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GL_n s_n(x) z^n = \frac{4-2i+(8i-6)xz+(6-8x^2+4i(x^2-2))z^2+(2-6i)xz^3}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} GL_n T_n(x) z^n = \frac{2-i+(4i-3)xz+(3-4x^2+2i(x^2-2))z^2+(1-3i)xz^3}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} \quad \text{(see [13])},$$

and according to the relationship (1.2), we get

$$\sum_{n=0}^{+\infty} GL_n s_n(x) z^n = \frac{4-2i+(8i-6)xz+(6-8x^2+4i(x^2-2))z^2+(2-6i)xz^3}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}.$$

This completes the proof. □

Theorem 4.7. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal numbers with Vieta Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GJ_n v_n(x) z^n = \frac{2i + (2-3i)xz - 4(4x + i(2+2x+x^2))z^2 - 4(2-i)xz^3}{2 - 2xz + (10-4x^2)z^2 + 2xz^3 + 8z^4}.$$

Proof . We know that

$$\sum_{n=0}^{+\infty} GJ_n T_n(x) z^n = \frac{i + (2-3i)xz - 4(4x + i(1+2x+2x^2))z^2 - 4(2-i)xz^3}{2 - 4xz + (10-16x^2)z^2 + 4xz^3 + 8z^4}, \text{ (see [13]).}$$

Then

$$\begin{aligned}\sum_{n=0}^{+\infty} GJ_n v_n(x) z^n &= 2\sum_{n=0}^{+\infty} GJ_n T_n\left(\frac{x}{2}\right) z^n \\ &= \frac{2\left(i + (2-3i)\left(\frac{x}{2}\right)z - 4(4\left(\frac{x}{2}\right) + i(1 + x + 2\left(\frac{x}{2}\right)^2))z^2 - 4(2-i)\left(\frac{x}{2}\right)z^3\right)}{2 - 2xz + (10-4x^2)z^2 + 2xz^3 + 8z^4} \\ &= \frac{2i + (2-3i)xz - 4(4x + i(2+2x+x^2))z^2 - 4(2-i)xz^3}{2 - 2xz + (10-4x^2)z^2 + 2xz^3 + 8z^4}.\end{aligned}$$

The proof is completed. □

Proposition 4.8. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal numbers with Vieta Pell polynomials $t_{n+1}(x)$ is given by*

$$\sum_{n=0}^{+\infty} GJ_n t_{n+1}(x) z^n = \frac{i + 2(2-i)xz + (3i-2)z^2}{2 - 4xz + (10-16x^2)z^2 + 4xz^3 + 8z^4}.$$

Proof . By referred to [13], we have $\sum\limits_{n=0}^{+\infty} GJ_n U_n(x) z^n = \frac{i+2(2-i)xz+(3i-2)z^2}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}$, and by using the relationship (1.3),we get

$$\sum_{n=0}^{+\infty} GJ_n t_{n+1}(x) z^n = \frac{i + 2(2-i)xz + (3i-2)z^2}{2 - 4xz + (10-16x^2)z^2 + 4xz^3 + 8z^4}.$$

The proof is completed. □

Proposition 4.9. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal numbers with Vieta Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GJ_n s_n(x) z^n = \frac{2i + (4-6i)xz - 8(4x + i(1+2x+2x^2))z^2 - 8(2-i)xz^3}{2 - 4xz + (10-16x^2)z^2 + 4xz^3 + 8z^4}.$$

Proof . By referred to [13], we have

$$\sum_{n=0}^{+\infty} GJ_n T_n(x) z^n = \frac{i + (2-3i)xz - 4(4x + i(1+2x+2x^2))z^2 - 4(2-i)xz^3}{2 - 4xz + (10-16x^2)z^2 + 4xz^3 + 8z^4},$$

and by using the relationship (1.2), we get

$$\sum_{n=0}^{+\infty} GJ_n s_n(x) z^n = \frac{2i + (4-6i)xz - 8(4x + i(1+2x+2x^2))z^2 - 8(2-i)xz^3}{2 - 4xz + (10-16x^2)z^2 + 4xz^3 + 8z^4}.$$

The proof is completed. □

Theorem 4.10. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal Lucas numbers with Vieta Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} Gj_n v_n(x) z^n = \frac{8 - 2i + 6(i-1)xz + 2((10-4x^2) + i(x^2-7))z^2 + 2(2-5i)xz^3}{2 - 2xz + (10-4x^2)z^2 + 2xz^3 + 8z^4}.$$

Proof . By [13], we have

$$\sum_{n=0}^{+\infty} Gj_n T_n(x) z^n = \frac{4 - i + 6(i-1)xz + ((10-16x^2) + i(4x^2-7))z^2 + 2(2-5i)xz^3}{2 - 4xz + (10-16x^2)z^2 + 4xz^3 + 8z^4}.$$

Then, we can see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} Gj_n v_n(x) z^n &= 2\sum_{n=0}^{+\infty} Gj_n T_n\left(\frac{x}{2}\right) z^n \\
&= \frac{2\left(4 - i + 3(i-1)xz + ((10-4x^2) + i(x^2-7))z^2 + (2-5i)xz^3\right)}{2 - 2xz + (10-4x^2)z^2 + 2xz^3 + 8z^4} \\
&= \frac{8 - 2i + 6(i-1)xz + 2((10-4x^2) + i(x^2-7))z^2 + 2(2-5i)xz^3}{2 - 2xz + (10-4x^2)z^2 + 2xz^3 + 8z^4}.
\end{aligned}$$

This completes the proof. □

Proposition 4.11. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal Lucas numbers with Vieta Pell polynomials $t_{n+1}(x)$ is given by*

$$\sum_{n=0}^{+\infty} Gj_n t_{n+1}(x) z^n = \frac{4 - i + 2(5i-2)xz + (10-7i)z^2}{2 - 4xz + (10-16x^2)z^2 + 4xz^3 + 8z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} Gj_n U_n(x) z^n = \frac{4 - i + 2(5i-2)xz + (10-7i)z^2}{2 - 4xz + (10-16x^2)z^2 + 4xz^3 + 8z^4},$$

(see [13]) and according the relationship (1.3), we get

$$\sum_{n=0}^{+\infty} Gj_n t_{n+1}(x) z^n = \frac{4 - i + 2(5i-2)xz + (10-7i)z^2}{2 - 4xz + (10-16x^2)z^2 + 4xz^3 + 8z^4}.$$

This completes the proof. □

Proposition 4.12. *For $n \in \mathbb{N}$, the new generating functions of the product of Gaussian Jacobsthal Lucas numbers with Vieta Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} Gj_n s_n(x) z^n = \frac{8-2i+12(i-1)xz+2((10-16x^2)+i(4x^2-7))z^2+4(2-5i)xz^3}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} Gj_n T_n(x) z^n = \frac{4-i+6(i-1)xz+((10-16x^2)+i(4x^2-7))z^2+2(2-5i)xz^3}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4} \text{ (see [13])} ,$$

and according the relationship (1.2), we get

$$\sum_{n=0}^{+\infty} Gj_n s_n(x) z^n = \frac{8-2i+12(i-1)xz+2((10-16x^2)+i(4x^2-7))z^2+4(2-5i)xz^3}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}.$$

This completes the proof. □

Theorem 4.13. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell numbers with Vieta Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GP_n v_n(x) z^n = \frac{2i+(1-4i)xz+(i(10-x^2)-4)z^2+(2i-1)xz^3}{1-2xz+(6-x^2)z^2+2xz^3+z^4}.$$

Proof . By [13], we have

$$\sum_{n=0}^{+\infty} GP_n T_n(x) z^n = \frac{i+(1-4i)xz+(i(5-2x^2)-2)z^2+(2i-1)xz^3}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

Then, we can see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} GP_n v_n(x) z^n &= 2\sum_{n=0}^{+\infty} GP_n T_n(\frac{x}{2}) z^n \\
&= \frac{2\left(i+(1-4i)\left(\frac{x}{2}\right)z+(i(5-2\left(\frac{x}{2}\right)^2)-2)z^2+(2i-1)\left(\frac{x}{2}\right)z^3\right)}{1-2xz+(6-x^2)z^2+2xz^3+z^4} \\
&= \frac{2i+(1-4i)xz+(i(10-x^2)-4)z^2+(2i-1)xz^3}{1-2xz+(6-x^2)z^2+2xz^3+z^4}.
\end{aligned}$$

This completes the proof. □

Proposition 4.14. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell numbers with Vieta Pell polynomials $t_{n+1}(x)$ is given by*

$$\sum_{n=0}^{+\infty} GP_n t_{n+1}(x) z^n = \frac{i+2(1-2i)xz+(5i-2)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} GP_n U_n(x) z^n = \frac{i + 2(1-2i)xz + (5i-2)z^2}{1 - 4xz + (6-4x^2)z^2 + 4xz^3 + z^4} \quad \text{(see [13])},$$

and according the relationship (1.3), we get

$$\sum_{n=0}^{+\infty} GP_n t_{n+1}(x) z^n = \frac{i + 2(1-2i)xz + (5i-2)z^2}{1 - 4xz + (6-4x^2)z^2 + 4xz^3 + z^4}.$$

This completes the proof. □

Proposition 4.15. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell numbers with Vieta Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GP_n s_n(x) z^n = \frac{2i + 2(1-4i)xz + (i(10-4x^2) - 4)z^2 + (4i-2)xz^3}{1 - 4xz + (6-4x^2)z^2 + 4xz^3 + z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} GP_n T_n(x) z^n = \frac{i + (1-4i)xz + (i(5-2x^2) - 2)z^2 + (2i-1)xz^3}{1 - 4xz + (6-4x^2)z^2 + 4xz^3 + z^4} \quad \text{(see [13])},$$

and according the relationship (1.2), we get

$$\sum_{n=0}^{+\infty} GP_n s_n(x) z^n = \frac{2i + 2(1-4i)xz + (i(10-4x^2) - 4)z^2 + (4i-2)xz^3}{1 - 4xz + (6-4x^2)z^2 + 4xz^3 + z^4}.$$

This completes the proof. □

Theorem 4.16. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell Lucas numbers with Vieta Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GQ_n v_n(x) z^n = \frac{4(1-i) + (10i-6)xz + (12 - 2x^2 + 2i(x^2-14))z^2 + (2-6i)xz^3}{1 - 2xz + (6-x^2)z^2 + 2xz^3 + z^4}.$$

Proof . By referred to [13], we have

$$\sum_{n=0}^{+\infty} GQ_n T_n(x) z^n = \frac{2(1-i) + (10i-6)xz + (6 - 4x^2 + i(4x^2-14))z^2 + (2-6i)xz^3}{1 - 4xz + (6-4x^2)z^2 + 4xz^3 + z^4}.$$

We see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} GQ_n v_n(x) z^n &= 2\sum_{n=0}^{+\infty} GQ_n T_n\left(\frac{x}{2}\right) z^n \\
&= \frac{2\left(2(1-i) + (5i-3)xz + (6 - x^2 + i(x^2-14))z^2 + (1-3i)xz^3\right)}{1 - 2xz + (6-x^2)z^2 + 2xz^3 + z^4} \\
&= \frac{4(1-i) + (10i-6)xz + (12 - 2x^2 + 2i(x^2-14))z^2 + (2-6i)xz^3}{1 - 2xz + (6-x^2)z^2 + 2xz^3 + z^4}.
\end{aligned}$$

This completes the proof. □

Proposition 4.17. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell Lucas numbers with Vieta Pell polynomials $t_{n+1}(x)$ is given by*

$$\sum_{n=0}^{+\infty} GQ_n t_{n+1}(x) z^n = \frac{2-2i+2(6i-2)xz+(6-14i)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

Proof . By referred to [13], we have

$$\sum_{n=0}^{+\infty} GQ_n U_n(x) z^n = \frac{2-2i+2(6i-2)xz+(6-14i)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4},$$

and according the relationship (1.3), we get

$$\sum_{n=0}^{+\infty} GQ_n t_{n+1}(x) z^n = \frac{2-2i+2(6i-2)xz+(6-14i)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

This completes the proof. □

Proposition 4.18. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell Lucas numbers with Vieta Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GQ_n s_n(x) z^n = \frac{4(1-i)+2(10i-6)xz+(12-8x^2+2i(4x^2-14))z^2+(4-12i)xz^3}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

Proof . By referred to [13], we have

$$\sum_{n=0}^{+\infty} GQ_n T_n(x) z^n = \frac{2(1-i)+(10i-6)xz+(6-4x^2+i(4x^2-14))z^2+(2-6i)xz^3}{1-4xz+(6-4x^2)z^2+4xz^3+z^4},$$

and according the relationship (1.2), we get

$$\sum_{n=0}^{+\infty} GQ_n s_n(x) z^n = \frac{4(1-i)+2(10i-6)xz+(12-8x^2+2i(4x^2-14))z^2+(4-12i)xz^3}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

This completes the proof. □

Theorem 4.19. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal polynomials with Vieta Fibonacci polynomials $V_{n+1}(y)$ is given by*

$$\sum_{n=0}^{+\infty} GJ_n(x) V_{n+1}(y) z^n = \frac{i+(2y-iy)z+((2x+1)i-2)z^2}{2-2yz-(4xy^2-8x-2)z^2+4xyz^3+8x^2z^4}.$$

Proof . By [9], we have

$$\sum_{n=0}^{+\infty} GJ_n(x) U_n(y) z^n = \frac{i+(4y-2iy)z+((2x+1)i-2)z^2}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}.$$

We use the change of variable $y = \frac{y}{2}$ and by relation (1.4) , we get

$$\begin{aligned}\sum_{n=0}^{+\infty} GJ_n(x)\, V_{n+1}(y)\, z^n &= \sum_{n=0}^{+\infty} GJ_n(x)\, U_n\left(\frac{y}{2}\right) z^n \\ &= \frac{i + (2y - iy)\, z + ((2x+1)\, i - 2)\, z^2}{2 - 2yz - (4xy^2 - 8x - 2)\, z^2 + 4xyz^3 + 8x^2z^4}.\end{aligned}$$

This completes the proof. □

Theorem 4.20. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal polynomials with Vieta Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GJ_n(x)v_n(y)z^n = \frac{2i + (2-2i)yz + ((2+4x-2xy^2)i - 4)z^2 + (2i-4)xyz^3}{2 - 2yz - (4xy^2 - 8x - 2)z^2 + 4xyz^3 + 8x^2z^4}.$$

Proof . By referred to [9], we have

$$\sum_{n=0}^{+\infty} GJ_n(x)T_n(y)z^n = \frac{i + (2-2i)yz + ((1+2x-4xy^2)i - 2)z^2 + (2i-4)xyz^3}{2 - 4yz - (16xy^2 - 8x - 2)z^2 + 8xyz^3 + 8x^2z^4}.$$

We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} GJ_n(x)v_n(y)z^n &= 2\sum_{n=0}^{+\infty} GJ_n(x)T_n(\frac{y}{2})z^n \\ &= \frac{2\left(i + (2-2i)\left(\frac{y}{2}\right)z + ((1+2x-xy^2)i - 2)z^2 + (2i-4)x\left(\frac{y}{2}\right)z^3\right)}{2 - 2yz - (4xy^2 - 8x - 2)z^2 + 4xyz^3 + 8x^2z^4} \\ &= \frac{2i + (2-2i)yz + ((2+4x-2xy^2)i - 4)z^2 + (2i-4)xyz^3}{2 - 2yz - (4xy^2 - 8x - 2)z^2 + 4xyz^3 + 8x^2z^4}.\end{aligned}$$

The proof is completed. □

Proposition 4.21. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal polynomials with Vieta Pell polynomials $t_{n+1}(y)$ is given by*

$$\sum_{n=0}^{+\infty} GJ_n(x)t_{n+1}(y)z^n = \frac{i + (4-2i)yz + ((2x+1)i - 2)z^2}{2 - 4yz - (16xy^2 - 8x - 2)z^2 + 8xyz^3 + 8x^2z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} GJ_n(x)U_n(y)z^n = \frac{i + (4-2i)yz + ((2x+1)i - 2)z^2}{2 - 4yz - (16xy^2 - 8x - 2)z^2 + 8xyz^3 + 8x^2z^4} \text{ (see [9])},$$

and according the relationship (1.3), we get

$$\sum_{n=0}^{+\infty} GJ_n(x)t_{n+1}(y)z^n = \frac{i + (4-2i)yz + ((2x+1)i - 2)z^2}{2 - 4yz - (16xy^2 - 8x - 2)z^2 + 8xyz^3 + 8x^2z^4}.$$

This completes the proof. □

Proposition 4.22. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal polynomials with Vieta Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GJ_n(x)s_n(y)z^n = \frac{2i+(4-4i)yz+((2+4x-8xy^2)i-4)z^2+(4i-8)xyz^3}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} GJ_n(x)T_n(y)z^n = \frac{i+(2-2i)yz+((1+2x-4xy^2)i-2)z^2+(2i-4)xyz^3}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4} \text{ (see [9])},$$

and according the relationship (1.2), we get

$$\sum_{n=0}^{+\infty} GJ_n(x)s_n(y)z^n = \frac{2i+(4-4i)yz+((2+4x-8xy^2)i-4)z^2+(4i-8)xyz^3}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}.$$

This completes the proof. □

Theorem 4.23. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal Lucas polynomials with Vieta Fibonacci polynomials $V_{n+1}(y)$ is given by*

$$\sum_{n=0}^{+\infty} Gj_n(x)V_{n+1}(y)z^n = \frac{4-i+((4xy+y)i-2y)z+((8x+2)-i(6x+1))z^2}{2-2yz-(4xy^2-8x-2)z^2+4xyz^3+8x^2z^4}.$$

Proof . By [9], we have

$$\sum_{n=0}^{+\infty} Gj_n(x)U_n(y)z^n = \frac{4-i+((8xy+2y)i-4y)z+((8x+2)-i(6x+1))z^2}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}.$$

We use the change of variable $y=\frac{y}{2}$ and by relationship (1.4), we get

$$\begin{aligned}\sum_{n=0}^{+\infty} Gj_n(x)V_{n+1}(y)z^n &= \sum_{n=0}^{+\infty} Gj_n(x)U_n\left(\frac{y}{2}\right)z^n \\ &= \frac{4-i+((4xy+y)i-2y)z+((8x+2)-i(6x+1))z^2}{2-2yz-(4xy^2-8x-2)z^2+4xyz^3+8x^2z^4}.\end{aligned}$$

This completes the proof. □

Theorem 4.24. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal Lucas polynomials with Vieta Lucas polynomials is given by*

$$\begin{aligned}\sum_{n=0}^{+\infty} Gj_n(x)v_n(y)z^n &= \frac{8-2i+((y+6xy)i-(8xy+2y))z}{2-2yz-(4xy^2-8x-2)z^2+4xyz^3+8x^2z^4} \\ &+\frac{((2xy^2-12x-2)i+(16x-8xy^2+2y))z^2}{2-2yz-(4xy^2-8x-2)z^2+4xyz^3+8x^2z^4} \\ &+\frac{(4xy-i(8x^2y+2xy))z^3}{2-2yz-(4xy^2-8x-2)z^2+4xyz^3+8x^2z^4}.\end{aligned}$$

Proof . We have

$$\begin{aligned}\sum_{n=0}^{+\infty} Gj_n(x)T_n(y)z^n &= \frac{4-i+((y+6xy)i-(8xy+2y))z}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}\\ &+\frac{((4xy^2-6x-1)i+(8x-16xy^2+2y))z^2}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}\\ &+\frac{(4xy-i(8x^2y+2xy))z^3}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}, \text{ (see [9])}.\end{aligned}$$

Then, we can see that

$$\begin{aligned}\sum_{n=0}^{+\infty} Gj_n(x)v_n(y)z^n &= 2\sum_{n=0}^{+\infty} Gj_n(x)T_n(\frac{y}{2})z^n\\ &= \frac{24-i+((\frac{y}{2}+3xy)i-(4xy+y))z}{2-2yz-(xy^2-8x-2)z^2+2xyz^3+8x^2z^4}\\ &+\frac{((xy^2-6x-1)i+(8x-4xy^2+y))z^2}{2-2yz-(xy^2-8x-2)z^2+2xyz^3+8x^2z^4}\\ &+\frac{2(2xy-i(4x^2y+xy))z^3}{2-2yz-(xy^2-8x-2)z^2+2xyz^3+8x^2z^4}\\ &= \frac{8-2i+((y+6xy)i-(8xy+2y))z}{2-2yz-(xy^2-8x-2)z^2+2xyz^3+8x^2z^4}\\ &+\frac{((2xy^2-12x-2)i+(16x-8xy^2+2y))z^2}{2-2yz-(xy^2-8x-2)z^2+2xyz^3+8x^2z^4}\\ &+\frac{(4xy-2i(4x^2y+xy))z^3}{2-2yz-(xy^2-8x-2)z^2+2xyz^3+8x^2z^4}.\end{aligned}$$

This completes the proof. □

Proposition 4.25. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal Lucas polynomials with Vieta Pell polynomials $t_{n+1}(y)$ is given by*

$$\sum_{n=0}^{+\infty} Gj_n(x)t_{n+1}(y)z^n = \frac{4-i+((8xy+2y)i-4y)z+((8x+2)-i(6x+1))z^2}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} Gj_n(x)U_n(y)z^n = \frac{4-i+((8xy+2y)i-4y)z+((8x+2)-i(6x+1))z^2}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4} \text{ (see [9])},$$

and according the relationship (1.3), we get

$$\sum_{n=0}^{+\infty} Gj_n(x)t_{n+1}(y)z^n = \frac{4-i+((8xy+2y)i-4y)z+((8x+2)-i(6x+1))z^2}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}.$$

This completes the proof. □

Proposition 4.26. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal Lucas polynomials with Vieta Pell Lucas polynomials is given by*

$$\begin{aligned}\sum_{n=0}^{+\infty} Gj_n(x)s_n(y)z^n &= \frac{8-2i+((2y+12xy)i-(16xy+4y))z}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}\\ &+\frac{((8xy^2-12x-2)i+(16x-32xy^2+4y))z^2}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}\\ &+\frac{(8xy-2i(8x^2y+2xy))z^3}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}.\end{aligned}$$

Proof . Recall that, we have

$$\begin{aligned}\sum_{n=0}^{+\infty} Gj_n(x)T_n(y)z^n &= \frac{4-i+((y+6xy)i-(8xy+2y))z}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}\\ &+\frac{((4xy^2-6x-1)i+(8x-16xy^2+2y))z^2}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}\\ &+\frac{(4xy-i(8x^2y+2xy))z^3}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4} \text{ (see [9])},\end{aligned}$$

and according the relationship (1.2) .

We get

$$\begin{aligned}\sum_{n=0}^{+\infty} Gj_n(x)s_n(y)z^n &= \frac{8-2i+((2y+12xy)i-(16xy+4y))z}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}\\ &+\frac{((8xy^2-12x-2)i+(16x-32xy^2+4y))z^2}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}\\ &+\frac{(8xy-2i(8x^2y+2xy))z^3}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}.\end{aligned}$$

This completes the proof. □

Theorem 4.27. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell polynomials with Vieta Fibonacci polynomials $V_{n+1}(y)$ is given by*

$$\sum_{n=0}^{+\infty} GP_n(x)\,V_{n+1}(y)\,z^n = \frac{i+(y-2ixy)\,z+(i\,(1+4x^2)-2x)\,z^2}{1-2xyz-(y^2-4x^2-2)\,z^2+2xyz^3+z^4}.$$

Proof . By [9], we have

$$\sum_{n=0}^{+\infty} GP_n(x)\,U_n(y)\,z^n = \frac{i+(2y-4ixy)\,z+(i\,(1+4x^2)-2x)\,z^2}{1-4xyz-(4\,(y^2-x^2)-2)\,z^2+4xyz^3+z^4}.$$

We use the change of variable $y = \frac{y}{2}$ and by relation (1.4) , we get

$$\begin{aligned}\sum_{n=0}^{+\infty} GP_n(x)\,V_{n+1}(y)\,z^n &= \sum_{n=0}^{+\infty} GP_n(x)\,U_n\left(\frac{y}{2}\right)z^n\\ &= \frac{i+(y-2ixy)\,z+(i\,(1+4x^2)-2x)\,z^2}{1-2xyz-(y^2-4x^2-2)\,z^2+2xyz^3+z^4}.\end{aligned}$$

This completes the proof. □

Theorem 4.28. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell polynomials with Vieta Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GP_n(x)v_n(y)z^n = \frac{2i + (y - 4ixy)z + ((8x^2 - y^2 + 2)i - 4x)z^2 + (2ixy - y)z^3}{1 - 2xyz - (y^2 - 4x^2 - 2)z^2 + 2xyz^3 + z^4}.$$

Proof . We have

$$\sum_{n=0}^{+\infty} GP_n(x)T_n(y)z^n = \frac{i + (y - 4ixy)z + ((4x^2 - 2y^2 + 1)i - 2x)z^2 + (2ixy - y)z^3}{1 - 4xyz - (4(y^2 - x^2) - 2)z^2 + 4xyz^3 + z^4}, \text{ (see [9]).}$$

Then, we can see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} GP_n(x)v_n(y)z^n &= 2\sum_{n=0}^{+\infty} GP_n(x)T_n(\frac{y}{2})z^n \\
&= \frac{2\left(i + (\frac{y}{2} - 2ixy)z + ((4x^2 - \frac{y^2}{2} + 1)i - 2x)z^2 + (ixy - \frac{y}{2})z^3\right)}{1 - 2xyz - (y^2 - 4x^2 - 2)z^2 + 2xyz^3 + z^4} \\
&= \frac{2i + (y - 4ixy)z + ((8x^2 - y^2 + 2)i - 4x)z^2 + (2ixy - y)z^3}{1 - 2xyz - (y^2 - 4x^2 - 2)z^2 + 2xyz^3 + z^4}.
\end{aligned}$$

The proof is completed. □

Proposition 4.29. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell polynomials with Vieta Pell polynomials $t_{n+1}(y)$ is given by*

$$\sum_{n=0}^{+\infty} GP_n(x)t_{n+1}(y)z^n = \frac{i + (2y - 4ixy)z + (i(1 + 4x^2) - 2x)z^2}{1 - 4xyz - (4(y^2 - x^2) - 2)z^2 + 4xyz^3 + z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} GP_n(x)U_n(y)z^n = \frac{i + (2y - 4ixy)z + (i(1 + 4x^2) - 2x)z^2}{1 - 4xyz - (4(y^2 - x^2) - 2)z^2 + 4xyz^3 + z^4} \text{ (see [9]),}$$

and according the relationship (1.3), we get

$$\sum_{n=0}^{+\infty} GP_n(x)t_{n+1}(y)z^n = \frac{i + (2y - 4ixy)z + (i(1 + 4x^2) - 2x)z^2}{1 - 4xyz - (4(y^2 - x^2) - 2)z^2 + 4xyz^3 + z^4}.$$

This completes the proof. □

Proposition 4.30. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell polynomials with Vieta Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} GP_n(x)s_n(y)z^n = \frac{2i + (2y - 8ixy)z + ((8x^2 - 4y^2 + 2)i - 4x)z^2 + (4ixy - 2y)z^3}{1 - 4xyz - (4(y^2 - x^2) - 2)z^2 + 4xyz^3 + z^4}.$$

Proof . Recall that, we have

$$\sum_{n=0}^{+\infty} GP_n(x)T_n(y)z^n = \frac{i+(y-4ixy)z+((4x^2-2y^2+1)i-2x)z^2+(2ixy-y)z^3}{1-4xyz-(4(y^2-x^2)-2)z^2+4xyz^3+z^4} \text{ (see[9])},$$

and according the relationship (1.2), we get

$$\sum_{n=0}^{+\infty} GP_n(x)s_n(y)z^n = \frac{2i+(2y-8ixy)z+((8x^2-4y^2+2)i-4x)z^2+(4ixy-2y)z^3}{1-4xyz-(4(y^2-x^2)-2)z^2+4xyz^3+z^4}.$$

This completes the proof. □

5. Conclusion

In this paper, by making use of the relation between Vieta polynomials and Chebyshev polynomials, we have derived some new generating functions of the products of Vieta Fibonacci and Vieta Lucas polynomials, Vieta Pell and Vieta Pell Lucas polynomials. and the products of Vieta polynomials with Gaussian numbers and polynomials.

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