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An Existence Result of Three Solutions for a $\mathbf{2n}$-th-Order Boundary-Value Problem

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Abstract

In this paper, we establish the existence of at least three weak solutions for some one-dimensional $2n$-th-order equations in a bounded domain. A particular case and a concrete example are then presented.

Keywords: Boundary value problem, Sobolev space, Critical point, Three solutions, Variational method
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1. Introduction

Let $n \in \mathbb{N} - \{1\}$. In this note, we consider the $2n$-th-order boundary-value problem

$$\begin{cases} [(-1)^n u^{(2n)} + (-1)^{n-1} u^{(2n-2)} + \cdots + u^{(4)}]h(x,u') - u'' \\ = [\lambda f(x,u) + \mu g(x,u) + p(u)]h(x,u'), & x \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0 = u^{(n)}(0) = u^{(n)}(1), \end{cases} \tag{1.1}$$

where λ is a positive parameter, μ is a nonnegative parameter, $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions, $p : \mathbb{R} \to (-\infty, 0]$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$ i.e., $|p(t_1) - p(t_2)| \leq L|t_1 - t_2|$ for every $t_1, t_2 \in \mathbb{R}$, with $p(0) = 0$, suppose that the Lipschitz constant L of the function p satisfies $0 < L < \pi^4$, and $h : [0,1] \times \mathbb{R} \to [0, +\infty)$ is a bounded and continuous function with $0 < m := \inf_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t) \leq h(x,t) \leq \sup_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t) = M < \infty$.

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Many researchers have studied the existance and multiplicity of solutions for such a problem. For example, authors in [2], using Ricceri's Variational Principle [9], established the existence of at least three weak solutions for the problem

$$\begin{cases} u'''' + \alpha u'' + \beta u = \lambda f(x,u) + \mu g(x,u), & x \in (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases}$$

where α, β are real constants, $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are L^2- Carethéodory functions and $\lambda, \mu > 0$. Also the authors in [6], employing Ricceri's Variational Principle [9], established the existence of at least three weak solutions for the problem

$$\begin{cases} u''''h(x,u') - u'' = [\lambda f(x,u) + \mu g(x,u) + p(u)]h(x,u'), & x \in (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases}$$

where $\lambda > 0$, $\mu \geq 0$ and f, g, p, h are functions with the same conditions in the problem (1.1).
We also refer the reader to the papers [1, 3, 7], in which existence results for boundary value problems with nonlinear derivative dependence were established.

2. Preliminaries

The aim of this paper is to establish the existence of a non-empty open interval $E \subseteq \mathbb{R}$ and a positive real number q with the following property: for each $\lambda \in E$ and for each Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ sach that $\sup_{|\zeta| \leq s} |g(.,\zeta)| \in L^1(0,1)$ for all $s > 0$, there is $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem (1.1) admits at least three solutions in $X = H^n([0,1]) \cap H_0^{n-1}([0,1])$ whose norms are less than q.
Our analysis is based on the following critical point theorem.

Theorem 2.1 ([9, Ricceri]). *Let X be a reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose derivative admits a continuous inverse on X^*, $J : X \to \mathbb{R}$ be a C^1 functional with compact derivative. Assume that $\lim_{\|x\| \to +\infty} (\Phi(x) + \lambda J(x)) = +\infty$ for all $\lambda \in I$, and there exists $\rho \in \mathbb{R}$ such that*

$$\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda(J(x) + \rho)) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda(J(x) + \rho)).$$

Then, there exist a non-empty open set interval $E \subseteq I$ and a positive real number q with the following property: for every C^1functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\tau > 0$ such that, for each $\mu \in [0, \tau]$, the equation

$$\Phi'(u) + \lambda J'(u) + \mu \Psi'(u) = 0$$

has at least three solutions in X whose norms less than q.
In the proof of our main result we also use the next result to verify the minimax inequality in Theorem2.1.

Theorem 2.2 ([4, Bonanno]). *Let X be a non- empty set and Φ, J two real functions on X. Assume that $\Phi(x) \geq 0$ for every $x \in X$ and there exists $u_0 \in X$ such that $\Phi(u_0) = J(u_0) = 0$. Further, assume that there exist $u_1 \in X$, $\quad r > 0$ such that*

(k_1) $\Phi(u_1) > r$, (k_2) $\sup_{\Phi(x)<r}(-J(x)) < r\dfrac{-J(u_1)}{\Phi(u_1)}$.

Then, for every $v > 1$ and for every $\rho \in \mathbb{R}$ satisfying

$$\sup_{\Phi(x)<r}(-J(x)) + \frac{r\frac{-J(u_1)}{\Phi(u_1)} - \sup_{\Phi(x)<r}(-J(x))}{v} < \rho < r\frac{-J(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda\in\mathbb{R}} \inf_{x\in X} (\Phi(x) + \lambda(J(x) + \rho)) < \inf_{x\in X} \sup_{\lambda\in[0,\sigma]} (\Phi(x) + \lambda(J(x) + \rho)),$$

where

$$\sigma = \frac{vr}{r\frac{-J(u_1)}{\Phi(u_1)} - \sup_{\Phi(x)<r}(-J(x))}.$$

Let us introduce some notations which will be used later. Define

$$H^n([0,1]) := \left\{u \in L^2([0,1]) : u', u'', \cdots, u^{(n)} \in L^2([0,1])\right\},$$
$$H_0^{n-1}([0,1]) := \left\{u \in L^2([0,1]) : u', u'', \cdots, u^{(n-1)} \in L^2([0,1]),\right.$$
$$\left. u(0) = u(1) = u'(0) = u'(1) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0\right\}.$$

Take $X = H^n([0,1]) \cap H_0^{n-1}([0,1]) = \left\{u \in L^2([0,1]) : u', u'', \cdots, u^{(n)} \in L^2([0,1]), u(0) = u(1) = u'(0) = u'(1) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0\right\}$,
endowed with the norm

$$|||u||| := \left(\|u''\|_2^2 + \|u'''\|_2^2 + \cdots + \|u^{(n)}\|_2^2\right)^{\frac{1}{2}}, \quad \text{where } \|u\|_2 := \left(\int_0^1 |u(x)|^2 dx\right)^{\frac{1}{2}}.$$

We recall the following Poincaré type inequalities ([8, Lemma 2.3]):

$$\|u\|_2 \le \frac{1}{\pi^2}\|u''\|_2, \tag{2.1}$$
$$\|u'\|_2 \le \frac{1}{\pi}\|u''\|_2, \tag{2.2}$$

for all $u \in X$. For the norm in $C^{n-1}([0,1])$,

$$\|u\|_\infty := \max\left\{\max_{x\in[0,1]} |u(x)|, \max_{x\in[0,1]} |u'(x)|, \cdots, \max_{x\in[0,1]} |u^{(n-1)}(x)|\right\},$$

since $C^{n-1}([0,1]) \subseteq C^1([0,1])$, we have the well- known inequality ([10]): $\|u\|_\infty \le \frac{1}{2}\|u'\|_2$, then, by (2.2)*, we have*

$$\max_{x\in[0,1]} |u(x)| \le \|u\|_\infty \le \frac{1}{2\pi}\|u''\|_2 \le \frac{1}{2\pi}|||u|||, \tag{2.3}$$

for all $u \in X$. The norm $|||\cdot|||$, is equivalent with the usual norm of Sobolev space $H^n((0,1)) = W^{n,2}((0,1))$:

$\|u\|_{W^{n,2}} := \left(\|u\|_2^2 + \|u'\|_2^2 + \|u''\|_2^2 + \cdots + \|u^{(n)}\|_2^2\right)^{\frac{1}{2}}$. *Because by* (2.1) *and* (2.2) *we have*

$$\begin{aligned}
|||u||| &= \left(\|u''\|_2^2 + \|u'''\|_2^2 + \cdots + \|u^{(n)}\|_2^2\right)^{\frac{1}{2}} \\
&\le \left(\|u\|_2^2 + \|u'\|_2^2 + \|u''\|_2^2 + \cdots + \|u^{(n)}\|_2^2\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{\pi^4}\|u''\|_2^2 + \frac{1}{\pi^2}\|u''\|_2^2 + \|u''\|_2^2 + \cdots + \|u^{(n)}\|_2^2\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{\pi^4} + \frac{1}{\pi^2} + 1\right)^{\frac{1}{2}} \left(\|u''\|_2^2 + \cdots + \|u^{(n)}\|_2^2\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{\pi^4} + \frac{1}{\pi^2} + 1\right)^{\frac{1}{2}} |||u|||.
\end{aligned}$$

We recall that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function if*

(a) the mapping $x \mapsto f(x,t)$ *is measurable for every* $t \in \mathbb{R}$;
(b) the mapping $t \mapsto f(x,t)$ *is continuous for almost every* $x \in [0,1]$.

Also if for every $\rho > 0$ *there exists a function* $\ell_\rho \in L^1([0,1])$ *such that*

$$\sup_{|t|\le\rho} |f(x,t)| \le \ell_\rho(x)$$

for almost every $x \in [0,1]$, *then the Carathéodory function* f *is called* L^1*-Carathéodory function. Corresponding to* f, g, p *and* h, *we introduce the functions* F, G, P *and* H, *respectively, as follows*

$$\begin{aligned}
F : [0,1] \times \mathbb{R} &\to \mathbb{R} & G : [0,1] \times \mathbb{R} &\to \mathbb{R} \\
(x,t) &\mapsto F(x,t) := \textstyle\int_0^t f(x,\zeta)\mathrm{d}\zeta, & (x,t) &\mapsto G(x,t) := \textstyle\int_0^t g(x,\zeta)\mathrm{d}\zeta,
\end{aligned}$$

$$\begin{aligned}
P : \mathbb{R} &\to [0,+\infty) & H : [0,1] \times \mathbb{R} &\to [0,+\infty) \\
t &\mapsto P(t) := -\textstyle\int_0^t p(\zeta)\mathrm{d}\zeta, & (x,t) &\mapsto H(x,t) := \textstyle\int_0^t \left(\int_0^\tau \frac{1}{h(x,\delta)}\mathrm{d}\delta\right)\mathrm{d}\tau,
\end{aligned}$$

for all $x \in [0,1]$, $t \in \mathbb{R}$.
If the parts of equation in (1.1) *divided by* $h(x,u')$ *and then multiplied by an arbitrary function* $v \in X$ *and then integrated in* $x \in [0,1]$ *then by* n *times integration by parts we have*

$$\begin{aligned}
&\int_0^1 u^{(n)}(x)v^{(n)}(x)\mathrm{d}x + \int_0^1 u^{(n-1)}(x)v^{(n-1)}(x)\mathrm{d}x + \cdots + \int_0^1 u''(x)v''(x)\mathrm{d}x \\
&+ \int_0^1 \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)}\mathrm{d}\tau\right) v'(x)\mathrm{d}x - \lambda \int_0^1 f(x,u(x))v(x)\mathrm{d}x \\
&- \mu \int_0^1 g(x,u(x))v(x)dx - \int_0^1 p(u(x))v(x)\mathrm{d}x = 0 \qquad (2.4)
\end{aligned}$$

for all $v \in X$. *Then we say that function* $u \in X$ *in* (2.4) *is a weak solution of* (1.1).

3. Main Results

Put $$A := \frac{\pi^4 - L}{2\pi^4}, \quad B := \frac{\pi^2 + m(\pi^4 + L)}{2m\pi^4}$$
and suppose that $B \leq 4A\pi^2$. We formulate our main result as follows.

Theorem 3.1. *Assume that there exist a positive constant r and a function $w \in X$ such that*

(i) $\frac{1}{2}|||w|||^2 + \int_0^1 [H(x, w'(x)) + P(w(x))]\mathrm{d}x > r$;

(ii) $\int_0^1 \sup_{t\in[-\sqrt{\frac{r}{B}},\sqrt{\frac{r}{B}}]} F(x,t) < r \frac{\int_0^1 F(x, w(x))\mathrm{d}x}{\frac{1}{2}|||w|||^2 + \int_0^1 [H(x, w'(x)) + P(w(x))]\mathrm{d}x}$;

(iii) $\frac{1}{\pi^4 A} \lim_{|t|\to+\infty} \sup \frac{F(x,t)}{t^2} < \frac{1}{\theta}$ *for almost every $x \in [0,1]$ and for all $t \in \mathbb{R}$, and for some θ satisfying*

$$\theta > \frac{1}{r\frac{\int_0^1 F(x,w(x))\mathrm{d}x}{\frac{1}{2}|||w|||^2+\int_0^1[H(x,w'(x))+P(w(x))]\mathrm{d}x} - \int_0^1 \sup_{t\in[-\sqrt{\frac{r}{B}},\sqrt{\frac{r}{B}}]} F(x,t)\mathrm{d}x}.$$

Then, there exist a non-empty open interval $E \subseteq (0, r\theta]$ and a number $q > 0$ with the following property: for each $\lambda \in E$ and for an arbitrary L^1-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, there is $\tau > 0$ such that, whenever $\mu \in [0, \tau]$, problem (1.1) *admits at least three weak solutions whose norms in X are less than q.*

Proof . Our aim is to apply Theorem 2.1 to problem (1.1). Taking $X = H^n([0,1]) \cap H_0^{n-1}([0,1])$ endowed with the norm

$$|||u||| = \left(\|u''\|_2^2 + \|u'''\|_2^2 + \cdots + \|u^{(n)}\|_2^2\right)^{\frac{1}{2}}, \quad \text{where} \quad \|u\|_2 = \left(\int_0^1 |u(x)|^2\mathrm{d}x\right)^{\frac{1}{2}},$$

for every $u \in X$. We introduce the following functionals:

$$\begin{aligned} &\Phi : X \to \mathbb{R} && J : X \to \mathbb{R} \\ &u \mapsto \Phi(u) := \tfrac{1}{2}|||u|||^2 + \textstyle\int_0^1 [H(x, u'(x)) + P(u(x))]\mathrm{d}x, && u \mapsto J(u) := -\textstyle\int_0^1 F(x, u(x))\mathrm{d}x. \end{aligned}$$

Since X is a reflexive real Banach space and X is compactly embedded into $C([0,1])$ then by classical results and that every norm in Banach space X, is a sequentially weakly lower semicontinuous functional, hence Φ is a sequentially weakly lower semicontinuous functional and Gâteaux differentiable with compact Gâteaux derivative hence by definition with continuous Gâteaux derivative, also $\Phi(u) \geq 0$, for every $u \in X$. By classical results, the functional J is well defined and Gâteaux differentiable whose Gâteaux derivative is compact hence by definition with continuous derivative. In particular, for each $u \in X$ one has $\Phi'(u) \in X^*$, $J'(u) \in X^*$ and

$$\begin{aligned} \Phi'(u)(v) &= \int_0^1 u^{(n)}(x)v^{(n)}(x)\mathrm{d}x + \cdots + \int_0^1 u''(x)v''(x)\mathrm{d}x \\ &\quad + \int_0^1 \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)}\mathrm{d}\tau\right) v'(x)\mathrm{d}x - \int_0^1 p(u(x))v(x)\mathrm{d}x, \\ J'(u)(v) &= -\int_0^1 f(x, u(x))v(x)\mathrm{d}x, \end{aligned}$$

for all $v \in X$.
Hence Φ' is a strongly monotone operator, because for every $u, v \in X$ we have:

$$\begin{aligned}
(\Phi'(u) - \Phi'(v), u - v) =& \Phi'(u)(u-v) - \Phi'(v)(u-v)\\
=& \int_0^1 (u^{(n)} - v^{(n)})(u^{(n)} - v^{(n)})\mathrm{d}x + \cdots + \int_0^1 (u'' - v'')(u'' - v'')\mathrm{d}x\\
&+ \int_0^1 \left(\int_{v'(x)}^{u'(x)} \frac{1}{h(x,\tau)}\mathrm{d}\tau \right)(u' - v')\mathrm{d}x - \int_0^1 (p(u) - p(v))(u - v)\mathrm{d}x\\
\geq& \left(\|u^{(n)} - v^{(n)}\|_2^2 + \cdots + \|u'' - v''\|_2^2\right) + \frac{1}{M}\|u' - v'\|_2^2 - L\|u - v\|_2^2\\
\geq& \left(\|u^{(n)} - v^{(n)}\|_2^2 + \cdots + \|u'' - v''\|_2^2\right) - L\|u - v\|_2^2\\
\geq& \left(\|u^{(n)} - v^{(n)}\|_2^2 + \cdots + \|u'' - v''\|_2^2\right) - \frac{L}{\pi^4}\|u'' - v''\|_2^2\\
\geq& (1 - \frac{L}{\pi^4})\left(\|u^{(n)} - v^{(n)}\|_2^2 + \cdots + \|u'' - v''\|_2^2\right)\\
=& 2A\left(\|u^{(n)} - v^{(n)}\|_2^2 + \cdots + \|u'' - v''\|_2^2\right)\\
=& 2A|||u - v|||^2.
\end{aligned}$$

That with the assumption $0 < L < \pi^4$ we have Φ' is a strongly monotone operator. Then by Minty-Browder theorem [11, Theorem 26.A], $\Phi' : X \to X^*$ admits a Lipschitz continuous inverse. Since p is Lipschitz continuous and satisfies $p(0) = 0$, while h is bounded away from zero, we have:

$$\begin{aligned}
|\Phi(u)| &= \left| \frac{1}{2}|||u|||^2 + \int_0^1 H(x, u'(x))\mathrm{d}x + \int_0^1 P(u(x))\mathrm{d}x \right|\\
&= \left| \frac{1}{2}|||u|||^2 + \int_0^1 \int_0^{u'(x)} \left(\int_0^{\tau} \frac{1}{h(x,\delta)}\mathrm{d}\delta \right)\mathrm{d}\tau\mathrm{d}x - \int_0^1 \left(\int_0^{u(x)} p(\zeta)\mathrm{d}\zeta \right)\mathrm{d}x \right|\\
&\geq \frac{1}{2}|||u|||^2 + \int_0^1 \frac{1}{2M}(u'(x))^2\mathrm{d}x - \int_0^1 \frac{L}{2}(u(x))^2\mathrm{d}x \geq \frac{1}{2}|||u|||^2 - \frac{L}{2}\|u\|_2^2\\
&\geq \frac{1}{2}|||u|||^2 - \frac{L}{2\pi^4}\|u''\|_2^2 \geq \left(\frac{1}{2} - \frac{L}{2\pi^4}\right)\left(\|u''\|_2^2 + \cdots + \|u^{(n)}\|_2^2\right)\\
&= A|||u|||^2.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|\Phi(u)| &= \left| \frac{1}{2}|||u|||^2 + \int_0^1 \int_0^{u'(x)} \left(\int_0^{\tau} \frac{1}{h(x,\delta)}\mathrm{d}\delta \right)\mathrm{d}\tau\mathrm{d}x - \int_0^1 \left(\int_0^{u(x)} p(\zeta)\mathrm{d}\zeta \right)\mathrm{d}x \right|\\
&\leq \frac{1}{2}|||u|||^2 + \int_0^1 \frac{1}{2m}(u'(x))^2\mathrm{d}x + \int_0^1 \frac{L}{2}(u(x))^2\mathrm{d}x\\
&= \frac{1}{2}|||u|||^2 + \frac{1}{2m}\|u'\|_2^2 + \frac{L}{2}\|u\|_2^2\\
&\leq \frac{1}{2}|||u|||^2 + \frac{1}{2m\pi^2}\|u''\|_2^2 + \frac{L}{2\pi^4}\|u''\|_2^2\\
&\leq \left(\frac{1}{2} + \frac{1}{2m\pi^2} + \frac{L}{2\pi^4}\right)|||u|||^2 = B|||u|||^2.
\end{aligned}$$

Since $\Phi(u) \geq 0$, for all $u \in X$, then we have:

$$A|||u|||^2 \leq \Phi(u) \leq B|||u|||^2. \tag{3.1}$$

Then Φ is bounded on each bounded subset of X. Furthermore from (iii) there exist two constants $\gamma, \tau \in \mathbb{R}$ with $0 < \gamma < \dfrac{1}{\theta}$ such that $\dfrac{1}{\pi^4 A} F(x,t) \leq \gamma t^2 + \tau$ for a.e. $x \in (0,1)$ and all $t \in \mathbb{R}$. Fix $u \in X$, then

$$F(x, u(x)) \leq \pi^4 A(\gamma |u(x)|^2 + \tau) \quad \text{for all } x \in (0,1) \tag{3.2}$$

Then, for any fixed $\lambda \in (0, \theta]$, from (3.1) , (3.2) and (2.1) we have

$$\begin{aligned}
\Phi(u) + \lambda J(u) &\geq A|||u|||^2 - \lambda \int_0^1 F(x, u(x))\mathrm{d}x \\
&\geq A|||u|||^2 - \pi^4 A\lambda \int_0^1 (\gamma|u(x)|^2 + \tau)\mathrm{d}x \geq A|||u|||^2 - \pi^4 A\lambda(\frac{\gamma}{\pi^4}\|u''\|_2^2 + \tau) \\
&\geq A|||u|||^2 - \pi^4 A\theta(\frac{\gamma}{\pi^4}\|u''\|_2^2 + \tau) \geq A|||u|||^2 - \pi^4 A\theta(\frac{\gamma}{\pi^4}|||u|||^2 + \tau) \\
&= A(1 - \theta\gamma)|||u|||^2 - \pi^4 A\theta\tau
\end{aligned}$$

for all $u \in X$ and so $\lim_{|||u|||\to+\infty}(\Phi(u) + \lambda J(u)) = +\infty$.
We claim that there exist $r > 0$ and $w \in X$ such that

$$\sup_{u \in \Phi^{-1}((-\infty, r))} (-J(u)) < r \frac{-J(w)}{\Phi(w)}.$$

From (3.1) and (2.3), we have

$$\begin{aligned}
\Phi^{-1}((-\infty, r)) &= \{u \in X : \Phi(u) < r\} \subseteq \{u \in X : A|||u|||^2 < r\} \\
&\subseteq \{u \in X : \frac{B}{4\pi^2}|||u|||^2 < r\} = \{u \in X : |||u||| < 2\pi\sqrt{\frac{r}{B}}\} \\
&\subseteq \{u \in X : |u(x)| < \sqrt{\frac{r}{B}}\}
\end{aligned}$$

and it follows that

$$\sup_{u \in \Phi^{-1}((-\infty, r))} (-J(u)) = \sup_{u \in \Phi^{-1}((-\infty, r))} \int_0^1 F(x, u(x))\mathrm{d}x \leq \int_0^1 \sup_{t \in [-\sqrt{\frac{r}{B}}, \sqrt{\frac{r}{B}}]} F(x, t)\mathrm{d}x.$$

Now from (ii) we have

$$\sup_{u \in \Phi^{-1}((-\infty, r))} (-J(u)) < r \frac{\int_0^1 F(x, w(x))\mathrm{d}x}{\frac{1}{2}|||w|||^2 + \int_0^1 [H(x, w'(x)) + P(w(x))]\mathrm{d}x} = r\frac{-J(w)}{\Phi(w)},$$

also from (i) we have $\Phi(w) > r$. Next recall from (iii) that

$$\theta > \frac{1}{r\frac{-J(w)}{\Phi(w)} - \sup_{u \in \Phi^{-1}((-\infty, r))}(-J(u))},$$

choose

$$\alpha = \theta \left(r \frac{-J(w)}{\Phi(w)} - \sup_{u\in\Phi^{-1}((-\infty,r))} (-J(u)) \right),$$

and note $\alpha > 1$, also, since

$$\theta > \frac{1}{r\frac{-J(w)}{\Phi(w)} - \sup_{u\in\Phi^{-1}((-\infty,r))}(-J(u))},$$

we have

$$\sup_{u\in\Phi^{-1}((-\infty,r))} (-J(u)) + \frac{1}{\theta} < r\frac{-J(w)}{\Phi(w)},$$

and so with our choice of α we have

$$\sup_{u\in\Phi^{-1}((-\infty,r))} (-J(u)) + \frac{r\frac{-J(w)}{\Phi(w)} - \sup_{u\in\Phi^{-1}((-\infty,r))}(-J(u))}{\alpha} < r\frac{-J(w)}{\Phi(w)}.$$

Now from Theorem 2.2 (with $u_0 = 0$ and $u_1 = w$) for every $\rho \in \mathbb{R}$ satisfying

$$\sup_{u\in\Phi^{-1}((-\infty,r))} (-J(u)) + \frac{r\frac{-J(w)}{\Phi(w)} - \sup_{u\in\Phi^{-1}((-\infty,r))}(-J(u))}{\alpha} < \rho < r\frac{-J(w)}{\Phi(w)},$$

with choice $\sigma = r\theta$ and $I = [0, r\theta]$, we have

$$\sup_{\lambda\in\mathbb{R}} \inf_{u\in X} (\Phi(u) + \lambda J(u) + \lambda\rho) < \inf_{u\in X} \sup_{\lambda\in[0,r\theta]} (\Phi(u) + \lambda J(u) + \lambda\rho).$$

For any fixed L^1- Carathéodory function $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$, set

$$\begin{aligned} \Psi : X &\to \mathbb{R} \\ u &\mapsto \Psi(u) = -\textstyle\int_0^1 \int_0^{u(x)} g(x,t)\mathrm{d}t\mathrm{d}x. \end{aligned}$$

Since X is a reflexive real Banach space and X is compactly embedded into $C([0, 1])$ then by classical results, the functional Ψ is well defined and Gâteaux differentiable whose Gâteaux derivative is compact and continuous, and $\Psi'(u) \in X^*$, at $u \in X$ is given by

$$\Psi'(u)(v) = -\int_0^1 g(x, u(x))v(x)\mathrm{d}x$$

for all $v \in X$. Now, all the assumptions of Theorem 2.1, are satisfied. Hence, applying Theorem 2.1 taking into account that the critical points of the functional $\Phi + \lambda J + \mu\Psi$ are exactly the weak solutions of the problem (1.1), we have that problem (1.1) admits at least three weak solutions in $X = W_0^{n-1,2}([0, 1]) \cap W^{n,2}([0, 1])$ whose norms in X are less than q. □

Remark 3.2. *In Theorem 3.1, the aim of taking p as a non-positive function, that's* $\Phi(u) = \frac{1}{2}|||u|||^2 + \int_0^1 [H(x, u'(x)) + P(u(x))]\mathrm{d}x$ *be nonnegative. Hence if $p : \mathbb{R} \to \mathbb{R}$ be such that $\Phi \geq 0$ then Theorem 3.1 is satisfied.*

The following lemma which is motivated from [5], will be used in the proof of next corollary.

Lemma 3.3. *Let $0<\alpha<\beta<1$ and assume that there exist two positive constants c and d satisfying $c<(n-1)\frac{d}{\pi}\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]^{\frac{1}{2}}$, such that*

(j) $F(x,t)\geq 0$ for each $(x,t)\in([0,\alpha]\cup[\beta,1])\times[0,d]$,

(jj)

$$\int_0^1 \sup_{t\in[-c,c]} F(x,t)\mathrm{d}x < \min\left\{ \frac{\pi^2c^2}{(n-1)^2d^2\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]}\int_\alpha^\beta F(x,d)\mathrm{d}x\ , \right.$$
$$\left. \frac{c^2}{(n-1)d^2\left(\frac{(2n-2)!}{(n-2)!}\right)^2\left[\frac{1}{\alpha^{2n-1}}+\frac{1}{(1-\beta)^{2n-1}}\right]}\int_\alpha^\beta F(x,d)\mathrm{d}x \right\}.$$

Then there exist $r>0$ and $w\in X$ such that $\frac{1}{2}|||w|||^2+\int_0^1[H(x,w'(x))+P(w(x))]\mathrm{d}x>r$ and

$$\int_0^1 \sup_{t\in[-\sqrt{\frac{r}{B}},\sqrt{\frac{r}{B}}]} F(x,t)\mathrm{d}x < r\frac{\int_0^1 F(x,w(x))\mathrm{d}x}{\frac{1}{2}|||w|||^2+\int_0^1[H(x,w'(x))+P(w(x))]\mathrm{d}x}.$$

Proof . We put $r=Bc^2$ and

$$w(x)=\begin{cases} d\sum_{i=0}^{n-1}(-1)^{n-1-i}\begin{pmatrix}2n-3-i\\ n-1-i\end{pmatrix}\begin{pmatrix}2n-2\\ i\end{pmatrix}\left(\frac{x}{\alpha}\right)^{2n-2-i}, & x\in[0,\alpha),\\ d, & x\in[\alpha,\beta],\\ \frac{d}{(1-\beta)^{2n-2}}\left[\begin{pmatrix}2n-3\\ n-1\end{pmatrix}(2n-2)\sum_{i=0}^{2n-3}\frac{(-1)^{n-1-i}}{2n-2-i}\right. & \\ \left(\sum_{j=\max\{0,-n+1+i\}}^{\min\{i,n-2\}}\begin{pmatrix}n-2\\ n-2-j\end{pmatrix}\begin{pmatrix}n-1\\ n-1-i+j\end{pmatrix}\beta^{i-j}\right) & \\ \left. x^{2n-2-i}+\sum_{i=n-1}^{2n-2}(-1)^i\begin{pmatrix}2n-2\\ i\end{pmatrix}\beta^{2n-2-i}\right], & x\in(\beta,1]. \end{cases}$$

It is easy to see that $w\in X$ and, in particular,

$$4(n-1)^2d^2\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]\leq |||w|||^2\leq (n-1)d^2\left(\frac{(2n-2)!}{(n-2)!}\right)^2\left[\frac{1}{\alpha^{2n-1}}+\frac{1}{(1-\beta)^{2n-1}}\right].$$

Since

$$w'(x)=\begin{cases}\frac{(-1)^{n-1}k_nd}{\alpha^{2n-2}}x^{n-2}(x-\alpha)^{n-1}, & x\in[0,\alpha),\\ 0, & x\in[\alpha,\beta],\\ \frac{(-1)^{n-1}k_nd}{(1-\beta)^{2n-2}}(x-1)^{n-2}(x-\beta)^{n-1}, & x\in(\beta,1],\end{cases}$$

that k_n is a real constant dependent on n, then $0 \le w(x) \le d$ for each $x \in [0,1]$. Hence taking into account that $c < (n-1)\frac{d}{\pi}\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]^{\frac{1}{2}}$ and (3.1), one has

$$\begin{aligned} r = Bc^2 &< \frac{Bd^2}{\pi^2}(n-1)^2\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right] \le \frac{B}{4\pi^2}|||w|||^2 \le A|||w|||^2 \\ &\le \frac{1}{2}|||w|||^2 + \int_0^1 [H(x,w'(x)) + P(w(x))]\mathrm{d}x \le B|||w|||^2 \\ &\le (n-1)Bd^2\left(\frac{(2n-2)!}{(n-2)!}\right)^2\left[\frac{1}{\alpha^{2n-1}}+\frac{1}{(1-\beta)^{2n-1}}\right]. \end{aligned}$$

Since $0 \le w(x) \le d$ for each $x \in [0,1]$, condition (j) ensures that

$$\int_0^\alpha F(x,w(x))\mathrm{d}x + \int_\beta^1 F(x,w(x))\mathrm{d}x \ge 0.$$

Moreover, if $\int_0^1 \sup_{t\in[-\sqrt{\frac{r}{B}},\sqrt{\frac{r}{B}}]} F(x,t)\mathrm{d}x \ge 0$, from (jj) and $r = Bc^2$ and the above inequality we have

$$\begin{aligned} 0 &\le \int_0^1 \sup_{t\in[-\sqrt{\frac{r}{B}},\sqrt{\frac{r}{B}}]} F(x,t)\mathrm{d}x \\ &< \frac{c^2}{(n-1)d^2\left(\frac{(2n-2)!}{(n-2)!}\right)^2\left[\frac{1}{\alpha^{2n-1}}+\frac{1}{(1-\beta)^{2n-1}}\right]}\int_\alpha^\beta F(x,d)\mathrm{d}x \\ &\le \frac{Bc^2}{(n-1)Bd^2\left(\frac{(2n-2)!}{(n-2)!}\right)^2\left[\frac{1}{\alpha^{2n-1}}+\frac{1}{(1-\beta)^{2n-1}}\right]}\int_0^1 F(x,w(x))\mathrm{d}x \\ &\le r\frac{\int_0^1 F(x,w(x))\mathrm{d}x}{\frac{1}{2}|||w|||^2 + \int_0^1 [H(x,w'(x)) + P(w(x))]\mathrm{d}x}. \end{aligned}$$

On the other hand, if $\int_0^1 \sup_{t\in[-\sqrt{\frac{r}{B}},\sqrt{\frac{r}{B}}]} F(x,t)\mathrm{d}x < 0$, from $B \le 4A\pi^2$ we have

$$\begin{aligned} \int_0^1 \sup_{t\in[-\sqrt{\frac{r}{B}},\sqrt{\frac{r}{B}}]} F(x,t)\mathrm{d}x &< \frac{\pi^2c^2}{(n-1)^2d^2\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]}\int_\alpha^\beta F(x,d)\mathrm{d}x \\ &\le \frac{\pi^2Bc^2}{4A\pi^2d^2(n-1)^2\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]}\int_\alpha^\beta F(x,d)\mathrm{d}x \\ &\le \frac{Bc^2\int_0^1 F(x,w(x))\mathrm{d}x}{4Ad^2(n-1)^2\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]} \\ &\le r\frac{\int_0^1 F(x,w(x))\mathrm{d}x}{\frac{1}{2}|||w|||^2 + \int_0^1 [H(x,w'(x)) + P(w(x))]\mathrm{d}x}. \end{aligned}$$

Thus

$$\int_0^1 \sup_{t\in[-\sqrt{\frac{r}{B}},\sqrt{\frac{r}{B}}]} F(x,t)\mathrm{d}x < r\frac{\int_0^1 F(x,w(x))\mathrm{d}x}{\frac{1}{2}|||w|||^2+\int_0^1[H(x,w'(x))+P(w(x))]\mathrm{d}x},$$

so the proof is complete. □ We prove the following corollary with help of the above lemma.

Corollary 3.4. *Let $0<\alpha<\beta<1$ and assume that there exist two positive constants c and d satisfying $c<(n-1)\frac{d}{\pi}\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]^{\frac{1}{2}}$, such that:*

(j) $F(x,t)\geq 0$ *for each* $(x,t)\in([0,\alpha]\cup[\beta,1])\times[0,d]$,

(jj)

$$\int_0^1 \sup_{t\in[-c,c]} F(x,t)\mathrm{d}x < \min\left\{\frac{\pi^2c^2}{(n-1)^2d^2\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]}\int_\alpha^\beta F(x,d)\mathrm{d}x,\right.$$
$$\left.\frac{c^2}{(n-1)d^2\left(\frac{(2n-2)!}{(n-2)!}\right)^2\left[\frac{1}{\alpha^{2n-1}}+\frac{1}{(1-\beta)^{2n-1}}\right]}\int_\alpha^\beta F(x,d)\mathrm{d}x\right\},$$

(jjj) $\frac{1}{\pi^4A}\lim_{|t|\to+\infty}\sup\frac{F(x,t)}{t^2}<\frac{1}{\theta}$ *for almost every* $x\in[0,1]$ *and for all* $t\in\mathbb{R}$, *and for some* θ *satisfying*

$$\theta>\frac{1}{\min\left\{\frac{\pi^2c^2}{(n-1)^2d^2\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]}\int_\alpha^\beta F(x,d)\mathrm{d}x,\ \frac{c^2\int_\alpha^\beta F(x,d)\mathrm{d}x}{(n-1)d^2\left(\frac{(2n-2)!}{(n-2)!}\right)^2\left[\frac{1}{\alpha^{2n-1}}+\frac{1}{(1-\beta)^{2n-1}}\right]}\right\}-\int_0^1\sup_{t\in[-c,c]}F(x,t)\mathrm{d}x}.$$

Then, there exist a non-empty open interval $E\subseteq(0,r\theta]$ and a number $q>0$ with the following property: for each $\lambda\in E$ and for an arbitrary L^1-Carathéodory function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$, there is $\tau>0$ such that, whenever $\mu\in[0,\tau]$, problem (1.1) *admits at least three weak solutions whose norms in X are less than q.*

Proof . From Lemma 3.3 we see that assumptions (i) and (ii) of Theorem 3.1 are fulfilled for w given in the first of proof of Lemma 3.3. Also from (jjj), one has that (iii) is satisfied. Hence, the conclusion follows directly from Theorem 3.1. □

Example 3.5. *Consider the problem*

$$\begin{cases}(-1)^nu^{(2n)}+(-1)^{n-1}u^{(2n-2)}+\cdots+u^{(4)}-u''\left[(2+x)\cos u'+\sin u'\right]+3u\\ =\lambda f(x,u)+\mu g(x,u), & x\in(0,1),\\ u(0)=u(1)=u'(0)=u'(1)=\cdots=u^{(n-2)}(0)=u^{(n-2)}(1)=0=u^{(n)}(0)=u^{(n)}(1),\end{cases}\tag{3.3}$$

where

$$\begin{aligned} f:[0,1]\times\mathbb{R}\ &\to\mathbb{R}\\ (x,t)\ &\mapsto f(x,t)=\begin{cases}-e^{-t}+e^{-1}, & (x,t)\in[0,1]\times(1,+\infty),\\ 0, & (x,t)\in[0,1]\times[0,1],\\ -e^tt^9(t+10), & (x,t)\in[0,1]\times(-\infty,0),\end{cases}\end{aligned}$$

and $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a fixed L^1-Carathéodory function.
Here, $p(t)=-3t$ and $h(x,t)=[(2+x)\cos t+\sin t]^{-1}$ for all $x\in[0,1]$ and $t\in\mathbb{R}$. Hence we have

$$\begin{aligned}\Phi(u)&=\frac{1}{2}|||u|||^2+\int_0^1[H(x,u'(x)+P(u(x))]\,\mathrm{d}x\\&=\frac{1}{2}|||u|||^2+\int_0^1\int_0^{u'(x)}\left(\int_0^{\tau}[(2+x)\cos t+\sin t]\,\mathrm{d}t\right)\mathrm{d}\tau\mathrm{d}x+\int_0^1\left(\int_0^{u(x)}3\zeta\mathrm{d}\zeta\right)\mathrm{d}x\\&=\frac{1}{2}|||u|||^2+\int_0^1\int_0^{u'(x)}[(2+x)\sin\tau-\cos\tau+1]\,\mathrm{d}\tau\mathrm{d}x+\int_0^1\frac{3}{2}(u(x))^2\mathrm{d}x\\&\geq\frac{1}{2}|||u|||^2+\int_0^1\int_0^{u'(x)}[(2+x)\sin\tau]\,\mathrm{d}\tau\mathrm{d}x+\frac{3}{2}\|u\|_2^2\\&=\frac{1}{2}|||u|||^2+\int_0^1[(2+x)(-\cos(u'(x))+1)]\,\mathrm{d}x+\frac{3}{2}\|u\|_2^2\\&\geq\frac{1}{2}|||u|||^2+\frac{3}{2}\|u\|_2^2\geq 0.\end{aligned}$$

Note that

$$F(x,t)=\int_0^t f(x,\zeta)\mathrm{d}\zeta=\begin{cases}e^{-t}+(t-2)e^{-1}, & (x,t)\in[0,1]\times(1,+\infty),\\ 0, & (x,t)\in[0,1]\times[0,1],\\ -e^tt^{10}, & (x,t)\in[0,1]\times(-\infty,0).\end{cases}$$

By choosing $c=1$ and $d=5$, it is clear that $F(x,t)\geq 0$ for all $0<\alpha<\beta<1$ and $(x,t)\in([0,\alpha]\cup[\beta,1])\times[0,d]$, i.e. (j) is satisfied. Also we have $c<(n-1)\frac{d}{\pi}\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]^{\frac{1}{2}}$, for all $n\in\mathbb{N}-\{1\}$. On the other hand

$$\begin{aligned}\int_0^1\sup_{t\in[-c,c]}F(x,t)\mathrm{d}x&=\int_0^1\max\left\{\sup_{t\in[0,c]}(0),\ \sup_{t\in[-c,0)}(-e^tt^{10})\right\}\mathrm{d}x\\&=\int_0^1 0\mathrm{d}x=0\\&<\frac{(e^{-5}+3e^{-1})(\beta-\alpha)}{25(n-1)\left(\frac{(2n-2)!}{(n-2)!}\right)^2\left[\frac{1}{\alpha^{2n-1}}+\frac{1}{(1-\beta)^{2n-1}}\right]}\\&=\min\left\{\frac{\pi^2c^2\int_\alpha^\beta F(x,d)\mathrm{d}x}{(n-1)^2d^2\left[\frac{1}{\alpha^3}+\frac{1}{(1-\beta)^3}\right]},\frac{c^2\int_\alpha^\beta F(x,d)\mathrm{d}x}{(n-1)d^2\left(\frac{(2n-2)!}{(n-2)!}\right)^2\left[\frac{1}{\alpha^{2n-1}}+\frac{1}{(1-\beta)^{2n-1}}\right]}\right\}.\end{aligned}$$

So (jj)is satisfied. Also since $\lim_{|t|\to+\infty}\sup\frac{F(x,t)}{t^2}=0$, then (jjj) holds. Now we can apply Corollary 3.4 for every

$$\theta>\frac{25(n-1)\left(\frac{(2n-2)!}{(n-2)!}\right)^2\left[\frac{1}{\alpha^{2n-1}}+\frac{1}{(1-\beta)^{2n-1}}\right]}{(e^{-5}+3e^{-1})(\beta-\alpha)}.$$

Then problem (3.3), admits at least three weak solutions.

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