An Existence Result of Three Solutions for a 2n-th-Order Boundary-Value Problem

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Abstract

In this paper, we establish the existence of at least three weak solutions for some one-dimensional 2n-th-order equations in a bounded domain. A particular case and a concrete example are then presented.

Keywords: Boundary value problem, Sobolev space, Critical point, Three solutions, Variational method

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1. Introduction

Let \( n \in \mathbb{N} - \{1\} \). In this note, we consider the 2n-th-order boundary-value problem

\[
\begin{cases}
[(−1)^nu^{(2n)} + (−1)^{n−1}u^{(2n−2)} + \cdots + u^{(4)}]|h(x, u') - u''
\ = [\lambda f(x, u) + \mu g(x, u) + p(u)]h(x, u'), \\
u(0) = u(1) = u'(0) = u'(1) = \cdots = u^{(n−2)}(0) = u^{(n−2)}(1) = 0 = u^{(n)}(0) = u^{(n)}(1),
\end{cases}
\]

where \( \lambda \) is a positive parameter, \( \mu \) is a nonnegative parameter, \( f, g : [0, 1] \times \mathbb{R} \to \mathbb{R} \) are two \( L^1 \)-Carathéodory functions, \( p : \mathbb{R} \to (−\infty, 0] \) is a Lipschitz continuous function with the Lipschitz constant \( L > 0 \) i.e., \(|p(t_1) - p(t_2)| \leq L|t_1 - t_2|\) for every \( t_1, t_2 \in \mathbb{R} \), with \( p(0) = 0 \), suppose that the Lipschitz constant \( L \) of the function \( p \) satisfies \( 0 < L < \pi^4 \), and \( h : [0, 1] \times \mathbb{R} \to [0, +\infty) \) is a bounded and continuous function with 0 < \( m := \inf_{(x,t)\in[0,1]\times\mathbb{R}} h(x, t) \leq \sup_{(x,t)\in[0,1]\times\mathbb{R}} h(x, t) = M < \infty \).

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Many researchers have studied the existence and multiplicity of solutions for such a problem. For example, authors in [2], using Ricceri’s Variational Principle [8], established the existence of at least three weak solutions for the problem

$$\begin{cases} u'''' + \alpha u''' + \beta u = \lambda f(x, u) + \mu g(x, u), & x \in (0, 1), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases}$$

where \(\alpha, \beta\) are real constants, \(f, g : [0, 1] \times \mathbb{R} \to \mathbb{R}\) are \(L^2\)-Carethéodory functions and \(\lambda, \mu > 0\). Also the authors in [9], employing Ricceri’s Variational Principle [8], established the existence of at least three weak solutions for the problem

$$\begin{cases} u''''h(x, u') - u'' = [\lambda f(x, u) + \mu g(x, u) + p(u)]h(x, u'), & x \in (0, 1), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases}$$

where \(\lambda > 0, \mu \geq 0\) and \(f, g, p, h\) are functions with the same conditions in the problem (1.1).

We also refer the reader to the papers [1, 3, 7], in which existence results for boundary value problems with nonlinear derivative dependence were established.

2. Preliminaries

The aim of this paper is to establish the existence of a non-empty open interval \(E \subseteq \mathbb{R}\) and a positive real number \(q\) with the following property: for each \(\lambda \in E\) and for each Carathéodory function \(g : [0, 1] \times \mathbb{R} \to \mathbb{R}\) such that \(\sup_{|t| \leq s} |g(\cdot, \zeta)| \in L^1(0, 1)\) for all \(s > 0\), there is \(\delta > 0\) such that for each \(\mu \in [0, \delta]\), the problem (1.1) admits at least three solutions in \(X = H^n([0, 1]) \cap H_0^{n-1}([0, 1])\) whose norms are less than \(q\).

Our analysis is based on the following critical point theorem.

**Theorem 2.1** (9, Ricceri). Let \(X\) be a reflexive real Banach space, \(I \subseteq \mathbb{R}\) an interval, \(\Phi : X \to \mathbb{R}\) be a sequentially weakly lower semicontinuous \(C^1\) functional, bounded on each bounded subset of \(X\), whose derivative admits a continuous inverse on \(X^*\), \(J : X \to \mathbb{R}\) be a \(C^1\) functional with compact derivative. Assume that \(\lim_{\|x\| \to +\infty} (\Phi(x) + \lambda J(x)) = +\infty\) for all \(\lambda \in I\), and there exists \(\rho \in \mathbb{R}\) such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda J(x) + \rho) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda J(x) + \rho).$$

Then, there exist a non-empty open set interval \(E \subseteq I\) and a positive real number \(q\) with the following property: for every \(C^1\) functional \(\Psi : X \to \mathbb{R}\) with compact derivative, there exists \(\tau > 0\) such that for each \(\mu \in [0, \tau]\), the equation

$$\Phi'(u) + \lambda J'(u) + \mu \Psi'(u) = 0$$

has at least three solutions in \(X\) whose norms less than \(q\).

In the proof of our main result we also use the next result to verify the minimax inequality in Theorem 2.1.

**Theorem 2.2** (4, Bonanno). Let \(X\) be a non-empty set and \(\Phi, J\) two real functions on \(X\). Assume that \(\Phi(x) \geq 0\) for every \(x \in X\) and there exists \(u_0 \in X\) such that \(\Phi(u_0) = J(u_0) = 0\). Further, assume that there exist \(u_1 \in X, \quad r > 0\) such that
(k_1) \Phi(u_1) > r, \quad (k_2) \sup_{\Phi(x) < r}(-J(x)) < r \frac{-J(u_1)}{\Phi(u_1)}.

Then, for every \( v > 1 \) and for every \( \rho \in \mathbb{R} \) satisfying

\[
\sup_{\Phi(x) < r} (-J(x)) + \frac{r \frac{-J(u_1)}{\Phi(u_1)} - \sup_{\Phi(x) < r}(-J(x))}{v} < \rho < r \frac{-J(u_1)}{\Phi(u_1)},
\]

one has

\[
\sup_{\lambda \in \mathbb{R}} \inf_{x \in X} (\Phi(x) + \lambda (J(x) + \rho)) < \inf_{x \in X} \sup_{\lambda \in [0, \sigma]} (\Phi(x) + \lambda (J(x) + \rho)),
\]

where

\[
\sigma = \frac{vr}{r \frac{-J(u_1)}{\Phi(u_1)} - \sup_{\Phi(x) < r}(-J(x))}.
\]

Let us introduce some notations which will be used later. Define

\[
H^n([0, 1]) := \{ u \in L^2([0, 1]) : u', u'', \ldots, u^{(n)} \in L^2([0, 1]) \},
\]

\[
H_0^{n-1}([0, 1]) := \{ u \in L^2([0, 1]) : u', u'', \ldots, u^{(n-1)} \in L^2([0, 1]), u(0) = u(1) = u'(0) = u'(1) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0 \}.
\]

Take \( X = H^n([0, 1]) \cap H_0^{n-1}([0, 1]) = \{ u \in L^2([0, 1]) : u', u'', \ldots, u^{(n)} \in L^2([0, 1]), u(0) = u(1) = u'(0) = u'(1) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0 \} \),

endowed with the norm

\[
|||u||| := \left( \|u''\|^2_2 + \|u'''\|^2_2 + \cdots + \|u^{(n)}\|^2_2 \right)^\frac{1}{2}, \quad \text{where } \|u\|_2 := \left( \int_0^1 |u(x)|^2 dx \right)^\frac{1}{2}.
\]

We recall the following Poincaré type inequalities (\ref{2.3}):

\[
\|u\|_2 \leq \frac{1}{\pi^2} \|u''\|_2, \quad (2.1)
\]

\[
\|u'\|_2 \leq \frac{1}{\pi} \|u''\|_2, \quad (2.2)
\]

for all \( u \in X \). For the norm in \( C^{n-1}([0, 1]) \),

\[
\|u\|_\infty := \max \left\{ \max_{x \in [0, 1]} |u(x)|, \max_{x \in [0, 1]} |u'(x)|, \cdots, \max_{x \in [0, 1]} |u^{(n-1)}(x)| \right\},
\]

since \( C^{n-1}([0, 1]) \subseteq C^1([0, 1]) \), we have the well-known inequality (\ref{2.2}): \( \|u\|_\infty \leq \frac{1}{2\pi} \|u'\|_2 \), then, by (2.2), we have

\[
\max_{x \in [0, 1]} |u(x)| \leq \|u\|_\infty \leq \frac{1}{2\pi} \|u''\|_2 \leq \frac{1}{2\pi} |||u|||,
\]

for all \( u \in X \). The norm \( ||| \cdot ||| \), is equivalent with the usual norm of Sobolev space \( H^n((0, 1)) = W^{n, 2}((0, 1)) \):
\[ \|u\|_{W^{n,2}} := \left( \|u\|^2_2 + \|u\'|^2_2 + \|u''\|^2_2 + \cdots + \|u^{(n)}\|^2_2 \right)^{\frac{1}{2}}. \]  
Because by \((2.1)\) and \((2.2)\) we have

\[
\||u|| = \left( \|u''\|^2_2 + \|u''\|^2_2 + \cdots + \|u^{(n)}\|^2_2 \right)^{\frac{1}{2}} 
\leq \left( \frac{1}{\pi^2} \|u''\|^2_2 + \frac{1}{\pi^2} \|u''\|^2_2 + \cdots + \|u^{(n)}\|^2_2 \right)^{\frac{1}{2}} 
\leq \left( \frac{1}{\pi^2} + \frac{1}{\pi^2} + 1 \right)^{\frac{1}{2}} \left( \|u''\|^2_2 + \cdots + \|u^{(n)}\|^2_2 \right)^{\frac{1}{2}} 
\leq \left( \frac{1}{\pi^2} + \frac{1}{\pi^2} + 1 \right)^{\frac{1}{2}} \|u\|. 
\]

We recall that \(f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}\) is a Carathéodory function if

(a) the mapping \(x \mapsto f(x, t)\) is measurable for every \(t \in \mathbb{R}\);

(b) the mapping \(t \mapsto f(x, t)\) is continuous for almost every \(x \in [0, 1]\).

Also if for every \(\rho > 0\) there exists a function \(\ell_\rho \in L^1([0, 1])\) such that

\[ \sup_{|t| \leq \rho} |f(x, t)| \leq \ell_\rho(x) \]

for almost every \(x \in [0, 1]\), then the Carathéodory function \(f\) is called \(L^1\)-Carathéodory function.

Corresponding to \(f, g, p\) and \(h\), we introduce the functions \(F, G, P\) and \(H\), respectively, as follows

\[
F : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}, \quad (x, t) \mapsto F(x, t) := \int_0^t f(x, \zeta) d\zeta, \\
G : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}, \quad (x, t) \mapsto G(x, t) := \int_0^t g(x, \zeta) d\zeta, \\
P : \mathbb{R} \rightarrow [0, +\infty), \quad t \mapsto P(t) := -\int_0^t p(\zeta) d\zeta, \\
H : [0, 1] \times \mathbb{R} \rightarrow [0, +\infty), \quad (x, t) \mapsto H(x, t) := \int_0^t \left( \int_0^\tau \frac{1}{h(x, \delta)} d\delta \right) d\tau, 
\]

for all \(x \in [0, 1]\) and \(t \in \mathbb{R}\).

If the parts of equation in \((1.1)\) divided by \(h(x, u')\) and then multiplied by an arbitrary function \(v \in X\) and then integrated in \(x \in [0, 1]\) then by \(n\) times integration by parts we have

\[
\int_0^1 u^{(n)}(x)v(x)dx + \int_0^1 u^{(n-1)}(x)v^{(n)}(x)dx + \cdots + \int_0^1 u''(x)v''(x)dx \\
+ \int_0^1 \left( \int_0^x \frac{1}{h(x, \tau)} d\tau \right) v'(x)dx - \lambda \int_0^1 f(x, u(x))v(x)dx \\
- \mu \int_0^1 g(x, u(x))v(x)dx - \int_0^1 p(u(x))v(x)dx = 0
\] (2.4)

for all \(v \in X\). Then we say that function \(u \in X\) in \((2.4)\) is a weak solution of \((1.1)\).
3. Main Results

Put \( A := \frac{\pi^4 - L}{2\pi^4} \), \( B := \frac{\pi^2 + m(\pi^4 + L)}{2m\pi^4} \)
and suppose that \( B \leq 4A\pi^2 \). We formulate our main result as follows.

**Theorem 3.1.** Assume that there exist a positive constant \( r \) and a function \( w \in X \) such that

(i) \( \frac{1}{2} \|u\|^2 + \int_0^1 [H(x, w'(x)) + P(w(x))]dx > r \);

(ii) \( \int_0^1 \sup_{t \in [-\sqrt{2}, \sqrt{2}]} F(x, t) < \frac{1}{r} \|u\|^2 + \int_0^1 [H(x, w'(x)) + P(w(x))]dx \);

(iii) \( \frac{1}{\pi^4 A} \lim_{t \to +\infty} \sup \frac{F(x, t)}{t^2} < \frac{1}{\theta} \) for almost every \( x \in [0, 1] \) and for all \( t \in \mathbb{R} \), and for some \( \theta > 0 \) satisfying

\[ \theta > \frac{1}{\frac{1}{r^2} \|u\|^2 + \int_0^1 [H(x, w'(x)) + P(w(x))]dx - \int_0^1 \sup_{t \in [-\sqrt{2}, \sqrt{2}]} F(x, t)dx} \]

Then, there exist a non-empty open interval \( E \subseteq (0, r\theta] \) and a number \( q > 0 \) with the following property: for each \( \lambda \in E \) and for an arbitrary \( L^1 \)-Carathéodory function \( g : [0, 1] \times \mathbb{R} \to \mathbb{R} \), there is \( \tau > 0 \) such that, whenever \( \mu \in [0, \tau] \), problem \([1.1]\) admits at least three weak solutions whose norms in \( X \) are less than \( q \).

**Proof.** Our aim is to apply Theorem 2.1 to problem (1.1). Taking \( X = H^n([0, 1]) \cap H_0^{n-1}([0, 1]) \)
endowed with the norm

\[ \|u\| = (\|u''\|^2 + \|u'''\|^2 + \cdots + \|u^{(n)}\|^2)^{\frac{1}{2}} \],

for every \( u \in X \). We introduce the following functionals:

\[ \Phi : X \to \mathbb{R}, \quad u \mapsto \Phi(u) := \frac{1}{2} \|u\|^2 + \int_0^1 [H(x, u'(x)) + P(u(x))]dx \]

and

\[ J : X \to \mathbb{R}, \quad u \mapsto J(u) := -\int_0^1 F(x, u(x))dx. \]

Since \( X \) is a reflexive real Banach space and \( X \) is compactly embedded into \( C([0, 1]) \) then by classical results and that every norm in Banach space \( X \), is a sequentially weakly lower semicontinuous functional, hence \( \Phi \) is a sequentially weakly lower semicontinuous functional and Gâteaux differentiable with compact Gâteaux derivative hence by definition with continuous Gâteaux derivative, also \( \Phi(u) \geq 0 \), for every \( u \in X \). By classical results, the functional \( J \) is well defined and Gâteaux differentiable whose Gâteaux derivative is compact hence by definition with continuous derivative. In particular, for each \( u \in X \) one has \( \Phi'(u) \in X^* \), \( J'(u) \in X^* \) and

\[ \Phi'(u)(v) = \int_0^1 u^{(n)}(x)v^{(n)}(x)dx + \cdots + \int_0^1 u''(x)v''(x)dx \]

\[ + \int_0^1 \left( \int_0^x \frac{1}{h(x, \tau)}d\tau \right) v'(x)dx - \int_0^1 p(u(x))v(x)dx, \]

\[ J'(u)(v) = -\int_0^1 f(x, u(x))v(x)dx, \]

 respectively.
for all $v \in X$. Hence $\Phi'$ is a strongly monotone operator, because for every $u, v \in X$ we have:

\[
(\Phi'(u) - \Phi'(v), u - v) = \Phi'(u)(u - v) - \Phi'(v)(u - v)
\]

\[
\begin{aligned}
&= \int_0^1 (u^{(n)} - v^{(n)})(u^{(n)} - v^{(n)}) \, dx + \cdots + \int_0^1 (u'' - v'')(u'' - v'') \, dx \\
&\quad + \int_0^1 \left( \int_{v(x)}^{u(x)} \frac{1}{h(x, \tau)} \, d\tau \right) (u' - v') \, dx - \int_0^1 (p(u) - p(v))(u - v) \, dx \\
\end{aligned}
\]

\[
\geq \left( \|u^{(n)} - v^{(n)}\|_2^2 + \cdots + \|u'' - v''\|_2^2 \right) + \frac{1}{M} \|u' - v'\|_2^2 - L\|u - v\|_2^2
\]

\[
\geq \left( \|u^{(n)} - v^{(n)}\|_2^2 + \cdots + \|u'' - v''\|_2^2 \right) - \frac{L}{\pi^4} \|u'' - v''\|_2^2
\]

\[
\geq (1 - \frac{L}{\pi^4}) \left( \|u^{(n)} - v^{(n)}\|_2^2 + \cdots + \|u'' - v''\|_2^2 \right)
\]

\[
= 2A \left( \|u^{(n)} - v^{(n)}\|_2^2 + \cdots + \|u'' - v''\|_2^2 \right)
\]

\[
= 2A\|u - v\|_2^2.
\]

That with the assumption $0 < L < \pi^4$ we have $\Phi'$ is a strongly monotone operator. Then by Minty-Browder theorem [1], Theorem 26.A), $\Phi : X \to X^*$ admits a Lipschitz continuous inverse. Since $p$ is Lipschitz continuous and satisfies $p(0) = 0$, while $h$ is bounded away from zero, we have:

\[
|\Phi(u)| = \left| \frac{1}{2} \|u\|_2^2 + \int_0^1 H(x, u'(x)) \, dx + \int_0^1 P(u(x)) \, dx \right|
\]

\[
= \left| \frac{1}{2} \|u\|_2^2 + \int_0^1 \int_0^1 \left( \int_0^\tau \frac{1}{h(x, \delta)} \, d\delta \right) \, d\tau \, dx - \int_0^1 \left( \int_0^u p(\zeta) \, d\zeta \right) \, dx \right|
\]

\[
\geq \frac{1}{2} \|u\|_2^2 + \int_0^1 \frac{1}{2M} (u'(x))^2 \, dx - \int_0^1 \frac{L}{2} (u(x))^2 \, dx \geq \frac{1}{2} \|u\|_2^2 - \frac{L}{2} \|u\|_2^2
\]

\[
\geq \frac{1}{2} \|u\|_2^2 - \frac{L}{2\pi^4} \|u''\|_2^2 \geq \left( \frac{1}{2} - \frac{L}{2\pi^4} \right) \left( \|u''\|_2^2 + \cdots + \|u^{(n)}\|_2^2 \right)
\]

\[
= A\|u\|_2^2.
\]

On the other hand, we have:

\[
|\Phi(u)| = \left| \frac{1}{2} \|u\|_2^2 + \int_0^1 \int_0^1 \left( \int_0^\tau \frac{1}{h(x, \delta)} \, d\delta \right) \, d\tau \, dx - \int_0^1 \left( \int_0^u p(\zeta) \, d\zeta \right) \, dx \right|
\]

\[
\leq \frac{1}{2} \|u\|_2^2 + \int_0^1 \frac{1}{2m} (u'(x))^2 \, dx + \int_0^1 \frac{L}{2} (u(x))^2 \, dx
\]

\[
= \frac{1}{2} \|u\|_2^2 + \frac{1}{2m} \|u'\|_2^2 + \frac{L}{2} \|u\|_2^2
\]

\[
\leq \frac{1}{2} \|u\|_2^2 + \frac{1}{2m\pi^2} \|u''\|_2^2 + \frac{L}{2\pi^4} \|u''\|_2^2
\]

\[
\leq \left( \frac{1}{2} + \frac{1}{2m\pi^2} + \frac{L}{2\pi^4} \right) \|u\|_2^2 = B\|u\|_2^2.
\]
Since $\Phi(u) \geq 0$, for all $u \in X$, then we have:
\[
A||u||^2 \leq \Phi(u) \leq B||u||^2.
\] (3.1)

Then $\Phi$ is bounded on each bounded subset of $X$. Furthermore from (iii) there exist two constants $\gamma, \tau \in \mathbb{R}$ with $0 < \gamma < \frac{1}{\theta}$ such that $\frac{1}{\pi^4 A} F(x, t) \leq \gamma t^2 + \tau$ for a.e. $x \in (0, 1)$ and all $t \in \mathbb{R}$. Fix $u \in X$, then
\[
F(x, u(x)) \leq \pi^4 A(\gamma |u(x)|^2 + \tau) \quad \text{for all } x \in (0, 1)
\] (3.2)

Then, for any fixed $\lambda \in (0, \theta]$, from (3.1), (3.2) and (2.1) we have
\[
\Phi(u) + \lambda J(u) \geq A||u||^2 - \pi^4 A\lambda \int_0^1 (\gamma |u(x)|^2 + \tau) dx \geq A||u||^2 - \pi^4 A\lambda \left( \frac{\gamma}{\pi^4} ||u''||_2^2 + \tau \right)
\]
\[
\geq A||u||^2 - \pi^4 A\theta \left( \frac{\gamma}{\pi^4} ||u''||_2^2 + \tau \right) \geq A||u||^2 - \pi^4 A\theta \left( \frac{\gamma}{\pi^4} ||u||^2 + \tau \right)
\]
\[
= A(1 - \theta\gamma) ||u||^2 - \pi^4 A\theta \tau
\]
for all $u \in X$ and so $\lim_{||u|| \to +\infty} (\Phi(u) + \lambda J(u)) = +\infty$.

We claim that there exist $r > 0$ and $w \in X$ such that
\[
\sup_{u \in \Phi^{-1}((\infty, r))} (-J(u)) < r - \frac{J(w)}{\Phi(w)}.
\]

From (3.1) and (2.3), we have
\[
\Phi^{-1}((\infty, r)) = \{ u \in X : \Phi(u) < r \} \subseteq \{ u \in X : 4\pi^2 ||u||^2 < r \}
\]
\[
\subseteq \{ u \in X : \frac{B}{4\pi^2} ||u||^2 < r \} = \{ u \in X : ||u|| < 2\pi \sqrt{\frac{r}{B}} \}
\]
\[
\subseteq \{ u \in X : |u(x)| < \sqrt{\frac{r}{B}} \}
\]
and it follows that
\[
\sup_{u \in \Phi^{-1}((\infty, r))} (-J(u)) = \sup_{u \in \Phi^{-1}((\infty, r))} \int_0^1 F(x, u(x)) dx \leq \int_0^1 \sup_{t \in [\sqrt{\frac{r}{B}}, \sqrt{\frac{r}{B}}]} F(x, t) dx.
\]

Now from (ii) we have
\[
\sup_{u \in \Phi^{-1}((\infty, r))} (-J(u)) \leq r \frac{\int_0^1 F(x, w(x)) dx}{\frac{1}{2} ||w||^2 + \int_0^1 [H(x, w'(x)) + P(w(x))] dx} = r - \frac{J(w)}{\Phi(w)}
\]
also from (i) we have $\Phi(w) > r$. Next recall from (iii) that
\[
\theta > \frac{1}{r - \frac{J(w)}{\Phi(w)} - \sup_{u \in \Phi^{-1}((\infty, r))} (-J(u))},
\]
choose
\[ \alpha = \theta \left( \frac{r - J(w)}{\Phi(w)} - \sup_{u \in \Phi^{-1}((-\infty, r))} (-J(u)) \right), \]
and note \( \alpha > 1 \), also, since
\[ \frac{1}{r - J(w)} \Phi(w) - \sup_{u \in \Phi^{-1}((-\infty, r))} (-J(u)), \]
we have
\[ \sup_{u \in \Phi^{-1}((-\infty, r))} (-J(u)) + \frac{1}{\theta} < r \frac{J(w)}{\Phi(w)}, \]
and so with our choice of \( \alpha \) we have
\[ \sup_{u \in \Phi^{-1}((-\infty, r))} (-J(u)) + \frac{r - J(w)}{\Phi(w)} - \sup_{u \in \Phi^{-1}((-\infty, r))} (-J(u)) < r \frac{J(w)}{\Phi(w)}. \]
Now from Theorem 2.2 (with \( u_0 = 0 \) and \( u_1 = w \)) for every \( \rho \in \mathbb{R} \) satisfying
\[ \sup_{u \in \Phi^{-1}((-\infty, r))} (-J(u)) + \frac{r - J(w)}{\Phi(w)} - \sup_{u \in \Phi^{-1}((-\infty, r))} (-J(u)) < \rho < r \frac{J(w)}{\Phi(w)}, \]
with choice \( \sigma = r \theta \) and \( I = [0, r \theta] \), we have
\[ \inf_{\lambda \in \mathbb{R}} \inf_{u \in X} (\Phi(u) + \lambda J(u) + \lambda \rho) < \inf_{u \in X} \sup_{\lambda \in [0, r \theta]} (\Phi(u) + \lambda J(u) + \lambda \rho). \]
For any fixed \( L^1 \)-Carathéodory function \( g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \), set
\[ \Psi : X \rightarrow \mathbb{R}, \quad u \mapsto \Psi(u) = -\int_0^t \int_0^1 g(x, t) dt dx. \]
Since \( X \) is a reflexive real Banach space and \( X \) is compactly embedded into \( C([0, 1]) \) then by classical results, the functional \( \Psi \) is well defined and Gâteaux differentiable whose Gâteaux derivative is compact and continuous, and \( \Psi'(u) \in X^* \), at \( u \in X \) is given by
\[ \Psi'(u)(v) = -\int_0^1 g(x, u(x)) v(x) dx \]
for all \( v \in X \). Now, all the assumptions of Theorem 2.1 are satisfied. Hence, applying Theorem 2.1 taking into account that the critical points of the functional \( \Phi + \lambda J + \mu \Psi \) are exactly the weak solutions of the problem (1.1), we have that problem (1.1) admits at least three weak solutions in \( X = W_0^{n-1, 2}([0, 1]) \cap W^{n, 2}(0, 1) \) whose norms in \( X \) are less than \( q \). \( \Box \)

**Remark 3.2.** In Theorem 3.1, the aim of taking \( p \) as a non-positive function, that’s \( \Phi(u) = \frac{1}{2} ||u||^2 + \int_1^1 [H(x, u'(x)) + P(u(x))] dx \) be nonnegative. Hence if \( p : \mathbb{R} \rightarrow \mathbb{R} \) be such that \( \Phi \geq 0 \) then Theorem 3.1 is satisfied.
The following lemma which is motivated from [3], will be used in the proof of next corollary.

Lemma 3.3. Let $0 < \alpha < \beta < 1$ and assume that there exist two positive constants $c$ and $d$ satisfying
\[
c < (n - 1) \frac{d}{\pi} \left[ \frac{1}{\alpha^3} + \frac{1}{(1 - \beta)^3} \right]^\frac{1}{2}, \]
such that
\[
(j) \quad F(x, t) \geq 0 \quad \text{for each} \quad (x, t) \in ([0, \alpha] \cup [\beta, 1]) \times [0, d],
\]
\[
(jj) \quad \int_0^1 \sup_{t \in [-c, c]} F(x, t) dx < \min \left\{ \frac{\pi^2 c^2}{(n - 1)^2 d^2 \left[ \frac{1}{\alpha^3} + \frac{1}{(1 - \beta)^3} \right]} \int_0^\beta F(x, d) dx, \frac{c^2}{(n - 1)^2 d^2 \left( \frac{2n - 2)!}{(n - 2)!} \right)^2 \left[ \frac{1}{\alpha^{2n-1}} + \frac{1}{(1 - \beta)^{2n-1}} \right] \int_0^\beta F(x, d) dx \right\}.
\]

Then there exist $r > 0$ and $w \in X$ such that $\frac{1}{2} ||w||^2 + \int_0^1 [H(x, w'(x)) + P(w(x))] dx > r$ and
\[
\int_0^1 \sup_{t \in [-\sqrt{n} \sqrt{n}]} F(x, t) dx < \frac{1}{2} ||w||^2 + \int_0^1 [H(x, w'(x)) + P(w(x))] dx.
\]

Proof. We put $r = Bc^2$ and
\[
w(x) = \begin{cases} d \sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{2n - 3 - i}{n - 1 - i} \binom{2n - 2}{i} \left( \frac{x}{\alpha} \right)^{2n-2-i}, & x \in [0, \alpha), \\ d, & x \in [\alpha, \beta], \\ \frac{d}{(1 - \beta)^{2n-2}} \left[ \binom{2n - 3}{n - 1} \binom{2n - 2}{n - 2} \sum_{i=0}^{2n-3} \frac{(-1)^{n-1-i}}{2n - 2 - i} \right. \\ \left. \sum_{j=\max(0, -n+1+i)}^{\min(i, n-2)} \binom{n - 2 - j}{n - 2 - j} \binom{n - 1 - i + j}{n - 1 - i + j} (\beta^i - j) \right) \\ x^{2n-2-i} + \sum_{i=n-1}^{2n-2} (-1)^i \binom{2n - 2}{i} (\beta^{2n-2-i}), & x \in (\beta, 1].
\end{cases}
\]

It is easy to see that $w \in X$ and, in particular,
\[
4(n - 1)^2 d^2 \left[ \frac{1}{\alpha^3} + \frac{1}{(1 - \beta)^3} \right] \leq ||w||^2 \leq (n - 1)^2 d^2 \left( \frac{(2n - 2)!}{(n - 2)!} \right)^2 \left[ \frac{1}{\alpha^{2n-1}} + \frac{1}{(1 - \beta)^{2n-1}} \right].
\]

Since
\[
w'(x) = \begin{cases} \frac{(-1)^{n-1-k}d}{\alpha^{2n-2}} x^{n-2} (x - \alpha)^{n-1}, & x \in [0, \alpha), \\ 0, & x \in [\alpha, \beta], \\ \frac{(-1)^{n-1-k}d}{(1 - \beta)^{2n-2}} (x - 1)^{n-2} (x - \beta)^{n-1}, & x \in (\beta, 1].
\end{cases}
\]
that $k_n$ is a real constant dependent on $n$, then $0 \leq w(x) \leq d$ for each $x \in [0, 1]$. Hence taking into account that $e < (n - 1) \frac{d}{\pi} \left[ \frac{1}{\alpha^3} + \frac{1}{(1 - \beta)^3} \right]^{\frac{1}{2}}$ and (3.1), one has

$$r = Bc^2 < \frac{Bd^2}{\pi^2} (n - 1)^2 \left[ \frac{1}{\alpha^3} + \frac{1}{(1 - \beta)^3} \right] \leq \frac{B}{4\pi^2} ||w||^2 \leq A||w||^2$$

$$\leq \frac{1}{2} ||w||^2 + \int_0^1 [H(x, w'(x)) + P(w(x))]dx \leq B||w||^2$$

$$\leq (n - 1)Bd^2 \left( \frac{(2n - 2)!}{(n - 2)!} \right)^2 \left[ \frac{1}{\alpha^{2n-1}} + \frac{1}{(1 - \beta)^{2n-1}} \right].$$

Since $0 \leq w(x) \leq d$ for each $x \in [0, 1]$, condition (j) ensures that

$$\int_0^\alpha F(x, w(x))dx + \int_\beta^1 F(x, w(x))dx \geq 0.$$

Moreover, if $\int_0^1 \sup_{t \in [-\sqrt{\pi}, \sqrt{\pi}]} F(x, t)dx \geq 0$, from (jj) and $r = Bc^2$ and the above inequality we have

$$0 \leq \int_0^1 \sup_{t \in [-\sqrt{\pi}, \sqrt{\pi}]} F(x, t)dx < \int_0^1 F(x, d)dx$$

$$\leq \frac{c^2}{(n - 1)d^2} \left( \frac{(2n - 2)!}{(n - 2)!} \right)^2 \left[ \frac{1}{\alpha^{2n-1}} + \frac{1}{(1 - \beta)^{2n-1}} \right] \int_\alpha^\beta F(x, w(x))dx$$

$$\leq \frac{Bc^2}{(n - 1)Bd^2} \left( \frac{(2n - 2)!}{(n - 2)!} \right)^2 \left[ \frac{1}{\alpha^{2n-1}} + \frac{1}{(1 - \beta)^{2n-1}} \right] \int_0^1 F(x, w(x))dx$$

$$\leq r \frac{1}{2} ||w||^2 + \int_0^1 [H(x, w'(x)) + P(w(x))]dx.$$

On the other hand, if $\int_0^1 \sup_{t \in [-\sqrt{\pi}, \sqrt{\pi}]} F(x, t)dx < 0$, from $B \leq 4A\pi^2$ we have

$$\int_0^1 \sup_{t \in [-\sqrt{\pi}, \sqrt{\pi}]} F(x, t)dx < \int_0^1 \frac{\pi^2c^2}{(n - 1)d^2} \left[ \frac{1}{\alpha^3} + \frac{1}{(1 - \beta)^3} \right] \int_\alpha^\beta F(x, d)dx$$

$$\leq \frac{\pi^2Bc^2}{4A\pi^2} \frac{d^2(n - 1)^2}{(n - 1)^2} \left[ \frac{1}{\alpha^3} + \frac{1}{(1 - \beta)^3} \right] \int_\alpha^\beta F(x, d)dx$$

$$\leq \frac{Bc^2\int_0^1 F(x, w(x))dx}{4Ad^2(n - 1)^2} \left[ \frac{1}{\alpha^3} + \frac{1}{(1 - \beta)^3} \right]$$

$$\leq \frac{1}{2} ||w||^2 + \int_0^1 [H(x, w'(x)) + P(w(x))]dx.$$
Thus
\[
\int_0^1 \sup_{t \in [-c,c]} F(x,t)dx < r \frac{\int_0^1 F(x,w(x))dx}{\frac{1}{2} \|w\|^2 + \int_0^1 [H(x,w'(x)) + P(w(x))]dx},
\]
so the proof is complete. ☐

We prove the following corollary with help of the above lemma.

**Corollary 3.4.** Let \(0 < \alpha < \beta < 1\) and assume that there exist two positive constants \(c\) and \(d\) satisfying \(c < (n-1) \frac{d}{\pi} \left[ \frac{1}{\alpha^3} + \frac{1}{(1-\beta)^3} \right]^{\frac{1}{2}}\), such that:

(i) \(F(x,t) \geq 0\) for each \((x,t) \in (0,\alpha] \cup [\beta,1] \times [0,d],\)

(ii)
\[
\int_0^1 \sup_{t \in [-c,c]} F(x,t)dx < \min \left\{ \frac{\pi^2 c^2}{(n-1)^2 d^2 \left[ \frac{1}{\alpha^3} + \frac{1}{(1-\beta)^3} \right]} \int_{\alpha}^{\beta} F(x,d)dx, \quad \frac{c^2}{(n-1)d^2 \left( \frac{(2n-2)!}{(2n-2)!} \right)^2 \left[ \frac{1}{\alpha^{2n-1}} + \frac{1}{(1-\beta)^{2n-1}} \right]} \int_{\alpha}^{\beta} F(x,d)dx \right\},
\]

(iii)
\[
\lim_{|t| \to \infty} \sup_{t \in [-c,c]} \frac{F(x,t)}{t^2} < \frac{1}{\theta} \quad \text{for almost every } x \in [0,1] \text{ and for all } t \in \mathbb{R}, \text{ and for some } \theta > \frac{1}{\min \left\{ \frac{\pi^2 c^2}{(n-1)^2 d^2 \left[ \frac{1}{\alpha^3} + \frac{1}{(1-\beta)^3} \right]} \int_{\alpha}^{\beta} F(x,d)dx, \quad \frac{c^2}{(n-1)d^2 \left( \frac{(2n-2)!}{(2n-2)!} \right)^2 \left[ \frac{1}{\alpha^{2n-1}} + \frac{1}{(1-\beta)^{2n-1}} \right]} \int_{\alpha}^{\beta} F(x,d)dx \right\} - \int_0^1 \sup_{t \in [-c,c]} F(x,t)dx}.
\]

Then, there exist a non-empty open interval \(E \subseteq (0,r\theta]\) and a number \(q > 0\) with the following property: for each \(\lambda \in E\) and for an arbitrary \(L^1\)-Carathéodory function \(g : [0,1] \times \mathbb{R} \to \mathbb{R}\), there is \(\tau > 0\) such that, whenever \(\mu \in [0,\tau]\), problem (1.1) admits at least three weak solutions whose norms in \(X\) are less than \(q\).

**Proof.** From Lemma 3.3 we see that assumptions (i) and (ii) of Theorem 3.1 are fulfilled for \(w\) given in the first of proof of Lemma 3.3. Also from (iii), one has that (iii) is satisfied. Hence, the conclusion follows directly from Theorem 3.1. ☐

**Example 3.5.** Consider the problem

\[
\begin{align*}
(-1)^n u^{(2n)} + (-1)^{n-1} u^{(2n-2)} + \cdots + u^{(4)} - u'' [(2 + x) \cos u' + \sin u'] + 3u &= \lambda f(x, u) + \mu g(x, u), \\
0 &= \lambda f(x, u) + \mu g(x, u), \\
0 &= \lambda f(x, u) + \mu g(x, u), \\
0 &= \lambda f(x, u) + \mu g(x, u), \\
0 &= \lambda f(x, u) + \mu g(x, u), \\
0 &= \lambda f(x, u) + \mu g(x, u), \quad x \in (0,1),
\end{align*}
\]

\[
\begin{align*}
0 &= u(1) = u'(0) = u'(1) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0 = u^{(n)}(0) = u^{(n)}(1),
\end{align*}
\]

(3.3)
where
\[
(f : [0, 1] \times \mathbb{R} \to \mathbb{R},
(x, t) \mapsto f(x, t) = \begin{cases} 
-e^{-t} + e^{-1}, & (x, t) \in [0, 1] \times [1, +\infty), \\
0, & (x, t) \in [0, 1] \times [0, 1], \\
-e^t(t + 10), & (x, t) \in [0, 1] \times (-\infty, 0),
\end{cases}
\]
and \(g : [0, 1] \times \mathbb{R} \to \mathbb{R}\) is a fixed \(L^1\)-Carathéodory function.

Here, \(p(t) = -3t\) and \(h(x, t) = [(2 + x) \cos t + \sin t]^{-1}\) for all \(x \in [0, 1]\) and \(t \in \mathbb{R}\). Hence we have
\[
\Phi(u) = \frac{1}{2}||u||^2 + \int_0^1 [H(x, u'(x) + P(u(x))]\, dx
= \frac{1}{2}||u||^2 + \int_0^1 \int_0^{u'(x)} \left( \int_0^\tau [(2 + x) \cos t + \sin t] \, dt \right) \, d\tau \, dx + \int_0^1 \left( \int_0^{u(x)} 3\zeta \, d\zeta \right) \, dx
= \frac{1}{2}||u||^2 + \int_0^1 \int_0^{u'(x)} [(2 + x) \sin \tau - \cos \tau + 1] \, d\tau \, dx + \int_0^1 \frac{3}{2}(u(x))^2 \, dx
\geq \frac{1}{2}||u||^2 + \int_0^1 [(2 + x)(-\cos(u'(x)) + 1)] \, dx + \frac{3}{2}||u||_2^2
\geq \frac{1}{2}||u||^2 + \frac{3}{2}||u||_2^2 \geq 0.
\]

Note that
\[
F(x, t) = \int_0^t f(x, \zeta) \, d\zeta = \begin{cases} 
e^{-t} + (t - 2)e^{-1}, & (x, t) \in [0, 1] \times (1, +\infty), \\
0, & (x, t) \in [0, 1] \times [0, 1], \\
e^{-t}10, & (x, t) \in [0, 1] \times (-\infty, 0).
\end{cases}
\]

By choosing \(c = 1\) and \(d = 5\), it is clear that \(F(x, t) \geq 0\) for all \(0 < \alpha < \beta < 1\) and \((x, t) \in ([0, \alpha] \cup [\beta, 1]) \times [0, d]\), i.e. \((j)\) is satisfied. Also we have \(c < (n - 1)\frac{d}{\pi} \left[ \frac{1}{\alpha^3} + \frac{1}{(1 - \beta)^3} \right]^2\), for all \(n \in \mathbb{N} - \{1\}\). On the other hand
\[
\int_0^1 \sup_{t \in [-c, c]} F(x, t) \, dx = \int_0^1 \max \left\{ \sup_{t \in [0, c]} \left( \sup_{t \in [-c, 0]} (-e^t10) \right) \right\} \, dx
= \int_0^1 0 \, dx = 0
< \frac{(e^{-5} + 3e^{-1})(\beta - \alpha)}{25(n - 1) \left( \frac{(2n - 2)!}{(n - 2)!} \right)^2 \left[ \frac{1}{\alpha^{2n - 1}} + \frac{1}{(1 - \beta)^{2n - 1}} \right]}
= \min \left\{ \frac{\pi^2e^2 \int_0^\beta F(x, d) \, dx}{(n - 1)^2d^2 \left[ \frac{1}{\alpha^3} + \frac{1}{(1 - \beta)^3} \right]}, \frac{e^2 \int_0^\beta F(x, d) \, dx}{(n - 1)d^2 \left( \frac{(2n - 2)!}{(n - 2)!} \right)^2 \left[ \frac{1}{\alpha^{2n - 1}} + \frac{1}{(1 - \beta)^{2n - 1}} \right]} \right\}.
\]
So (jj) is satisfied. Also since \( \lim_{|t| \to \infty} \sup \frac{F(x, t)}{t^2} = 0 \), then (jjj) holds. Now we can apply Corollary 3.4 for every

\[
\theta > \frac{25(n-1) \left( \frac{(2n-2)!}{(n-2)!} \right)^2 \left[ \frac{1}{\alpha^{2n-1}} + \frac{1}{(1-\beta)^{2n-1}} \right]}{(e^{-5} + 3e^{-1})(\beta - \alpha)}.
\]

Then problem (3.3), admits at least three weak solutions.

References


