On new classes of neutrosophic continuous and contra mappings in neutrosophic topological spaces

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Abstract
The aim of this paper is to investigate some new types of neutrosophic continuous mappings like, neutrosophic \(\alpha^*\)-continuous mapping \((N\alpha^* - CM)\), neutrosophic irresolute \(\alpha^*\)-continuous mapping \((NI\alpha^* - CM)\), and neutrosophic strongly \(\alpha^*\)-continuous mapping \((NS\alpha^* - CM)\) are given and some of their properties are studied. Moreover, new kind of neutrosophic contra continuous mappings is investigated in this work, it is called neutrosophic contra \(\alpha^*\)-continuous mapping \((NC\alpha^* - CM)\).

Keywords: neutrosophic sets, neutrosophic topological space, neutrosophic \(\alpha\)-open sets, neutrosophic \(\alpha^*\)-open set.

1. Introduction
In 1998, the connotation of Contra continuity is investigated by Dontchev \([6]\). Also, the connotation of \(\alpha^*\)-open set \((\alpha^* - OS)\) is shown \([7]\). The idea of neutrosophic sets is presented by Smarandache \([35]\), in 2014, the connotations of “neutrosophic closed set “and” neutrosophic continuous function” are given.

The neutrosophic set is studied in topology, algebra and other fields. It is one of the non-classical sets, such as soft set, fuzzy sets, nano set, permutation sets and so on, see \([1, 3, 4, 6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 33, 36]\). In this research, we introduce a new types of neutrosophic mappings, they are said neutrosophic \(\alpha^*\)-continuous and neutrosophic contra \(\alpha^*\)-continuous mappings. Next, we studied and discussed their basic properties.

2. Preliminaries
Here basic definitions and notations, which are used in this section are referred from the references \([2, 5, 9, 32, 34]\).

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Definition 2.1. Assume that $\Psi \neq \emptyset$. A neutrosophic set (NS) $\theta$ is defined as

$$\theta = \langle \alpha, \partial_\omega(\alpha), \omega_\theta(\alpha), \ell_\theta(\alpha) : \alpha \in \Psi \rangle,$$

where $\partial_\omega(\alpha)$ is the degree of membership, $\omega_\theta(\alpha)$ is the degree of indeterminacy and $\ell_\theta(\alpha)$ is the degree of nonmembership, for all $\alpha \in \Psi$.

Definition 2.2. We say $(\Psi, \tau)$ is a neutrosophic topological space (NTS) if and only if $\tau$ is a collection of (NSs) in $\Psi$ and it such that:

1. $1_N, 0_N \in \tau$, where $0_N = \{\langle \alpha, (0, 1, 1) \rangle : \alpha \in \Psi\}$ and $1_N = \{\langle \alpha, (1, 0, 0) \rangle : \alpha \in \Psi\}$,
2. $A \cap \beta \in \tau$ for any $\theta, \beta \in \tau$,
3. $\bigcup_{i \in I} A_i \in \tau$ for any arbitrary family $\{A_i \mid i \in I\} \subseteq \tau$.

Moreover, any $A \in \tau$ is called neutrosophic open set (NOS) and we say neutrosophic closed set (NCS) for its complement.

Definition 2.3. Assume $A$ is a neutrosophic set in (NTS) $X$.

(i) The neutrosophic closure (resp., neutrosophic $\alpha$-closure) of $A$ is the intersection of all neutrosophic closed (resp., neutrosophic $\alpha$-closed) sets containing $A$ and is denoted by $\text{Ncl}(A)$ (resp., $\text{Ncl}_\alpha(A)$).

(ii) The neutrosophic interior (resp., neutrosophic $\alpha$-interior) of $A$ is the union of all neutrosophic open (resp., neutrosophic $\alpha$-open) sets contained in $A$ and is denoted by $\text{Nint}(A)$ (resp., $\text{Nint}_\alpha(A)$), where $A$ is neutrosophic $\alpha$-open set ($N\alpha - O\alpha$) (resp., neutrosophic semi $\alpha$-open set ($NSe\alpha - OS\alpha$), neutrosophic $\alpha^*$-open set ($N\alpha^* - OS\alpha$) if $A \subseteq \text{Nint}(\text{Ncl}(\text{Nint}(A)))$ (resp., $A \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(\text{Nint}(A))))$ or equivalently $A \subseteq \text{Ncl}(\text{Nint}(A)), A \subseteq \text{Nint}_\alpha(\text{Ncl}(\text{Nint}_\alpha(A)))$.

The symbols of the above neutrosophic sets and their complements are referred as $N\alpha - O(X)$ (resp., $NSe\alpha - O(X), N\alpha^* - O(X)$), $N\alpha - C(X)$ (resp., $SNe\alpha - C(X), N\alpha^* - C(X)$).

Proposition 2.4. (1) If $A$ is ($N\alpha^* - OS\alpha$) and $B$ is (NOS), then $A \cap B$ is ($N\alpha^* - OS\alpha$).

(2) If $\{G_\lambda\}_{\lambda \in \Gamma}$ is a collection of ($N\alpha^* - OS\alpha$s), then their union is also ($N\alpha^* - OS\alpha$s).

Theorem 2.5. Assume that $X_1$ and $X_2$ are two neutrosophic topological spaces (NTSs), $A_1 \subseteq X_1$ and $A_2 \subseteq X_1$. Then $A_1$ and $A_2$ are ($N\alpha^* - OS\alpha$s) (resp., ($N\alpha^* - CS\alpha$s)) in $X_1$ and $X_2$, respectively if and only if $A_1 \times A_2$ is ($N\alpha^* - OS\alpha$) (resp., ($N\alpha^* - CS\alpha$s)) in $X_1 \times X_2$.

Theorem 2.6. Assume that $W$ is a subspace of $Z$ satisfies $G \subseteq W \subseteq Z$. The following assertions hold.

(i) If $G \in N\alpha^* - O(Z)$, then $G \in N\alpha^* - O(W)$.

(ii) If $G \in N\alpha^* - O(W)$, then $G \in N\alpha^* - O(Z)$, where $W$ is a neutrosophic closed subspace of $Z$.

Proposition 2.7. (1) Every (NOS) (resp., $N\alpha$-open, $Ncl$-open) set is ($N\alpha^* - OS\alpha$).
(2) Every \((N\alpha^* - OS)\) is \((NS\alpha - OS)\).

**Definition 2.8.** A (NTS) \(X\) is called a

(i) neutrosophic ultra-\(T_2\) (\(N-ultra-T_2\)) if for any \(t \neq h \in Z\), there are two disjoint neutrosophic closed sets (NDCSs) \(T, H\) satisfy \(t \in T, h \in H\).

(ii) neutrosophic ultra normal, if for all neutrosophic closed sets (NCSs) \(T, F\) with \(T \neq \emptyset \neq F\) and \(T \cap F = \emptyset\), there are two (NCSs) \(D, H\) with \(D \cap H = \emptyset\) and \(T \subseteq D, F \subseteq H\).

(iii) If \(h\) is (NOS) in \(W\).

Thus by Theorem 2.6, \((h|_G)^{-1}(B) \cap G\) is \((N\alpha^* - OS)\) in \(W_1\).

**Theorem 3.3.** Assume that \(W_1\) and \(W_2\) are NTSs and \(h : W_1 \to W_2\) is any map from \(W_1\) into \(W_2\).

(1) Every \((N\alpha^* - CM)\) is \((NI\alpha^* - CM)\).

(2) Every \((NI\alpha^* - CM)\) is \((NS\alpha^* - CM)\).

**Proof.** It follows from Proposition 2.7. \(\square\)

**Theorem 3.4.** Suppose that \(h : W_1 \to W_2\) is any mapping and \(W_1 = T \cup H\), where \(T, H\) are disjoint neutrosophic sets in \(W_1\). Then,
(i) $h$ is $(\text{N} \alpha^* - \text{CM})$ if and only if $h|_T$ and $h|_H$ are also, where $T$ and $H$ are neutrosophic open sets.

(ii) $h$ is $(\text{NI} \alpha^* - \text{CM})$ if and only if $h|_T$ and $h|_H$ are also, where $T$ and $H$ are neutrosophic open sets.

(iii) $h$ is $(\text{NS} \alpha^* - \text{CM})$ if and only if $h|_T$ and $h|_H$ are also, where $T, H$ are neutrosophic $\alpha^*$-open sets.

**Proof.** (i) Suppose that $G$ is $(\text{NOS})$ in $W_2$, since $h|_T$ and $h|_H$ are $(\text{N} \alpha^* - \text{CM})$, $(h|_T)^{-1}(G)$ and $(h|_H)^{-1}(G)$ are $(\text{N} \alpha^* - \text{OS})$ in $W_1$. So, their union is also, see Proposition 2.4. However, $h^{-1}(G) = (h|_T)^{-1}(G) \cup (h|_H)^{-1}(G)$ and hence $h^{-1}(G)$ is $(\text{N} \alpha^* - \text{OS})$ in $W_1$. Thus $h$ is $(\text{N} \alpha^* - \text{CM})$. Sufficiency, follows by using Theorem 3.3. The proofs of (i) and (iii) are the same way of proof (i).

□

**Theorem 3.5.** Suppose $h : W_1 \to W_2$ is any mapping and $h_T : h^{-1}(T) \to T$ is defined as $h_T(t) = h(t)$, for any neutrosophic set $T$ in $W_2$ and $t \in h^{-1}(T)$.

(i) If $h$ is $(\text{N} \alpha^* - \text{CM})$, then $h_T$ is also, where $T$ is $(\text{NOS})$ in $W_2$.

(ii) If $h$ is $(\text{NI} \alpha^* - \text{CM})$ (resp., $(\text{NS} \alpha^* - \text{CM})$), then $h_T$ is also, where $T$ is neutrosophic closed set (NCS) in $W_2$.

**Proof.** We shall prove the second case. The first case is similar to (ii). Suppose that $B$ is $(\text{N} \alpha^* - \text{OS})$ in $T$. Since $T$ is (NCS) in $W_2$, $B$ is $(\text{N} \alpha^* - \text{OS})$ in $W_2$, see Theorem 2.6(ii). Also, since $h$ is $(\text{NI} \alpha^* - \text{CM})$ (resp., $(\text{NS} \alpha^* - \text{CM})$), $h^{-1}(B)$ is $(\text{N} \alpha^* - \text{OS})$ (resp., (NOS)) in $W_1$. Therefore, $h^{-1}(B)$ is $(\text{N} \alpha^* - \text{OS})$ (resp., (NOS)) in $h^{-1}(T)$, see Theorem 2.6(i). □

**Theorem 3.6.** Suppose $X_1, X_2, X_3$ are three (NTSs) $L : X_1 \to X_2$ and $X_2 \subseteq X_3$. If $L : X_1 \to X_2$ is $(\text{N} \alpha^* - \text{CM})$ (resp., $(\text{NI} \alpha^* - \text{CM})$, $(\text{NS} \alpha^* - \text{CM})$), then $L : X_1 \to X_3$ is also.

**Proof.** Assume that $A$ is(NOS) (resp., $(\text{N} \alpha^* - \text{OS})$) in $X_3$, then $A$ is (NOS) (resp., $(\text{N} \alpha^* - \text{OS})$ in $X_2$, see Theorem 2.6(i) and hence $L^{-1}(A)$ is a neutrosophic $\alpha^*$-open set $(\text{N} \alpha^* - \text{OS}, \text{neutrosophic open})$ in $X_1$. Now, we recall that the set $\{(x, L(x)), x \in X\} \subseteq X \times Y$ is called the neutrosophic graph of the mapping $L : X \to Y$ and is denoted by $\text{NG}(L)$. □

**Theorem 3.7.** Suppose that $W_1$ and $W_2$ are two (NTSs), $h : W_1 \to W_2$ is any mapping and $L : W_1 \to W_1 \times W_2$ is a neutrosophic graph mapping of $h$ defined by $L(t) = (t, h(t))$, for all $t \in W_1$. If $L$ is $(\text{N} \alpha^* - \text{CM})$ (resp., $(\text{NI} \alpha^* - \text{CM})$, $(\text{NS} \alpha^* - \text{CM})$), then $h$ is also.

**Proof.** Assume that $K$ is (NOS) (resp., $(\text{N} \alpha^* - \text{OS})$) in $W_2$. Since $W_1$ is (NOS) (resp., $(\text{N} \alpha^* - \text{OS})$) in any NTS, $W_1 \times K$ is (NOS) (resp., $(\text{N} \alpha^* - \text{OS})$) in $W_1 \times W_2$, see Theorem 2.5. Therefore, $L^{-1}(W_1 \times K) = h^{-1}(K)$ is a neutrosophic $\alpha^*$-open (resp., $(\text{N} \alpha^* - \text{OS})$, (NOS)) in $W_1$. Hence, the proof is complete. □

4. Neutrosophic contra $\alpha^*$-continuity:

In this section, we define a new type of neutrosophic $\alpha^*$-continuity that we call it a neutrosophic contra $\alpha^*$-continuous mapping $(\text{NC} \alpha^*\text{-CM})$ and several propositions related to this new notion are investigated.
Definition 4.1. Assume that \( W_1 \) and \( W_2 \) are two (NTSs) and \( h : W_1 \to W_2 \) is a mapping, then \( h \) is called a neutrosophic contra \( \alpha^* \)-continuous mapping (NC\( \alpha^* \)-CM). If \( h^{-1}(K) \) is (\( \alpha^* \)-CS) in \( W_1 \), for any (NOS) \( K \) in \( W_2 \).

Theorem 4.2. Let \( h : W_1 \to W_2 \) be a mapping. The following statements are equivalent:

(i) \( h \) is (NC\( \alpha^* \)-CM),

(ii) for each \( t \in W_1 \) and each (NCS) \( K \) in \( W_2 \) containing \( h(t) \), there exists (\( \alpha^* \)-OS) \( B \) in \( W_1 \), such that \( B, h(B) \subseteq K \),

(iii) for every (NCS) \( K \) of \( W_2 \), \( h^{-1}(K) \) is (\( \alpha^* \)-OS) of \( W_1 \).

Proof. (i) \( \to \) (ii) Assume that \( \in W_1 \) and \( K \) is any (NCS) in \( W_2 \), then \( K^c \) is (NOS) in \( W_2 \). Thus \( h^{-1}(K^c) \) is (\( \alpha^* \)-CS) in \( W_1 \), but \( h^{-1}(K^c) = [h^{-1}(K)]^c \). Hence \( h^{-1}(K) \) is (\( \alpha^* \)-OS) in \( W_1 \), and \( t \in h^{-1}(K) \). Put \( B = h^{-1}(K) \), thus \( h(B) \subseteq K \).

(ii) \( \to \) (iii) Assume that \( K \) is a neutrosophic closed set in \( W_2 \) and \( t \in h^{-1}(K) \), then \( h(t) \in K \) and hence there exists (\( \alpha^* \)-OS) \( B \) containing \( t, h(B) \subseteq K \), thus \( t \in B = h^{-1}(K) \). So \( h^{-1}(K) = \bigcup \{B_t \mid t \in h^{-1}(K)\} \). Hence by Proposition 2.4, (1), we get \( h^{-1}(K) \) is (\( \alpha^* \)-OS) in \( W_1 \).

(iii) \( \to \) (i) Obviously holds. \( \square \)

Theorem 4.3. The restriction \( L_A \) of (NC\( \alpha^* \)-CM) \( L : X \to Y \) to (\( \alpha^* \)-CS) \( A \subseteq X \) is also (NC\( \alpha^* \)-CM).

Proof. Assume that \( B \) is (NOS) in \( Y \), thus \( L^{-1}(B) \) is (\( \alpha^* \)-CS) in \( X \). Since \( A \) is (\( \alpha^* \)-CS) in \( X \), \( L^{-1}(B) \cap A \) is also (\( \alpha^* \)-CS) in \( X \) and hence it is also (\( \alpha^* \)-CS) in \( A \), see Theorem 2.6(i), but \( (L|_A)^{-1}(B) = L^{-1}(B) \cap A \), hence the proof is complete. \( \square \)

Theorem 4.4. If \( L : X \to Y \) is (NC\( \alpha^* \)-CM), then \( L_A : L^{-1}(A) \to A \) is also, where \( A \) is (NCS) in \( Y \).

Proof. Assume that \( B \) is (NCS) in \( A \). Since \( A \) is (NCS) in \( Y \), \( B \) is (NCS) in \( Y \). Then \( L^{-1}(B) \) is (\( \alpha^* \)-OS) in \( X \). Since \( L^{-1}(B) \subseteq L^{-1}(A) \subseteq X \), \( L^{-1}(B) \) is (\( \alpha^* \)-OS) in \( L^{-1}(A) \), see Theorem 2.6(i). \( \square \)

Theorem 4.5. Assume that \( X \) and \( Y \) are two (NTSs), \( L : X \to Y \) is a mapping and \( X = A \cup B \), where \( A, B \) are disjoint (\( \alpha^* \)-CSs) in \( X \). Then \( L|_A \) and \( L|_B \) are (NC\( \alpha^* \)-CMs) if and only if \( L \) is (NC\( \alpha^* \)-CM).

Proof. Necessity follows by using Theorem 4.3. Assume that \( G \) is (NCS) in \( Y \). Since \( L|_A \) and \( L|_B \) are (NC\( \alpha^* \)-CMs), \( (L|_A)^{-1}(G) \) and \( (L|_B)^{-1}(G) \) are (\( \alpha^* \)-OS) in \( X \). So, their union is also, see Proposition 2.4. But \( L^{-1}(G) = (L|_A)^{-1}(G) \cup (L|_B)^{-1}(G) \) and hence the proof is complete. \( \square \)

Definition 4.6. An (NTS) \( W \) is called:

(i) an \( N - \alpha^* \)-T2 space (resp., \( N \)-ultra-\( \alpha^* \)-T2) space if, for each \( t \neq d \in W \), there exist two disjoint (\( \alpha^* \)-OSs) (resp., (\( \alpha^* \)-CSs)) \( T, D \) satisfy \( t \in T, d \in D \).

(ii) an \( N - \alpha^* \)-ultra normal space if for each pair nonempty (NDCSs) can be separated by disjoint \( N \)-clopen).
• (iii) a neutrosophic $\alpha^*$-compact space ($Na^C$-space) if for each $Na^*$-open cover of $W$ has a finite subcover.

**Theorem 4.7.** Suppose that $h : W_1 \to W_2$ is injective ($NC\alpha^* - CM$) and $W_2$ is $N - T_2-$ space. Then $W_1$ is $N$-$\alpha^*$-$\cdot T_2$ space.

**Proof.** Assume that $t \neq d \in W_1$. Since $h$ is injective, $h(t) \neq h(d)$ in $W_2$ and since $W_2$ is $N - T_2-$ space, there exist two (NDOSs) $T,D$ satisfy $h(t) \in T, h(d) \in D$. Since $h$ is ($NC\alpha^* - CM$), $h^{-1}(T),h^{-1}(D)$ are ($Na^* - CS$) in $W_1$ containing $t,d$ and $h^{-1}(T) \cap h^{-1}(D) = \emptyset = h^{-1}(T \cap D)$. Hence $W_1$ is $N$-$\alpha^*$-$\cdot T_2$ space. \hfill $\square$

**Theorem 4.8.** Suppose that $L : X \to Y$ is injective ($NC\alpha^* - CM$) and $Y$ is an $N$-$\alpha^*$ $T_2$-space. Then $X$ is an $N - \alpha^*$ $T_2$ space.

**Proof.** Take $x \neq y$ in $X$. Since $L$ is injective, $f(x) \neq f(y)$ in $Y$. Since $Y$ is an $N$-$\alpha^*$ $T_2$- space, there exist two (NDCSs) $A,B$ satisfy $L(x) \in A, L(y) \in B$. Moreover, from $L$ is ($NC\alpha^* - CM$), we have $L^{-1}(A), L^{-1}(B)$ are ($Na^* - OSs$) in $X$ containing $x,y$ and $L^{-1}(A) \cap L^{-1}(B) = \emptyset$. Then $X$ is an $N - \alpha^*$-$T_2$ space. \hfill $\square$

**Theorem 4.9.** Suppose that $h : W_1 \to W_2$ is a neutrosophic closed injective ($NC\alpha^* - CM$) and $W_2$ is a neutrosophic ultra normal space. Then $W_1$ is $N - \alpha^*$ - is an ultra normal space.

**Proof.** Assume that $A_1, A_2$ are two (NCSs) in $W_1$ with $A_1 \cap A_2 = \emptyset$. Since $h$ is a neutrosophic closed mapping, $h(A_1), h(A_2)$ are (NCSs) in $W_2$. Since, $W_2$ is a neutrosophic ultra normal space, there exist two disjoint neutrosophic clopen sets $B_1, B_2$ in $W_2$ satisfy $h(A_1) \subseteq B_1, h(A_2) \subseteq B_2$. Hence $A_1 \subseteq h^{-1}(B_1), A_2 \subseteq h^{-1}(B_2)$. From injectivity of $h$, we get $h^{-1}(B_1), h^{-1}(B_2)$ are disjoint neutrosophic $\alpha^*$-clopen sets. Thus $W_1$ is a neutrosophic $\alpha^*$-ultra normal space. \hfill $\square$

**Theorem 4.10.** Suppose that $h : W_1 \to W_2$ is a neutrosophic closed surjective ($NC\alpha^* - CM$) and $W_1$ is ($Na^*C-$ space). Then $W_2$ is a neutrosophic strongly closed space.

**Proof.** Assume that $\{V_i : i \in I\}$ is any neutrosophic closed cover of $W_2$. Since $h$ is ($NC\alpha^* - CM$), $h^{-1}(V_i) : i \in I\}$ is a neutrosophic $\alpha^*$-open cover of $W_1$, but $W_1$ is ($Na^*C-$ space), thus $W_1$ has finite subcover. This means that $W_1 = \bigcup_{i=1}^{n} h^{-1}(V_i)$, where $I_0 = \{1,\ldots,n\}$. Since $h$ is neutrosophic surjective, we have

$$h(W_1) = h \left( \bigcup_{j=1}^{n} h^{-1}(V_i) \right) = \bigcup_{j=1}^{n} hh^{-1}(V_i).$$

Hence, $W_2 = \bigcup_{i \in I_0} V_i$. Thus $W_2$ is a neutrosophic strongly closed space. \hfill $\square$

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