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Coupled fixed point theorems in partially ordered complex valued metric spaces with application

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Abstract

In this paper, we prove some coupled fixed point theorems for nonlinear contraction type mappings in complete complex valued metric spaces endowed with partial order. We support our results by establishing an illustrative example. Also we give an application of this results to the solution of the Urysohn type integral equations.

Keywords: complex valued metric spaces, coupled fixed point, partially ordered set, Urysohn integral equations.
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1. Introduction and Preliminaries

There are a lot of generalizations of the Banach contraction mapping principle in the literature. One of the most interesting of them is the result of Bhaskar and Lakshmikantham [1]. Indeed they introduced the notion of coupled fixed point. Afterwards a wide discussion on coupled fixed point theorems and it's various generalizations and extensions aimed the interest of many scientists because of their important role in the study of nonlinear analysis (see, for instance, [2]-[6]). We begin by recalling some needed definitions and results. let $(X, \leq, d)$ be a partially ordered complete metric space. Further, the product space $X^2 = X \times X$ has the following partial order:

$$(u, v) \leq (x, y) \Leftrightarrow x \geq u, \quad y \leq v, \quad \text{for all} \quad (x, y), (u, v) \in X \times X.$$

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Definition 1.1. *[1] A mapping $F : X \times X \to X$ is said to have the mixed monotone property if $F(x, y)$ is monotone non–decreasing in x and is monotone non–increasing in y, that is for any $x, y \in X$,*

$$x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y) \quad for \quad x_1, x_2 \in X,$$
$$y_1 \leq y_2 \Rightarrow F(x, y_2) \leq F(x, y_1) \quad for \quad y_1, y_2 \in X.$$

Definition 1.2. *[1] An element $(x, y) \in X \times X$ is said to be coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Theorem 1.3. *[1] Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}(d(x, u) + d(y, v)) \quad for\ each \quad x \leq u, \quad y \geq v. \tag{1.1}$$

Also suppose either
(a) F is continuous, or
(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq x$, for all n,
(ii) if a non-increasing sequence $\{y_n\} \to y$ then $y_n \geq y$, for all n.

If there exist $x_0, y_0 \in X$, such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

The purpose of this article is to present a version of Theorem 1.3 in the context of complex valued metric spaces. Recently, Azam et al [7] introduced then notion of complex valued metric spaces and also presented common fixed point theorems for mappings satisfying generalized contraction condition. Afterwards some of mathematicians investigated fixed points in complex valued metric spaces (see [8]-[12]). We summarize in the following the basic notions and results established in [7].

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \Leftrightarrow Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

$$\begin{aligned} &(c1) \quad Re(z_1) = Re(z_2), Im(z_1) < Im(z_2),\\ &(c2) \quad Re(z_1) < Re(z_2), Im(z_1) = Im(z_2),\\ &(c3) \quad Re(z_1) < Re(z_2), Im(z_1) < Im(z_2),\\ &(c4) \quad Re(z_1) = Re(z_2), Im(z_1) = Im(z_2). \end{aligned}$$

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (c1), (c2) and (c3) is satisfied and we will write $z_1 \prec z_2$ if only (c3) is satisfied. Note that
(1) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.
(2) $z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.
(3) $a, b \in \mathbb{R}, a \leq b \Rightarrow az \precsim bz$, for all $z \in \mathbb{C}$.

Definition 1.4. *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:*
(i) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1.5. *[8] Let $X = \mathbb{C}$, define the mapping $d : X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{ik}|z_1 - z_2|$ where $k \in \mathbb{R}$. Then (X, d) is a complex valued metric space.*

Definition 1.6. *Let (X, d) be a complex valued metric space.*
(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A.$$

(ii) A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$, $B(x, r) \cap (A - \{x\}) \neq \phi$.
(iii) A is called open whenever each element of A is an interior point of A.
(iv) A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.
(v) The family $F = \{B(x, r) : x \in X, 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on X.

Definition 1.7. *Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X and $x \in X$.*
(i) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$, as $n \to \infty$.
(ii) If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d).
(iii) If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space.

Lemma 1.8. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.9. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$ where $m \in \mathbb{N}$.*

Theorem 1.10. *Let (X, d) be a complete complex valued metric space and $S, T : X \to X$. If S and T satisfy*

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx) d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ, μ are nonnegative real numbers with $\lambda + \mu < 1$. Then S and T have a common fixed point.

2. Main results

Let $(X, \leq)$ be a partially ordered set and d be a complex valued metric on X such that (X, d) is a complete complex valued metric space. Also the product space $X \times X$ endowed with the following partial order:

$$(u, v) \leq (x, y) \Leftrightarrow x \geq u, \quad y \leq v$$

for $(u, v), (x, y) \in X \times X$.

Theorem 2.1. *Let $T : X \times X \to X$ has mixed monotone property on X satisfying :*

$$d(T(x,y), T(u,v)) \precsim \frac{\lambda}{2}(d(x,u) + d(y,v)) + \frac{\mu}{2}\Big(\frac{d(x, T(x,y))d(u, T(u,v))}{1 + d(x,u)} + \frac{d(y, T(y,x)d(v, T(v,u))}{1 + d(y,v)}\Big) \tag{2.1}$$

for all $(u, v) \leq (x, y)$ where λ, μ are nonnegative real numbers with $\lambda + \mu < 1$. Also suppose either
(a) T is continuous, or
(b) X has the following properties:

$$(i) \text{ if a non-decreasing sequence}\{x_n\} \to x \text{ then } x_n \leq x, \text{ for all } n,$$
$$(ii) \text{ if a non-decreasing sequence}\{y_n\} \to y \text{ then } y_n \geq y, \text{ for all } n.$$

If there exist $x_0, y_0 \in X$ such that $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$, then T has a coupled fixed point (x, y) in $X \times X$.

Proof . Let $x_0, y_0 \in X$ be such that $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$. We denote $x_1 = T(x_0, y_0)$, $y_1 = T(y_0, x_0)$ and

$$T^2(x_0, y_0) = T(T(x_0, y_0), T(y_0, x_0)) = T(x_1, y_1) = x_2$$
$$T^2(y_0, x_0) = T(T(y_0, x_0), T(x_0, y_0)) = T(y_1, x_1) = y_2.$$

Due to the mixed monotone property of T,

$$x_2 = T^2(x_0, y_0) = T(x_1, y_1) \geq T(x_0, y_0) = x_1$$
$$y_2 = T^2(y_0, x_0) = T(y_1, x_1) \leq T(y_0, x_0) = y_1.$$

Further for $n \in \{1, 2, \cdots\}$, we let

$$T^{n+1}(x_0, y_0) = T(T^n(x_0, y_0), T^n(y_0, x_0)) = T(x_n, y_n) = x_{n+1}$$
$$T^{n+1}(y_0, x_0) = T(T^n(y_0, x_0), T^n(x_0, y_0)) = T(y_n, x_n) = y_{n+1}.$$

It is easy to check that

$$x_0 \leq T(x_0, y_0) = x_1 \leq T^2(x_0, y_0) = x_2 \leq \cdots \leq T^{n+1}(x_0, y_0) \leq \cdots$$
$$\textit{and } y_0 \geq T(y_0, x_0) = y_1 \leq T^2(y_0, x_0) = y_2 \geq \cdots \geq T^{n+1}(y_0, x_0) \geq \cdots$$

Letting $h = \frac{\lambda}{(1-\mu)}$, we claim that

$$d(T^{n+1}(x_0, y_0), T^n(x_0, y_0)) \precsim \frac{h^n}{2}(d(x_0, T(x_0, y_0)) + d(y_0, T(y_0, x_0)), \tag{2.2}$$

$$d(T^{n+1}(y_0, x_0), T^n(y_0, x_0)) \precsim \frac{h^n}{2}(d(y_0, T(y_0, x_0)) + d(x_0, T(x_0, y_0)). \tag{2.3}$$

We will prove (2.2) by induction. For $n = 1$, using $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$, we get from (2.1);

$$\begin{aligned} d(T^2(x_0,y_0),T(x_0,y_0)) &= d(T(T(x_0,y_0),T(y_0,x_0)),T(x_0,y_0)) \\ &\precsim \frac{\lambda}{2}(d(T(x_0,y_0),x_0)+d(T(y_0,x_0),y_0)) \\ &+ \frac{\mu}{2}(\frac{d(T(x_0,y_0),T(T(x_0,y_0),T(y_0,x_0))d(x_0,T(x_0,y_0)}{1+d(T(x_0,y_0),x_0)}) \\ &+ \frac{\mu}{2}(\frac{d(T(y_0,x_0),T(T(y_0,x_0),T(x_0,y_0))d(y_0,T(y_0,x_0)}{1+d(T(y_0,x_0),y_0)}). \end{aligned}$$

On the other hand, we have $d(x_0, T(x_0, y_0)) \precsim 1 + d(x_0, T(x_0, y_0))$ and $d(y_0, T(y_0, x_0)) \precsim 1 + d(y_0, T(y_0, x_0))$, then we get

$$\begin{aligned} d(T^2(x_0,y_0),T(x_0,y_0)) &\precsim \frac{\lambda}{2(1-\frac{\mu}{2})}\big(d(T(x_0,y_0),x_0)+d(T(y_0,x_0),y_0)\big) \\ &+ \frac{\mu}{2(1-\frac{\mu}{2})}\big(d(T(y_0,x_0),T^2(y_0,x_0)\big). \end{aligned} \tag{2.4}$$

Also

$$\begin{aligned} d(T^2(y_0,x_0),(T(y_0,x_0)) &\precsim \frac{\lambda}{2(1-\frac{\mu}{2})}\big(d(T(x_0,y_0),x_0)\big)+d\big(T(y_0,x_0),y_0)\big) \\ &+ \frac{\mu}{2(1-\frac{\mu}{2})}\big(d(T(x_0,y_0),T^2(x_0,y_0)\big). \end{aligned} \tag{2.5}$$

Using (2.4) and (2.5) to obtain

$$d(T^2(x_0,y_0),T(x_0,y_0)) \precsim \frac{\lambda}{2(1-\mu)}(d(x_0,T(x_0,y_0))+d(y_0,T(y_0,x_0)),$$

$$d(T^2(y_0,x_0),T(y_0,x_0)) \precsim \frac{\lambda}{2(1-\mu)}(d(y_0,T(y_0,x_0))+d(x_0,T(x_0,y_0)).$$

Thus (2.2) and (2.3) are true for $n = 1$. Suppose that (2.2) and (2.3) are true for some n. Using $T^{n+1}(x_0, y_0) \geq T^n(x_0, y_0)$ and $T^{n+1}(y_0, x_0) \leq T^n(y_0, x_0)$, to get

$$\begin{aligned} &d(T^{n+2}(x_0,y_0),T^{n+1}(x_0,y_0)) = d(T(T^{n+1}(x_0,y_0),T^{n+1}(y_0,x_0)),T(T^n(x_0,y_0),T^n(y_0,x_0)) \\ &\precsim \frac{\lambda}{2}\big(d(T^{n+1}(x_0,y_0),T^n(x_0,y_0))+d(T^{n+1}(y_0,x_0),T^n(y_0,x_0)\big) \\ &\quad + \frac{\mu}{2}\left(\frac{d(T^{n+1}(x_0,y_0),T(T^{n+1}(x_0,y_0),T^{n+1}(y_0,x_0))d(T^n(x_0,y_0),T(T^n(x_0,y_0),T^n(y_0,x_0))}{1+d(T^{n+1}(x_0,y_0),T^n(x_0,y_0))}\right) \\ &\quad + \frac{\mu}{2}\left(\frac{0d(T^{n+1}(y_0,x_0),T(T^{n+1}(y_0,x_0),T^{n+1}(x_0,y_0))d(T^n(y_0,x_0),T(T^n(y_0,x_0),T^n(x_0,y_0))}{1+d(T^{n+1}(y_0,x_0),T^n(y_0,x_0))}\right). \end{aligned}$$

Similar relation are satisfied for $d(T^{n+2}(y_0, x_0), T^{n+1}(y_0, x_0))$. Using (2.2) and (2.3), we obtain

$$\begin{aligned} d(T^{n+2}(x_0,y_0),T^{n+1}(x_0,y_0)) &\precsim \frac{\lambda}{2(1-\mu)}(\frac{h^n}{2}(d(x_0,T(x_0,y_0))+d(y_0,T(y_0,x_0)) \\ &+ \frac{h^n}{2}(d(y_0,T(y_0,x_0)+d(x_0,T(x_0,y_0)))\big) \\ &= \frac{h^{n+1}}{2}(d(x_0,T(x_0,y_0))+d(y_0,T(y_0,x_0)). \end{aligned}$$

Similarly, one can show that

$$d(T^{n+2}(y_0,x_0),T^{n+1}(y_0,x_0)) \precsim \frac{h^{n+1}}{2}(d(y_0,T(y_0,x_0))+d(x_0,T(x_0,y_0)).$$

So for any $m>n$,

$$\begin{aligned} d(T^m(x_0,y_0),T^n(x_0,y_0)) &\precsim d(T^m(x_0,y_0),T^{m-1}(x_0,y_0))+\cdots+d(T^{n+1}(x_0,y_0),T^n(x_0,y_0)) \\ &\precsim \frac{h^{m-1}+\cdots+h^n}{2}\big(d(x_0,T(x_0,y_0))+d(y_0,T(y_0,x_0)\big) \\ &\precsim \frac{h^n}{2(1-h)}\big(d(x_0,T(x_0,y_0))+d(y_0,T(y_0,x_0)\big). \end{aligned}$$

This implies that $\{T^n(x_0,y_0)\}$ is Cauchy sequence. Similarly, we can verify that $\{T^n(y_0,x_0)\}$ is also Cauchy sequence. Since X is complete metric space, there exist $x,y\in X$ such that

$$\lim_{n\to\infty}T^n(x_0,y_0)=x \quad \text{and} \quad \lim_{n\to\infty}T^n(y_0,x_0)=y. \tag{2.6}$$

Now we claim that $T(x,y)=x$ and $T(y,x)=y$. We consider two cases:
Case1: The assumption (a) holds, thus given $\epsilon\succ 0$ there exists $\delta\succ 0$ such that

$$d(x,u)+d(y,v)\prec\delta \implies d(T(x,y),T(u,v))\prec\epsilon.$$

For $\eta=\min\{\epsilon,\frac{\delta}{2}\}$, from (2.6) there exists $n_0\in\mathbb{N}$ such that, for $n\geq n_0$

$$d(T^n(x_0,y_0),x)\prec\eta, \quad d(T^n(y_0,x_0),y)\prec\eta.$$

Therefore

$$\begin{aligned} d(T(x,y),x) &\precsim d(T(x,y),T^{n+1}(x_0,y_0))+d(T^{n+1}(x_0,y_0),x) \\ &= d(T(x,y),T(T^n(x_0,y_0),T^n(y_0,x_0)))+d(T^{n+1}(x_0,y_0),x) \\ &\prec \epsilon+\eta\precsim 2\epsilon. \end{aligned}$$

Thus $T(x,y)=x$.
Case2: The assumption (b) holds. Let $0\prec\epsilon\prec 1$ be an arbitrary number and $A:=\frac{1-\mu}{1-\lambda}$, from (2.6) there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, we have

$$d(T^n(x_0,y_0),x)\prec\epsilon A, d(T^n(y_0,x_0),y)\prec\epsilon A.$$

Using $T^n(x_0,y_0)\leq x, T^n(y_0,x_0)\geq y$ to get

$$\begin{aligned} d(T(x,y),x) &\precsim d(T(x,y),T^{n+1}(x_0,y_0))+d(T^{n+1}(x_0,y_0),x) \\ &= d(T(x,y),T(T^n(x_0,y_0),T^n(y_0,x_0))+d(T^{n+1}(x_0,y_0),x) \\ &\precsim d(T^{n+1}(x_0,y_0),x)+\frac{\lambda}{2}\big(d(x,T^n(x_0,y_0))+d(y,T^n(y_0,x_0)\big) \\ &+\frac{\mu}{2}\left(\frac{d(x,T(x,y))d(T^n(x_0,y_0),T^{n+1}(x_0,y_0))}{1+d(x,T^n(x_0,y_0))}+\frac{d(y,T(y,x))d(T^n(y_0,x_0),T^{n+1}(y_0,x_0))}{1+d(y,T^n(y_0,x_0))}\right). \end{aligned}$$

Therefore

$$d(T(x,y),x)\prec\frac{\epsilon A(1+\lambda)}{1-\frac{\mu}{2}}+\frac{\mu}{2(1-\frac{\mu}{2})}d(y,T(y,x)).$$

Similarly

$$d(T(y,x),y) \prec \frac{\epsilon A(1+\lambda)}{1-\frac{\mu}{2}} + \frac{\mu}{2(1-\frac{\mu}{2})} d(x,T(y,x)).$$

Thus we get

$$d(T(x,y),x) \prec \frac{\epsilon A(1+\lambda)}{1-\frac{\mu}{2}} + \frac{\mu}{2(1-\frac{\mu}{2})}\big(\frac{\epsilon A(1+\lambda)}{1-\frac{\mu}{2}} + \frac{\mu}{2(1-\frac{\mu}{2})} d(x,T(x,y))\big)$$

$$\Rightarrow d(T(x,y),x) \prec \frac{\epsilon A(1-\lambda)}{1-\mu} = \epsilon.$$

This implies that $T(x,y)=x$. Similarly one can show that $T(y,x)=y$, i.e, (x,y) is a coupled fixed point of T. □

Corollary 2.2. *Let $(X,\leq)$ be a partially ordered set and d be a complex valued metric on X such that (X,d) is a complete complex valued metric space. Suppose for some $n\in\mathbb{N}$, $T^n : X\times X\to X$ has mixed monotone property on X satisfying:*

$$\begin{aligned} d(T^n(x,y),T^n(u,v)) &\precsim \frac{\lambda}{2}(d(x,u)+d(y,v)) \\ &+ \frac{\mu}{2}\left(\frac{d(x,T^n(x,y)d(u,T^n(u,v))}{1+d(x,u)} + \frac{d(y,T^n(y,x)d(v,T^n(v,u))}{1+d(y,v)}\right), \end{aligned}$$

for all $(u,v)\leq(x,y)$, where λ,μ are nonnegative real numbers with $\lambda+\mu<1$. Also suppose either
(a) T^n is continuous, or
(b) X has the following properties:

$$(i) \text{ if a non-decreasing sequence}\{x_n\}\to x \text{ then } x_n\leq x, \text{ for all } n,$$
$$(ii) \text{ if a non-increasing sequence}\{y_n\}\to y \text{ then } y_n\geq y,$$

$textforall\ n$.

If there exists $x_0,y_0\in X$ such that $x_0\leq T^n(x_0,y_0)$ and $y_0\geq T^n(y_0,x_0)$ then T has a coupled fixed point (x,y) in $X\times X$.

Proof . By Theorem 2.1, there exists (u,v) in $X\times X$,

$$T^n(u,v)=u, \quad T^n(v,u)=v.$$

It is easy to show that

$$T(T^n(u,v),T^n(v,u)) = T^{n+1}(u,v) = T^n(T(u,v),T(v,u)).$$

Thus

$$\begin{aligned} d(T(u,v),u) &= d(T(T^n(u,v),T^n(v,u)),T^n(u,v)) = d(T^n(T(u,v),T(v,u)),T^n(u,v)) \\ &\precsim \frac{\lambda}{2}(d(T(u,v),u)+d(T(v,u),v)) + \frac{\mu}{2}\left(\frac{d(T(u,v),T^n(T(u,v),T(v,u))d(u,T^n(u,v))}{1+d(T(u,v),u)}\right. \\ &\left.+\frac{d(T(v,u),T^n(T(v,u),T(u,v))d(v,T^n(v,u))}{1+d(T(v,u),v)}\right). \end{aligned}$$

Since $d(T^n(v,u),v) = 0 = d(T^n(u,v),u)$, we get

$$d(T(u,v),u) \precsim \frac{\lambda}{2}(d(T(u,v),u) + d(T(v,u),v)).$$

Similarly we can obtain

$$d(T(v,u),v) \precsim \frac{\lambda}{2}(d(T(v,u),v) + d(T(u,v),u)).$$

Therefore

$$d(T(u,v),u) \precsim \frac{\lambda^2}{(2-\lambda)^2} d(T(u,v),u).$$

Thus $d(T(u,v),u) = 0$, also we get $d(T(v,u),v) = 0$. □

Example 2.3. *Let* $X = \{x : x \in \mathbb{C}, 0 \precsim x \precsim 1\}$*, define* $T : X \times X \to X$ *by* $T(x,y) = \dfrac{x+iy}{2}$ *consider* $d_c : X \times X \to \mathbb{C}$*, as follows:*

$$d_c(x,y) = |x-y| + i|x-y|.$$

Then (X, d_c) *is a complete complex valued metric space, also*

$$d_c(T(x,y), T(u,v)) \precsim \frac{1}{2}\big(d_c(x,u) + d_c(y,v)\big)$$

for all $(x,y) \leq (u,v)$*. Thus, inequality (2.1) satisfies with constant* $\lambda = \frac{1}{2}$ *and* $\mu = 0$*. Other conditions in Theorem 2.1 are satisfied. It follows that* T *has a coupled fixed point* $(0,0)$ *in* $X \times X$*.*

3. Urysohn integral equations

In this section, we apply Theorem 2.1 to show the existence of solutions of the following Uryshon integral equations:

$$\begin{aligned} x(t) &= \textstyle\int_a^b K(t,s,(x-y)(s))ds + g(t), \\ y(t) &= \textstyle\int_a^b K(t,s,(y-x)(s))ds + g(t). \end{aligned} \tag{3.1}$$

where $t \in [a,b] \subseteq \mathbb{R}$, $x, y, g \in C([a,b], \mathbb{R}^n)$, $x \neq y$ and $K : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$. Define the following order

$$x \leq y \Leftrightarrow |x(t)| \leq |y(t)|, \qquad t \in [a,b)$$

for all $x,\ y \in C([a,b], \mathbb{R}^n)$.

Theorem 3.1. *Let* $0 < a < b$*,* $X = C([a,b], \mathbb{R}^n)$ *and* $d : X \times X \to \mathbb{C}$ *is defined by*

$$d(x,y) = \|x-y\|_\infty \sqrt{1+a^2}\, e^{i \tan^{-1} a},$$

where $\|x-y\|_\infty = \sup\{|x(t)-y(t)|;\ t \in [a,b]\}$*. Let*

$$F(u,v)(t) = F(u(t), v(t)) = \int_a^b K(t,s,(u-v)(s))ds.$$

Suppose $K : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping such that $F(x(t), y(t)),\ F(y(t), x(t)) \in X$ for all $t \in [a,b]$ and $x,\ y \in X, \quad x \neq y$. If there exist non-negative real numbers λ, μ with $\lambda + \mu < 1$ such that

$$\|F(x,y) - F(u,v)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a} \precsim \frac{\lambda}{2} A((x,y),(u,v)) + \frac{\mu}{2} B((x,y),(u,v)),$$

for all $x,\ y,\ u, v\ \in X$ with $(x,y) \leq (u,v)$, where

$$A((x,y),(u,v)) = \big(\|x-u\|_\infty + \|y-v\|_\infty\big)\sqrt{1+a^2} e^{i \tan^{-1} a},$$

$$\begin{aligned} B((x,y),(u,v)) = & \frac{\|x - F(x,y) - g\|_\infty \|u - F(u,v) - g\|_\infty (1+a^2) e^{2i \tan^{-1} a}}{1 + \big(\|x-u\|_\infty\big)\sqrt{1+a^2} e^{i \tan^{-1} a}} \\ & + \frac{\|y - F(y,x) - g\|_\infty \|v - F(v,u) - g\|_\infty (1+a^2) e^{2i \tan^{-1} a}}{1 + \big(\|y-v\|_\infty\big)\sqrt{1+a^2} e^{i \tan^{-1} a}}. \end{aligned}$$

If there exist $x_0,\ y_0 \in X$ such that $x_0 - g \leq F(x_0, y_0)$ and $y_0 - g \geq F(y_0, x_0)$, then the integral equations (2.1) have a solution.

Proof . First define $T : X \times X \to X$ by

$$T(x,y) = F(x,y) + g.$$

Then $x_0 \leq T(x_0, y_0)$, $y_0 \geq T(y_0, x_0)$ and T is continuous. Also,

$$d(T(x,y), T(u,v)) = \|F(x,y) - F(u,v)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a} \precsim \frac{\lambda}{2} A((x,y),(u,v)) + \frac{\mu}{2} B((x,y),(u,v)).$$

It follows that

$$d(T(x,y), T(u,v)) \precsim \frac{\lambda}{2}(d(x,u) + d(y,v)) + \frac{\mu}{2}\left(\frac{d(x, T(x,y) d(u, T(u,v))}{1 + d(x,u)} + \frac{d(y, T(y,x) d(v, T(v,u))}{1 + d(y,v)}\right),$$

for all $(x,y) \leq (u,v)$. Theorem 2.1 implies that there exists $(x*, y*) \in X \times X$ such that

$$T(x*, y*) = x*,\ T(y*, x*) = y*.$$

That is $(x*, y*)$ is a solution of integral equation (3.1). □

References

[1] T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. 65 (2006), 1379–1393.

[2] V. Lakshmikantham and L. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. 70 (2009), 4341–4349.

[3] B. Samet and C. Vetro, *Coupled fixed point, f-invariant set and fixed point of $N-$order*, Ann. Funct. Anal. 1 (2) (2010), 46–56.

[4] M. Eshaghi Gordji, H. Baghani and Y. J. Cho, *Coupled fixed point theorems for contractions in intuitionistic fuzzy normed spaces*, Math. Comput. Modelling, 54 (9) (2011), 1897–1906.

[5] M. Borcut and V. Berinde, *Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces*, Appl. Math. Comput. 218 (2012), 5929–5936.

[6] M. Eshaghi Gordji, M. Paknazar, M. De La Sen and S. M. Vaezpour, *N-fixed point theorem for nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory Appl. 111(2013).

[7] A. Azam, F. Brian and M. Khan, *Common fixed point theorems in complex valued metric spaces*, Numer. Func. Anal. Optim. 32 (3) (2011), 243–253.

[8] W. Sintunavarat and P. Kumam, *Generalized common fixed point theorems in complex valued metric spaces and applications*, J. Inequal. Appl. 84 (2012).

[9] S. K. Mohanta and R. Maitra, *Common fixed points of three self mapping in complex valued metric spaces*, Int. J. Math. Arch. 3 (8) (2012), 2946–2953.

[10] F. Rouzkard and M. Imdad, *Some common fixed point theorems on complex valued metric spaces*, Comput. Math. Appl. 64 (6) (2012), 1866–1874.

[11] M. Abbas, M. Arshad and A. Azam, *Fixed points of asymptotically regular mappings in complex valued metric spaces*, Georgian Math. J. 20(2) (2013), 213–221.

[12] A. Azam, J.Ahmad and P. Kumam, *Common fixed point theorems for multi valued mappings in complex valued metric spaces*, J. Inequal. Appl. 578(2013).