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Fixed point of set-valued graph contractions in metric spaces

Masoud Hadian Dehkordi[1], Masoud Ghods[1,*]

[a]Department of Mathematics, Faculty of Basic Science,Iran University of Science and Technology, Narmak, Tehran,Iran.

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Abstract

In this paper, we introduce the (G-ψ) contraction in a metric space by using a graph. Let T be a multivalued mappings on X. Among other things, we obtain a fixed point of the mapping T in the metric space X endowed with a graph G such that the set of vertices of G, $V(G) = X$ and the set of edges of G, $E(G) \subseteq X \times X$.

Keywords: Fixed point, multivalued, $(G - \psi)$ contraction, directed graph.
2010 MSC: Primary 47H10; Secondary 47H09.

1. Introduction and preliminaries

For a given metric space (X, d), let T denote a selfmap. According to Petrusel and Rus [9], T is called a Picard operator (PO) if it has a unique fixed point x^* and $\lim_{n\to\infty} T^n x = x^*$, for all $x \in X$, and is a weakly Picard operator (WPO) if for all $x \in X$, $\lim_{n\longrightarrow\infty}(T^n x)$ exists (which may depend on x) and is a fixed point of T. Let (X, d) be a metric space and G be a directed graph with set $V(G)$ of its vertices coincides with X, and the set of its edges $E(G)$ is such that $(x, x) \notin E(G)$. Assume that G has no parallel edges, we can identify G with the pair $(V(G), E(G))$, and can treat it as a weighted graph by assigning to each edge, the distance between its vertices. By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of the edges. Thus we can write

$$E(G^{-1}) = \{(x, y)|(y, x) \in E(G)\}. \tag{1.1}$$

Let $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}). \tag{1.2}$$

*Corresponding author
Email addresses: mhadian@iust.ac.ir (Masoud Hadian Dehkordi), mghods@iust.ac.ir (Masoud Ghods)

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We point out the followings:

(i) $G' = (V', E')$ is called a subgraph of G if $V' \subseteq V(G)$ and $E' \subseteq E(G)$, for all $(x, y) \in E'$, $x, y \in V'$.

(ii) If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, ..., N$.

(iii) Graph G is connected if there is a path between any two vertices, and is weakly connected if $\tilde{G}$ is connected.

(iv) Assume that G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting x is called component of G, if it consists all edges and vertices which are contained in some path beginning at x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule: yRz if there is a path in G from y to z. Clearly, G_x is connected.

(v) The sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, included in X, are Cauchy equivalent if each of them is a Cauchy sequence and $d(x_n, y_n) \longrightarrow 0$.

Let (X, d) be a complete metric space and let $CB(X)$ be a class of all nonempty closed and bounded subset of X. For $A, B \in CB(X)$, let

$$H(A, B) := \max\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\},$$

where

$$d(a, B) := \inf_{b\in B} d(a, b).$$

The mapping H is said to be a Hausdorff metric induced by d.

Definition 1.1. *Let $T : X \longrightarrow CB(X)$ be a mappings, a point $x \in X$ is said to be a fixed point of the set-valued mapping T if $x \in T(x)$*

Definition 1.2. *A metric space (X, d) is called a $\epsilon-$chainable metric space for some $\epsilon > 0$ if given $x, y \in X$, there is $n \in \mathbb{N}$ and a sequence $\{x_i\}_{i=0}^{n}$ such that $x_0 = x$, $x_n = y$ and $d(x_{i-1}, x_i) < \epsilon$, for $i = 1, \ldots, n$.*

Property A[6]. For any sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \longrightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$.

Lemma 1.3. *[1] Let (X, d) be a complete metric space and $A, B \in CB(X)$. Then for all $\epsilon > 0$ and $a \in A$ there exists a point $b \in B$ such that $d(a, b) \le H(A, B) + \epsilon$.*

Lemma 1.4. *[1] Let $\{A_n\}$ be a sequence in $CB(X)$ and $\lim_{n\longrightarrow\infty} H(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n\longrightarrow\infty} d(x_n, x) = 0$, then $x \in A$.*

Lemma 1.5. *Let $A, B \in CB(X)$ with $H(A, B) < \epsilon$, then for each $a \in A$ there exists an element $b \in B$ such that $d(a, b) < \epsilon$.*

Definition 1.6. *Let us define the class* $\Psi = \{\psi : [0,+\infty) \longrightarrow [0,+\infty) |\ \psi$ *is nondecreasing*$\}$ *which satisfies the following conditions:*

(i) *for every* $(t_n) \in \mathbb{R}^+$, $\psi(t_n) \longrightarrow 0$ *if and only if* $t_n \longrightarrow 0$;

(ii) *for every* t_1,$t_2 \in \mathbb{R}^+$, $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$;

(iii) *for any* $t > 0$ *we have* $\psi(t) \leq t$.

Lemma 1.7. *Let* $A, B \in CB(X)$, $a \in A$ *and* $\psi \in \Psi$. *Then for each* $\epsilon > 0$, *there exists* $b \in B$ *such that* $\psi(d(a, b)) \leq \psi(H(A, B)) + \epsilon$.

2. Main results

We begin with the following theorem the gives the existence of a fixed point for set valued mappings (not necessarily unique) in metric spaces endowed with a graph.

Definition 2.1. *Let* (X, d) *be a complete metric space and* $T : X \longrightarrow CB(X)$ *be a mappings,* T *is said to be a (G-*ψ*) contraction if there exists* $k \in (0, 1)$ *such that*

$$\psi(H(T(x), T(y)) \leq k\psi(d(x, y)) \text{ for all } (x, y) \in E(G), \tag{2.1}$$

and for all $(x, y) \in E(G)$ *if* $u \in T(x)$ *and* $v \in T(y)$ *are such that* $\psi(d(u, v)) \leq k\psi(d(x, y)) + \epsilon$, *for each* $\epsilon > 0$, *then* $(u, v) \in E(G)$.

Theorem 2.2. *Let* (X, d) *be a complete metric space and suppose that the triple* (X, d, G) *have the property A. Let* $F : X \longrightarrow CB(X)$ *be a (G-*ψ*) contraction and* $X_F = \{x \in X : (x, u) \in E(G)$ *for some* $u \in F(x)\}$. *Then the following statements hold:*

1. *for any* $x \in X_F$, $F|_{[x]_G}$ *has a fixed point.*
2. *if* $X_F \neq \emptyset$ *and* G *is weakly connected, then* F *has a fixed point in* X.
3. *if* $F \subseteq E(G)$, *then* F *has a fixed point.*
4. Fix $F \neq \emptyset$ *if and only if* $x \in X_F \neq \emptyset$.

Proof . *1. Let* $x_0 \in X_F$. *Then there exists* $x_1 \in F(x_0)$ *for which* $(x_0, x_1) \in E(G)$. *Since* F *is a (G-*ψ*) contraction, we should have*

$$\psi(H(F(x_0), F(x_1))) \leq k\psi d(x_0, x_1).$$

By Lemma 1.4, it ensures that there exists $x_2 \in F(x_1)$ *such that*

$$\psi(d(x_1, x_2) \leq \psi(H(F(x_0), F(x_1))) + k \leq k\psi d(x_0, x_1) + k. \tag{2.2}$$

Using the property of F *being a (G-*ψ*) contraction* $(x_1, x_2) \in E(G)$, *we obtain*

$$\psi(H(F(x_1), F(x_2))) \leq k\psi d(x_1, x_2)$$

and then Lemma 1.4 shows the existence of an $x_3 \in F(x_2)$ *such that*

$$\psi(d(x_2, x_3)) \leq \psi(H(F(x_1), F(x_2))) + k^2. \tag{2.3}$$

By inequalities (2.2) *and* (2.3), *we have*

$$\psi(d(x_2,x_3)) \leq k\psi(d(x_1,x_2)) + k^2 \leq k^2\psi(d(x_0,x_1)) + 2k^2. \tag{2.4}$$

By a similar method, in general we can prove $x_{n+1} \in F(x_n)$ *such that* $(x_n, x_{n+1}) \in E(G)$ *and*

$$\psi(d(x_n, x_{n+1})) \leq k^n\psi(d(x_0,x_1) + nk^n.$$

We can easily show by following that (x_n) *is a Cauchy sequence in* X*:*

$$\sum_{n=0}^{\infty} \psi(d(x_n,x_{n+1})) \leq \psi(d(x_0,x_1)) \sum_{n=0}^{\infty} k^n + \sum_{n=0}^{\infty} nk^n < \infty,$$

hence (x_n) *converges to some point* x *in* X*. Next step is to show that* x *is a fixed point of the mapping* F*. Using the property (**A**) and the fact of* F *being a (G-*ψ*) contraction, we have*

$$\psi(H(F(x_n), F(x)) \leq k\psi(d(x_n, x)).$$

Since $x_{n+1} \in F(x_n)$ *and* $x_n \longrightarrow x$*, by Lemma 1.3,* $x \in F(x)$*. We conclude that* $(x_n, x) \in E(G)$*, for* $n \in \mathbb{N}$*, then* $(x_0, x_1, ..., x_n, x)$ *is a path in* G *and so* $x \in [x_0]_G$*.*

2. For $X_F \neq \emptyset$*, there exists* $x_0 \in X_F$*, and since* G *is weakly connected,* $[x_0]_G = X$ *and by* 1*,* F *has a fixed point.*

3. $F \subseteq E(G)$ *implies that all* $x \in X$ *are such that there exist some* $u \in F(x)$ *with* $(x, u) \in E(G)$*, so* $X_F = X$ *by* 2*,* F *has a fixed point.*

4. Let Fix $F \neq \emptyset$*; this implies that exists* $x \in$ Fix F *such that* $x \in F(x)$*. Since* $\Delta \subseteq E(G)$*,* $(x, x) \in E(G)$ *which implies that* $x \in X_F$*, so* $X_F \neq \emptyset$*. If* $X_F \neq \emptyset$ *then* $FixF \neq \emptyset$*.* □

Example 2.3. *Let* $X = \{0\} \cup \{\frac{1}{2^n} : n \in \mathbb{N} \cup \{0\}\}$*. Consider the undirected graph* G *such that* $V(G) = X$ *and* $E(G) = \{(\frac{1}{2^n}, 0), (0, \frac{1}{2^n}), (\frac{1}{2^n}, \frac{1}{2^{n+1}}), (\frac{1}{2^{n+1}}, \frac{1}{2^n}) : n \in \mathbb{N} \cup \{0\}\} \cup \Delta$*. Let* $F : X \longrightarrow CB(X)$ *be defined by*

$$F(x) = \begin{cases} \{0\} & x = 0, \\ \{\frac{1}{2}\} & x = 1, \\ \{\frac{1}{2^{n+1}}, 0\} & x = \frac{1}{2^n}, n \in \mathbb{N}. \end{cases} \tag{2.5}$$

Then F *is a (G-*ψ*) contraction and* $0 \in F(0)$ *where* $\psi(t) = \frac{t}{t+1}$*.*

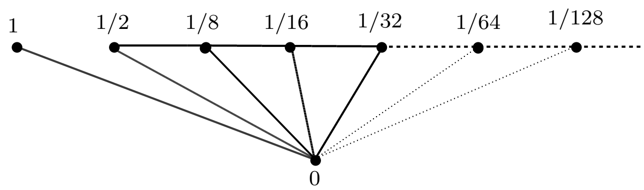
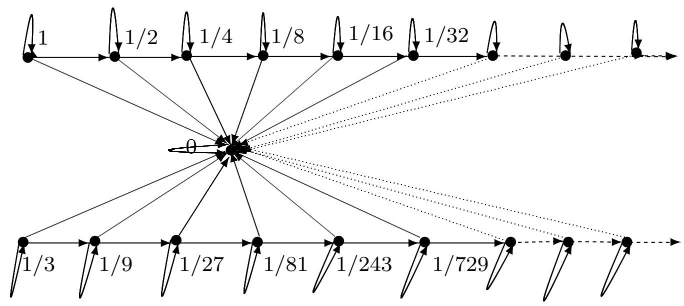


Figure 1

Example 2.4. *Let* $X = \{0\} \cup \{\frac{1}{2^n}, \frac{1}{3^n} : n \in \mathbb{N} \cup \{0\}\}$. *Consider the directed graph* G *such that* $V(G) = X$ *and* $E(G) = \{(\frac{1}{2^n}, 0), (\frac{1}{2^n}, \frac{1}{2^{n+1}}) : n \in \mathbb{N} \cup \{0\}\} \cup \{(\frac{1}{3^n}, 0), (\frac{1}{3^n}, \frac{1}{3^{n+1}}) : n \in \mathbb{N}\} \cup \Delta$. *Let* $F : X \longrightarrow CB(X)$ *be defined by*

$$F(x) = \begin{cases} \{0\} & x = 0, \\ \{\frac{1}{3^{n+1}}, 0\} & x = \frac{1}{3^n}, n \in \mathbb{N}, \\ \{\frac{1}{3^n}\} & x = \frac{1}{2^n}, n \in \mathbb{N} \cup \{0\}. \end{cases} \tag{2.6}$$

Then F *is a (G-*ψ*) contraction and* $0 \in F(0)$ *with* $\psi(t) = \frac{t}{2}$.



Property A′: For any sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \longrightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is subsequence $(x_{n_k})_{n_k\in\mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for $n_k \in \mathbb{N}$. If We have property $A^{'}$, then improve the result of this paper as follows:

Theorem 2.5. *Let* (X, d) *be a complete metric space and suppose that the triple* (X, d, G) *have the property* $A^{'}$. *Let* $F : X \longrightarrow CB(X)$ *be a (G-*ψ*) contraction and* $X_F = \{x \in X : (x, u) \in E(G)$ *for some* $u \in F(x)\}$. *Then the following statements hold.*

1. *for any* $x \in X_F$, $F|_{[x]_G}$ *has a fixed point.*
2. *If* $X_F \neq \emptyset$ *and* G *is weakly connected, then* F *has a fixed point in* X.
3. *If* $F \subseteq E(G)$, *then* F *has a fixed point.*
4. $FixF \neq \emptyset$ *if and only if* $x \in X_F \neq \emptyset$.

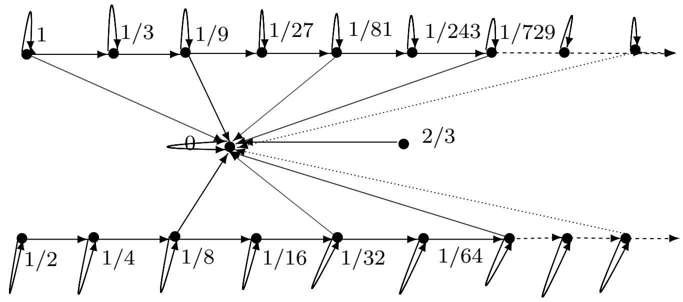
See the following example.

Example 2.6. *Let* $X = \{0, \frac{2}{3}, 1\} \cup \{\frac{1}{2^n}, \frac{1}{3^n} : n \in \mathbb{N}\}$. *Consider the directed graph* G *such that* $V(G) = X$ *and*

$$E(G) = (\frac{2}{3}, 0) \cup (1, 0) \cup (1, \frac{1}{3}) \cup \{(\frac{1}{3^{2n}}, 0), (\frac{1}{3^n}, \frac{1}{3^{n+1}}), (\frac{1}{2^{2n+1}}, 0), (\frac{1}{2^n}, \frac{1}{2^{n+1}}) : n \in \mathbb{N}\} \cup \Delta.$$

Let $F : X \longrightarrow CB(X)$ be defined by

$$F(x) = \begin{cases} \{0\} & x = 0, 1 \\ \{\frac{1}{2}\} & x = \frac{2}{3}, \\ \{\frac{1}{3^{n+1}}, 0\} & x = \frac{1}{3^n}, n \in \mathbb{N}, \\ \{\frac{1}{2^{n+1}}, 0\} & x = \frac{1}{2^n}, n \in \mathbb{N}. \end{cases} \tag{2.7}$$



Corollary 2.7. *Let (X, d) be a complete metric space and suppose that the triple (X, d, G) have the property $\boldsymbol{A}$. If G is weakly connected, then (G-ψ) contraction mapping $T : X \longrightarrow CB(X)$ such that $(x_0, x_1) \in E(G)$ for some $x_1 \in T_{x_0}$ has a fixed point.*

Corollary 2.8. *Let (X, d) be a ϵ-chainable complete metric space for some $\epsilon > 0$. Let $T : X \longrightarrow CB(X)$ be a such that there exists $k \in (0, 1)$ with*

$$0 < d(x, y) < \epsilon \Longrightarrow \psi(H(T(x), T(y)) \leq k\psi(d(x, y)).$$

Then T has a fixed point.

Proof . *Consider the G as $V(G) = X$ and*

$$E(G) := (x, y) \in X \times X : 0 < d(x, y) < \epsilon.$$

The ϵ-chainability of (X, d) means G is connected. If $(x, y) \in E(G)$, then

$$\psi(H(T(x), T(y)) \leq k\psi(d(x, y)) \leq k(d(x, y) < k\epsilon < \epsilon,$$

and by using Lemma 1.5, for each $u \in T(x)$, we have the existence of $v \in T(y)$, such that $d(u, v) < \epsilon$, which implies $(u, v) \in E(G)$. Therefore T is (G-ψ) contraction mapping. Also, (X, d, G) has property $\boldsymbol{A}$. Indeed, if $x_n \longrightarrow x$ and $d(x_n, x_{n+1}) < \epsilon$, for $\in \mathbb{N}$, then $d(x_n, x) < \epsilon$ for sufficiently large n, hence $(x_n, x) \in E(G)$. So, by Theorem 2.2 T has a fixed point. $\square$

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