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On complex valued G_b-metric spaces and related fixed point theorems

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Abstract

In this paper, we establish complex valued G_b-metric spaces and introduced the notion of G_b-Banach Contraction, G_b-Kannan mapping and prove fixed point theorems in the such spaces.

Keywords: fixed point, complex valued G_b-metric spaces, complex valued G-metric spaces, G-metric spaces, G_b-metric spaces, G_b-Banach contraction and G_b-Kannan mapping
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1. Introduction and Preliminaries

Fixed point theory became one of the most interesting area of research in the last fifty years. Banach contraction principle was introduced in 1922 by Banach [1] as follows:
(i) Let (X, d) be a metric space and let $T : X \to X$. Then T is called a Banach contraction mapping if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$.

The concept of Kannan mapping was introduced in 1969 by Kannan [2] as follows:
(ii) T is called a Kannan mapping if there exists $r \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq rd(x, Tx) + rd(y, Ty),$$

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for all $x, y \in X$.
If (X, d) is complete metric spaces, at least one of (i) and (ii) holds, then have a unique fixed point; see [1]-[2]. Next, we discuss the development of spaces. In 2011 Azam et el. [3], introduced complex valued metric space as a generalized the idea of metric space. The class of complex valued metric spaces is larger than the class of metric spaces since any metric space must be a complex valued metric space. Therefore, it is obvious that complex valued metric spaces generalize metric spaces. Moreover, these authors introduced basis definitions and generally the result of Banach [1] as follows: Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$, we define a partial order $\prec$ and $\precsim$ on $\mathbb{C}$ as follows:

(i) $z_1 \prec z_2$ if and only if Re(z_1) <Re(z_2) and Im(z_1) <Im(z_2)

(ii) $z_1 \precsim z_2$ if and only if Re(z_1) ≤Re(z_2) and Im(z_1) ≤Im(z_2).

Remark 1.1. *We obtained that following statements hold:*

(i) If $z_1 \precsim z_2$ and $z_2 \precsim z_3$, then $z_1 \precsim z_3$.

(ii) If $z \in \mathbb{C}$, $a, b \in \mathbb{R}$ and $a \leq b$, then $az \precsim bz$.

(iii) If $0 \precsim z_1 \precsim z_2$, then $|z_1| \leq |z_2|$.

Definition 1.2. *[3] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:*

(d_1) $0 \precsim d(x, y)$, *for all* $x, y \in X$;

(d_2) $d(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$;

(d_3) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(d_4) $d(x, y) \precsim d(x, z) + d(z, y)$, *for all* $x, y, z \in X$.

Then d is called a complex valued metric *on X and (X, d) is called a* complex valued metric space.

Theorem 1.3. *([3], Theorem 4) If S and T are self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition*

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx)d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ and μ are nonnegative with $\lambda + \mu < 1$, then S and T have a unique common fixed point, (i.e., there exists $z_0 \in \mathbb{Z}$ such that $Sz_0 = Tz_0 = z_0$).

Moreover, Klin-eam and Suanoom [4], Sintunavarat et el. [5], Rouzkard et el. [6] make the results of the Azam et el. is known more in 2012 and 2013.

On the other hand, Bakhtin [7] introduced b-metric space as a generalized the idea of metric space. Finally in many other generalized b-metric space, such as, quasi b-metric space [8], b-metric-like space [9], quasi b-metric-like space [10] and dislocated quasi-b-metric spaces [11]. The concept of complex valued b-metric spaces was introduced in 2013 by Rao et al. [12], which was more general than the well-known complex valued metric spaces that were introduced in 2011 by Azam et al. Later, Ghaler [13] generalized the idea of metric space and introduced a 2-metric space. In 1992 Dhage [14], to introduce a new class of generalized metrics called D-metrics as a generalized the idea of the results of Ghaler. In 2004 Mustafa and Sims [16], introduced a new concept in the area, called G-metric space as follows:

Definition 1.4. *Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following axioms:*

(G_1) $G(x, y, z) = 0$ iff $x = y = z$,

(G_2) $0 < G(x, x, y)$, *for all* $x, y \in X$, *with* $x \neq y$,
(G_3) $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$, *with* $z \neq y$,
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (*symmetry in all three variables*),
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$, (*rectangle inequality*).
Then the function G *is called a* generalized metric, *or more specifically a* G-metric *on* X, *and the pair* (X, G) *is called a* G-metric space.

Moreover, these authors make some remarks concerning D-metric spaces, and present some examples which show that many of the basic claims concerning the topological structure of such spaces are incorrect, thus nullifying many of the results claimed for these spaces.

Next, Agarwal and Karapinar [15] studied many fixed point theorems for mappings satisfying general contractive conditions on complete G-metric spaces as follows :

Theorem 1.5. *([15]) Let* (X, G) *be a G-metric space. Let* $T : X \to X$ *and* $g : X \to X$ *be two mappings such that*

$$G(Tx, Ty, Tz) \leq kG(gx, gy, gz),$$

for all $x, y, z \in X$ *Assume that* T *and* g *satisfy the following conditions :*
(1) $T(X) \subseteq g(X)$,
(2) $g(X)$ *is complete,*
(3) g *is G-continuous and commutes with* T.
If $k \in [0, 1)$, *then there is a unique* $x \in X$ *such that* $gx = Tx = x$.

Recently, Mustafa et.al. studied many fixed point theorems for mappings satisfying various contractive conditions on complete G-metric spaces; see ([16]-[20]).

Afterwards, Aghajani et al. [21], extended the notion of G-metric space to the concept of G_b-metric space as follows:

Definition 1.6. *Let* X *be a nonempty set, and* $s \geq 1$ *be a given real number. Suppose that a mapping* $G : X \times X \times X \to \mathbb{R}^+$ *satisfying :*
(G_b1) $G(x, y, z) = 0$ *iff* $x = y = z$,
(G_b2) $0 < G(x, x, y)$, *for all* $x, y \in X$, *with* $x \neq y$,
(G_b3) $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$, *with* $y \neq z$,
(G_b4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (*symmetry in all three variables*),
(G_b5) $G(x, y, z) \leq s(G(x, a, a) + G(a, y, z))$, *for all* $x, y, z, a \in X$, (*rectangle inequality*).
Then G *is called a* generalized b-metric, *and pair* (X, G) *is called a* generalized b-metric space *or* G_b-metric space.

Moreover, these authors studied many fixed point theorems as follows:

Theorem 1.7. ([21]) *Let* (X, G) *be a complete* G_b-*metric space and let* $A, B, C : X \to X$ *satisfy the following condition :*

$$\psi(2s^4G(Ax, By, Cz)) \leq \psi(M(x, y, z)) - \varphi(M(x, y, z))$$

for all $x, y, z \in X$, *where* $\psi, \varphi, : [0, \infty) \to [0, \infty)$ *are two mappings such that* ψ *is continuous non-decreasing,* φ *is a lower semi-continuous function with* $\psi(t) = \varphi(t) = 0$ *if and only if* $t = 0$ *and* $M(x, y, z) = \max\{G(x, y, z), G(x, Ax, By), G(y, By, Cz), G(z, Cz, Ax)\}$. *Then, either one of* A, B *and* C *has a fixed point, or, the maps* A, B *and* C *have a unique common fixed point.*

The concept of complex valued G-metric spaces was introduced in 2013 by Kang et.al. [22], as follows:

Definition 1.8. *Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{C}$ be a function satisfying the following axioms:*
(CG_1) $G(x, y, z) = 0 \ iff \ x = y = z,$
(CG_2) $0 \prec G(x, x, y), \ for \ all \ x, y \in X, \ with \ x \neq y,$
(CG_3) $G(x, x, y) \preceq G(x, y, z), \ for \ all \ x, y, z \in X, \ with \ z \neq y,$
(CG_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots (symmetry \ in \ all \ three \ variables),$
(CG_5) $G(x, y, z) \preceq G(x, a, a) + G(a, y, z), \ for \ all \ x, y, z, a \in X, \ (rectangle \ inequality).$
Then the function G is called a complex valued generalized metric *or, more specifically a* complex valued G-metric *on* , X, *and the pair (X, G) is called a* complex valued G-metric space*, which was more general than the results of the Azam et al., Mustafa and Sims: see ([3], [16]).*

Moreover, these authors studied many fixed point theorems as follows :

Theorem 1.9. ([22]) *Let (X, G) be a complete complex valued G_b-metric space. Let $T : X \to X$ be a contraction mappings on X, i.e.,*

$$G(Tx, Ty, Tz) \preceq kG(x, y, z)$$

for all $x, y, z \in X$, where $k \in [0, 1)$. Then T has a unique fixed point.

Motivation by this, we introduce the notion of complex valued G_b-metric spaces as a generalization of G_b-metric space with complex valued G-metric spaces, which was more general than the results of the Azam et al. and Aghajani et al.: see ([3], [21]), and introduced the notion of G_b-Banach Contraction, G_b-Kannan mapping as a generalized the idea for the results of Banach [1] and Kannan [2]. Moreover, we prove fixed point theorems in the such spaces. Moreover, The author has continued to develop research on the fixed point theory, see ([23]-[36]).

2. Main results

In this section, we begin with introducing the notion of a complex valued G_b-metric spaces.

Definition 2.1. *Let X be a nonempty set, and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{C}$ satisfying :*
(CG_b1) $G(x, y, z) = 0 \ if \ x = y = z,$
(CG_b2) $0 \prec G(x, x, y), \ for \ all \ x, y \in X, \ with \ x \neq y,$
(CG_b3) $G(x, x, y) \preceq G(x, y, z), \ for \ all \ x, y, z \in X, \ with \ y \neq z,$
(CG_b4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots (symmetry \ in \ all \ three \ variables),$
(CG_b5) $G(x, y, z) \preceq s(G(x, a, a) + G(a, y, z)), \ for \ all \ x, y, z, a \in X, \ (rectangle \ inequality).$
Then G is called a complex valued generalized b-metric *and pair (X, G) is called a* complex valued generalized b-metric space *or* complex valued G_b-metric space.

Remark 2.2. *Indeed, each complex valued G-metric space is a complex valued G_b-metric space with $s = 1$. But, conversely is not true.*

Lemma 2.3. *Let c, d be nonnegative real numbers. Then $(c + d)^p \leq 2^{p-1}(c^p + d^p)$ for all $p \in \mathbb{N}$*

Proof . Let c, d be nonnegative real numbers. By induction proof, let Clearly, $P(1)$ is true.
Since $(c-d)^2 \geq 0$, we have $2cd \leq c^2 + d^2$. So, $(c+d)^2 = c^2 + 2cd + d^2 \leq 2c^2 + 2d^2 = 2(c^2 + d^2)$, and then $P(2)$ is true. Assume that $P(k)$ is true, that is $(c+d)^k \leq 2^{k-1}(c^k + d^k)$. We will to show that $P(k+1)$ is true.
Since $(c-d)(c^k - d^k) \geq 0$, we have $c^k d + cd^k \leq c^{k+1} + d^{k+1}$. Thus,

$$\begin{aligned}(c+d)^{p+1} &= (c+d)^p(c+d) \\ &\leq 2^{k-1}(c^k + d^k)(c+d) \\ &= 2^{k-1}(c^{k+1} + c^k d + cd^k + d^{k+1}) \\ &= 2^{k-1}(2c^{k+1} + 2d^{k+1}) \\ &= 2^k(c^{k+1} + d^{k+1}).\end{aligned}$$

Therefore, $(c+d)^p \leq 2^{p-1}(c^p + d^p)$. □

Example 2.4. *Let $X = \mathbb{R}$. Defined*

$$G(x,y,z) = (|x-y| + |y-z| + |x-z|)^p + i(|x-y| + |y-z| + |x-z|)^p.$$

For every $x, y, z, a \in \mathbb{R}$ and all $p \in \mathbb{N}$
(CG_b1) If $G(x,y,z) = 0$, than

$$G(x,y,z) = (|x-y| + |y-z| + |x-z|)^p + i(|x-y| + |y-z| + |x-z|)^p$$

Thus $(|x-y| + |y-z| + |x-z|)^p = 0$, and so $(|x-y| + |y-z| + |x-z|) = 0$.
Hence $x = y = z$.
If $x = y = z$, than

$$G(x,y,z) = (|x-y| + |y-z| + |x-z|)^p + i(|x-y| + |y-z| + |x-z|)^p = 0.$$

(CG_b2) Assumes that $x \neq y$.
Then,

$$G(x,y,z) = (|x-y| + |y-z| + |x-z|)^p + i(|x-y| + |y-z| + |x-z|)^p \succeq |x-y|^p + i|x-y|^p \succ 0.$$

(CG_b3) Since $|x-y| \leq |x-z| + |z-y|$, we have

$$\begin{aligned}G(x,x,y) &= (|x-x| + |x-y| + |x-y|)^p + i(|x-x| + |x-y| + |x-y|)^p \\ &= (|x-y| + |x-y|)^p + i(|x-y| + |x-y|)^p \\ &\preceq (|x-y| + |y-z| + |x-z|)^p + i(|x-y| + |y-z| + |x-z|)^p \\ &= G(x,y,z).\end{aligned}$$

(CG_b4) It is easy to see that $G(x,y,z) = G(\pi\{x,z,y\})$, where π is a permutation.
(CG_b5) By Lemma 2.3, we get

$$\begin{aligned}G(x,y,z) &= (|x-y| + |y-z| + |x-z|)^p + i(|x-y| + |y-z| + |x-z|)^p \\ &\preceq (|x-a| + |a-y| + |y-z| + |z-a| + |a-z|)^p \\ &\quad + i(|x-a| + |a-y| + |y-z| + |z-a| + |a-z|)^p \\ &\preceq 2(|x-a| + |x-a|)^p + p(|a-y| + |y-z| + |a-z|)^p \\ &\quad + 2i(|x-a| + |x-a|)^p + 2^{p-1}i(|a-y| + |y-z| + |a-z|)^p \\ &= 2(|x-a| + |a-a| + |x-a|)^p + 2^{p-1}i(|x-a| + |a-a| + |x-a|)^p \\ &\quad + 2(|a-y| + |y-z| + |a-z|)^p + 2^{p-1}i(|a-y| + |y-z| + |a-z|)^p \\ &= 2(G(x,a,a) + G(a,y,z)),\end{aligned}$$

where $s = 2^{p-1}$.

Thus, G *is complex valued* G_b*-metric on* $\mathbb{R}$ *with* $s = 2^{p-1}$.
Note that, if $p = 1$*, then* (X, G) *is a complex valued* G*-metric spaces.*

Example 2.5. *Let* $X = [0, 1]$*. Defined*

$$G(x, y, z) = \max\{|x-y|, |y-z|, |x-z|\}^p + i \max\{|x-y|, |y-z|, |x-z|\}^p.$$

For every $x, y, z, a \in [0, 1]$ *and all* $p \in \mathbb{N}$
(CG_b1) *If* $G(x, y, z) = 0$*, than*

$$\max\{|x-y|, |y-z|, |x-z|\}^p + i \max\{|x-y|, |y-z|, |x-z|\}^p = 0$$

Thus $\max\{|x-y|, |y-z|, |x-z|\}^p = 0$*, and so* $\max\{|x-y|, |y-z|, |x-z|\} = 0$*. Hence* $x = y = z$*.*
If $x = y = z$*, then*

$$G(x, y, z) = \max\{|x-y|, |y-z|, |x-z|\}^p + i \max\{|x-y|, |y-z|, |x-z|\}^p = 0.$$

(CG_b2) *Assumes that* $x \neq y$*. Then,*

$$\begin{aligned} G(x, y, z) &= \max\{|x-y|, |y-z|, |x-z|\}^p + i \max\{|x-y|, |y-z|, |x-z|\}^2 \\ &\succeq |x-y|^2 + i|x-y|^p \succ 0 \end{aligned}$$

(CG_b3)

$$\begin{aligned} G(x, x, y) &= \max\{|x-x|, |x-y|, |x-y|\}^p + i \max\{|x-x|, |x-y|, |x-y|\}^p \\ &= |x-y|^2 + i|x-y|^p \\ &\preceq \max\{|x-y|, |y-z|, |x-z|\}^p + i \max\{|x-y|, |y-z|, |x-z|\}^p \\ &= G(x, y, z). \end{aligned}$$

(CG_b4) *It is easy to see that* $G(x, y, z) = G(\pi\{x, z, y\})$*, where* π *is a permutation.*
(CG_b5) *By Lemma 2.3, we get*

$$\begin{aligned} G(x, y, z) &= \max\{|x-y|, |y-z|, |x-z|\}^p + i \max\{|x-y|, |y-z|, |x-z|\}^p \\ &\preceq 2^{p-1}(|x-a|^p + \max\{|a-y|, |y-z|, |a-z|\}^p) \\ &\quad + 2^{p-1} i(|x-a|^p + \max\{|a-y|, |y-z|, |a-z|\}^p) \\ &= 2^{p-1} \max\{|x-a|, |a-a|, |x-a|\}^p + 2^{p-1} i \max\{|x-a|, |a-a|, |x-a|\}^p \\ &\quad + 2^{p-1} \max\{|a-y|, |y-z|, |a-z|\}^p + 2^{p-1} i \max\{|a-y|, |y-z|, |a-z|\}^p \\ &= 2(G(x, a, a) + G(a, y, z)), \end{aligned}$$

where $s = 2^{p-1}$*. Thus,* G *is complex valued* G_b*-metric on* X *with* $s = 2^{p-1}$*.*
Moreover, (X, G) *is not complex valued* G*-metric spaces, if* $p \neq 1$*, indeed,* $x = \frac{1}{2}$*,* $y = \frac{1}{7}$*,* $z = \frac{1}{4}$*,*

$a = \frac{1}{3}$ *and fixed* $p = 2$, *we have*

$$\begin{aligned}
G(x,y,z) &= \max\{|x-y|,|y-z|,|x-z|\}^2 + i\max\{|x-y|,|y-z|,|x-z|\}^2 \\
&= \max\{|\frac{1}{2}-\frac{1}{7}|,|\frac{1}{7}-\frac{1}{4}|,|\frac{1}{2}-\frac{1}{4}|\}^2 + i\max\{|\frac{1}{2}-\frac{1}{7}|,|\frac{1}{7}-\frac{1}{4}|,|\frac{1}{2}-\frac{1}{4}|\}^2 \\
&= \max\{|\frac{5}{14}|,|\frac{3}{28}|,|\frac{1}{4}|\}^2 + i\max\{|\frac{5}{14}|,|\frac{3}{28}|,|\frac{1}{4}|\}^2 \\
&= \frac{25}{196} + i\frac{25}{196} \\
&\succ \frac{1,017}{15,876} + i\frac{1,017}{15,876} \\
&= \frac{1}{36} + i\frac{1}{36} + \frac{16}{441} + i\frac{16}{441} \\
&= (\frac{1}{36} + i\frac{1}{36}) + \max\{\frac{4}{21},\frac{3}{28},\frac{1}{12}\}^2 + i\max\{\frac{4}{21},\frac{3}{28},\frac{1}{12}\}^2 \\
&= (|\frac{1}{2}-\frac{1}{3}|^2 + i|\frac{1}{2}-\frac{1}{3}|^2) \\
&\quad + \max\left\{|\frac{1}{3}-\frac{1}{7}|,|\frac{1}{7}-\frac{1}{4}|,|\frac{1}{3}-\frac{1}{4}|\right\}^2 + i\max\left\{|\frac{1}{3}-\frac{1}{7}|,|\frac{1}{7}-\frac{1}{4}|,|\frac{1}{3}-\frac{1}{4}|\right\}^2 \\
&= \max\{|x-a|,|a-a||x-a|\}^2 + i\max\{|x-a|,|a-a||x-a|\}^2 \\
&\quad + \max\{|a-y|,|y-z|,|a-z|\}^2 + i\max\{|a-y|,|y-z|,|a-z|\}^2 \\
&= G(x,a,a) + G(a,y,z).
\end{aligned}$$

Definition 2.6. *Let* (X, G) *be a complex valued* G_b*-metric space let* $\{x_n\}$ *be a sequence in* X, *we say that* $\{x_n\}$ *is call* complex valued G_b-convergent *to* x *if for every* $c \in \mathbb{C}$ *with* $0 \prec c$, *there exists* $k \in \mathbb{N}$ *such that* $G(x, x_n, x_m) \prec c$ *for all* $n, m \geq k$. *We refer to* x *as the limit of sequence* $\{x_n\}$ *and we write* $x_n \to x$.

Proposition 2.7. *Let* (X, G) *be a complex valued* G_b*-metric space and* $\{x_n\}$ *be a sequence in* X. *Then* $\{x_n\}$ *is complex valued* G_b*-convergent to* x *if and only if* $|G(x, x_n, x_m)| \to 0$ *as* $n, m \to \infty$.

Proof . *Suppose that* $\{x_n\}$ *is complex valued* G_b*-convergent to* x. *For a given real number* $\epsilon > 0$, *let*

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Then $0 \prec c \in C$ *and there is a natural number* k *such that* $G(x, x_n, x_m) \prec c$ *for all* $n, m \geq k$. *Therefore,* $|G(x, x_n, x_m)| < |c| = \epsilon$ *for all* $n, m \geq k$. *It follows that* $|G(x, x_n, x_m)| \to 0$ *as* $n, m \to \infty$.

Conversely, suppose that $|G(x, x_n, x_m)| \to 0$ *as* $n, m \to \infty$. *Then given* $c \in C$ *with* $0 \prec c$, *there exists a real number* $\delta > 0$ *such that for* $z \in \mathbb{C}$

$$|z| < \delta \text{ implies } z \prec c.$$

For this δ, *there is a natural number* k *such that* $|G(x, x_n, x_m)| < \delta$ *for all* $n, m \geq k$. *This means that* $G(x, x_n, x_m) \prec c$ *for all* $n, m \geq k$. *Hence* $\{x_n\}$ *is complex valued* G_b*-convergent to* x. □

Definition 2.8. *Let* (X, G) *be a complex valued* G_b*-metric space, Then a sequence* $\{x_n\}$ *is called* complex valued G_b-Cauchy *if for every* $c \in \mathbb{C}$ *with* $0 \prec c$, *there exists* $k \in \mathbb{N}$ *such that* $G(x_n, x_m, x_l) \prec c$ *for all* $n, m, l \geq k$.

Proposition 2.9. *Let (X, G) be a complex valued G_b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued G_b-Cauchy sequence if and only if $|G(x_n, x_m, x_l)| \to 0$ as $n, m, l \to \infty$.*

Proof . *Suppose that $\{x_n\}$ is complex valued G_b-Cauchy sequence. For a given real number $\epsilon > 0$, let*

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Then $0 \prec c \in C$ and there is a natural number k such that $G(x_n, x_m, x_l) \prec c$ for all $n, m, l \geq k$. Therefore, $|G(x_n, x_m, x_l)| < |c| = \epsilon$ for all $n, m, l \geq k$. It follows that $|G(x_n, x_m, x_l)| \to 0$ as $n, m, l \to \infty$.

Conversely, suppose that $|G(x_n, x_m, x_l)| \to 0$ as $n, m, l \to \infty$. Then given $c \in C$ with $0 \prec c$, there exists a real number $\delta > 0$ such that for $z \in \mathbb{C}$

$$|z| < \delta \text{ implies } z \prec c.$$

For this δ, there is a natural number k such that $|G(x_n, x_m, x_l)| < \delta$ for all $n, m, l \geq k$. This means that $G(x_n, x_m, x_l) \prec c$ for all $n, m, l \geq k$. Hence $\{x_n\}$ is complex valued G_b-Cauchy sequence. □

Definition 2.10. *Let A, B be a subset of X. A complex valued G_b-metric space (X, G) is said to be complex valued G_b-complete if every complex valued G_b-Cauchy sequence is complex valued G_b-convergent in (X, G).*

Definition 2.11. *Let X be nonempty subsets of a complete complex valued G_b-metric space (X, G). A map $T : X \to X$ is said to be a G_b-*Banach Contraction mappings *and if there exists $k \in [0, 1)$ such that*

$$G(Tx, Ty, Tz) \preceq kG(x, y, z). \tag{2.1}$$

for all $x, y, z \in X$ and $s \geq 1$ and $sk \leq 1$.

Theorem 2.12. *Let (X, G) be a complete complex valued G_b-metric space. Let T be a G_b-Banach Contraction mappings on X, i.e.,*

$$G(Tx, Ty, Tz) \preceq kG(x, y, z)$$

for all $x, y, z \in X$ and $s \geq 1$ and $sk \leq 1$. Then T has a unique fixed point.

Proof . *Suppose that T satisfies condition (2.1). Let $x_0 \in X$ be an arbitrary point, and define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then by (2.1), we have*

$$G(x_n, x_{n+1}, x_{n+1}) \preceq kG(x_{n-1}, x_n, x_n). \tag{2.2}$$

Again by (2.1), we have

$$G(x_{n-1}, x_n, x_n) \preceq kG(x_{n-2}, x_{n-1}, x_{n-1}).$$

Since (2.2), we have

$$G(x_n, x_{n+1}, x_{n+1}) \preceq k^2 G(x_{n-2}, x_{n-1}, x_{n-1}).$$

Continuing in the same way, we get

$$G(x_n, x_{n+1}, x_{n+1}) \preceq k^n G(x_0, x_1, x_1). \tag{2.3}$$

Then, for all $n, m \in \mathbb{N}$ with $m > n$, we have by using of (CG_b5) and (2.4) that

$$\begin{aligned} G(x_n, x_m, x_m) &\preceq sG(x_n, x_{n+1}, x_{n+1}) + s^2 G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + s^3 G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + s^{m-n} G(x_{m-1}, x_m, x_m) \\ &\preceq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) + s^{m-n-2} k^{m-3} G(x_0, x_1, x_1) + \cdots \\ &\quad + s^{m-n-1} k^{m-2} + s^{m-n} k^{m-1} G(x_0, x_1, x_1) \\ &\preceq (k^{n-1} + k^{n-1} + k^{n-1} + \ldots + k^{n-1} + k^{n-1}) G(x_0, x_1, x_1) \\ &= (k^{n-1})(m - n + 1) G(x_0, x_1, x_1). \end{aligned}$$

Thus,

$$|G(x_n, x_m, x_m)| \leq (k^{n-1})(m - n + 1)|G(x_0, x_1, x_1)|.$$

Take $n \to \infty$, we get $|G(x_n, x_m, x_m)| \to 0$. So, by Proposition 2.7, $\{x_n\}$ is complex valued G_b-Cauchy sequence.
Since(X, d) is complete, we have $\{x_n\}$ is complex valued G_b-convergent to some $z \in X$.

Now, we will to show that $Tz = z$. Assume that $Tz \neq z$. Then, we get

$$G(x_{n+1}, Tz, Tz) \preceq kG(x_n, z, z),$$

and so,

$$|G(x_{n+1}, Tz, Tz)| \leq k|G(x_n, z, z)|.$$

Taking $n \to \infty$, we have

$$|G(z, Tz, Tz)| \leq k|G(z, z, z)|,$$

which is a contradiction to $k \in [0, 1)$. Thus $Tz = z$.

Finally, to prove the uniqueness of fixed point, let $z^* \in X$ be another fixed point of and T such that $Tz^* = z^*$. Then by (2.1),

$$G(z, z^*, z^*) = G(Tz, Tz^*, Tz^*) \preceq kG(z, z^*, z^*).$$

Hence,

$$|G(z, z^*, z^*)| \leq k|G(z, z^*, z^*)|.$$

Since $k \in [01,)$, we have $|G(z, z^*, z^*)| = 0$. Therefore, $z^* = z$. Therefore z is a unique fixed point of T. □

Example 2.13. *Let $X = [0, 1]$ and $G : X \times X \times X \to \mathbb{C}$ be complex valued G_b-metric space defined as follows:*

$$G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}^2 + i \max\{|x - y|, |y - z|, |x - z|\}^2$$

for all $x, y, z \in X$, with $s = 2$. Define $T : X \to X$ as $Tx = \frac{x}{2}$. Then T satisfy $G(Tx, Ty, Tz) \preceq kG(x, y, z)$ holds for all $x, y, z \in X$, where $\frac{1}{4} \leq k < 1$. Hence $x = 0$ is the unique fixed point of T.

Definition 2.14. *Let X be nonempty subsets of a complete complex valued G_b-metric space (X, G). A map $T : X \to X$ is said to be a G_b-*Kannan mapping *and if there exists $r \in [0, \frac{1}{2})$ such that*

$$G(Tx, Ty, Tz) \preceq r(G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz))$$

for all $x, y, z \in X$ and $s \geq 1$ and $rk \leq 1$.

Theorem 2.15. *Let (X, G) be a complete complex valued G_b-metric space. Let T be a G_b-Kannan mapping on X, i.e.,*

$$G(Tx, Ty, Tz) \preceq r(G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)).$$

for all $x, y, z \in X$ and $s \geq 1$ and $sk \leq 1$. Then T has a unique fixed point.

Proof . *Suppose that T satisfies G_b-Kannan mapping. Let $x_0 \in X$ be an arbitrary point, and define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then by T satisfies G_b-Kannan mapping, we have*

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\preceq G(Tx_{n-1}, Tx_n, Tx_n) \\ &\preceq r(G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G(x_n, Tx_n, Tx_n) + G(x_n, Tx_n, Tx_n)). \\ &\preceq r(G(x_{n-1}, x_n, x_n)r + G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1})). \end{aligned} \tag{2.4}$$

So,

$$G(x_n, x_{n+1}, x_{n+1}) \preceq (\frac{r}{1-2r})G(x_{n-1}, x_n, x_n)$$

Again by T satisfies G_b-Kannan mapping, we get

$$G(x_{n-1}, x_n, x_n) \preceq (\frac{r}{1-2r})G(x_{n-2}, x_{n-1}, x_{n-1}).$$

Continuing in the same way, we get

$$G(x_n, x_{n+1}, x_{n+1}) \preceq (\frac{r}{1-2r})^n G(x_0, x_1, x_1). \tag{2.5}$$

Then, for all $n, m \in \mathbb{N}$ with $m > n$, we have by using of (CG_b5) and (2.4) that

$$\begin{aligned} G(x_n, x_m, x_m) &\preceq sG(x_n, x_{n+1}, x_{n+1}) + s^2 G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + s^3 G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + s^{m-n} G(x_{m-1}, x_m, x_m) \\ &\preceq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) + s^{m-n-2} k^{m-3} G(x_0, x_1, x_1) + \cdots \\ &\quad + s^{m-n-1} k^{m-2} + s^{m-n} k^{m-1} G(x_0, x_1, x_1) \\ &\preceq (k^{n-1} + k^{n-1} + k^{n-1} + ... + k^{n-1} + k^{n-1}) G(x_0, x_1, x_1) \\ &= (k^{n-1})(m - n + 1) G(x_0, x_1, x_1), \end{aligned}$$

where $k = \frac{r}{1-2r}$. Thus,

$$|G(x_n, x_m, x_m)| \leq k^{n-1}(m - n + 1)|G(x_0, x_1, x_1)|.$$

Take $n \to \infty$, we get $|G(x_n, x_m, x_m)| \to 0$. So, by Proposition 2.7, $\{x_n\}$ is complex valued G_b-Cauchy sequence.

Since(X, d) is complete, we have $\{x_n\}$ is complex valued G_b-convergent to some $z \in X$.

Now, we will to show that $Tz = z$. Assume that $Tz \neq z$. Then, we get

$$\begin{aligned} G(x_{n+1}, Tz, Tz) &\preceq G(Tx_n, Tz, Tz) \\ &\preceq r(G(x_n, Tx_n, Tx_n) + G(z, Tz, Tz) + G(z, Tz, Tz)), \\ &\preceq r(G(x_n, x_{n+1}, x_{n+1}) + G(z, Tz, Tz) + G(z, Tz, Tz)), \end{aligned}$$

and so,

$$|G(x_{n+1}, Tz, Tz)| \leq r|G(x_n, x_{n+1}, x_{n+1})| + 2r|G(z, Tz, Tz)|.$$

Taking $n \to \infty$, we get

$$|G(z, Tz, Tz)| \leq \frac{r}{1-2r}|G(z, z, z)|,$$

which is a contradiction to $\frac{r}{1-2r} \in [0, 1)$. Thus $Tz = z$.

Finally, to prove the uniqueness of fixed point, let $z^* \in X$ be another fixed point of and T such that $Tz^* = z^*$. Then by T satisfies G_b-Kannan mapping,

$$G(z, z^*, z^*) = G(Tz, Tz^*, Tz^*) \preceq r(G(z, Tz, Tz) + G(z^*, Tz^*, Tz^*) + G(z^*, Tz^*, Tz^*)).$$

Hence,

$$0 \leq |G(z, z^*, z^*)| \leq r|G(z, z, z)| + r|G(z^*, z^*, z^*)| + r|G(z^*, z^*, z^*)| \leq 0.$$

Thus, $|G(z, z^*, z^*)| = 0$, and then $z^* = z$. Therefore z is a unique fixed point of T. □

Example 2.16. *Let $X = [0, 1]$ and $G : X \times X \times X \to \mathbb{C}$ be complex valued G_b-metric space defined as follows:*

$$G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}^2 + i \max\{|x - y|, |y - z|, |x - z|\}^2$$

for all $x, y, z \in X$, with $s = 2$. Define $T : X \to X$ as $Tx = \frac{x}{4}$.
Assumes that $x \geq y \geq z$. Then T satisfy

$$\begin{aligned}
G(Tx, Ty, Tz) &= \max\{|\frac{x}{4} - \frac{y}{4}|, |\frac{y}{4} - \frac{z}{4}|, |\frac{x}{4} - \frac{z}{4}|\}^2 + i \max\{|\frac{x}{4} - \frac{y}{4}|, |\frac{y}{4} - \frac{z}{4}|, |\frac{x}{4} - \frac{z}{4}|\}^2 \\
&= \frac{1}{16}(|x - z|^2 + i|x - z|^2) \\
&\preceq \frac{1}{8}(|x - \frac{x}{4}|^2 + |\frac{x}{4} - z|^2 + i|x - \frac{x}{4}|^2 + i|\frac{x}{4} - z|^2) \\
&\preceq \frac{1}{8}(|x - \frac{x}{4}|^2 + |y - \frac{y}{4}|^2 + |z - \frac{y}{4}|^2) + i\frac{1}{8}(|x - \frac{x}{4}|^2 + |y - \frac{y}{4}|^2 + |z - \frac{y}{4}|^2) \\
&= \frac{1}{8}(|x - Tx|^2 + i|x - Tx|^2 + |y - Ty|^2 + i|y - Ty|^2 + |z - Tz|^2 + i|z - Tz|^2) \\
&\preceq kG(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz),
\end{aligned}$$

holds for all $x, y, z \in X$, where $\frac{1}{8} \leq k < 1$. Hence $x = 0$ is the unique fixed point of T.

Competing interests

The authors declare that they have no competing interests.

Authorś contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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