Int. J. Nonlinear Anal. Appl. 12 (2021) No. 1, 812-820
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/IJNAA.2021.4936

Natural homotopy perturbation method for solving nonlinear fractional gas dynamics equations

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, we investigate solutions of nonlinear fractional differential equations by using Natural homotopy perturbation method (NHPM). This method is coupled by the Natural transform (NT) and homotopy perturbation method (HPM). The method in general is easy to implement and yields good results. Illustrative examples are included to demonstrate the validity and applicability of the presented method.

Keywords: Local fractional RDTM; Fractional gas dynamics equation; Natural transform; homotopy perturbation method.
2010 MSC: 35R11; 74H10.

1. Introduction

In recent years, many researchers have paid attention to study the behavior of physical problems by using various analytical and numerical techniques which are not described by the common observations, such as the FVIM [1, 2, 3, 4, 5], FDTM [6, 7], FSEM [8, 9], FSTM [10], FLTM [11, 12], FHPM [13], FLDM [14, 15], FFSM [16], FLVIM [17, 18, 19, 20] and another methods [21, 22, 23, 24, 25, 26, 27, 28, 29].

Nonlinear gas dynamics equations are the mathematical expressions of conservation laws that exist in engineering practices such as conservation of mass, momentum, energy and etc. Several different types of these equations have been solved in physics by Jafari et al. [30], Baleanu et al. [31], Kumar et al. [32], Jassim et al. [33] and Das et al. [34] via several analytical and numerical methods.

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Received: 10 2020 *Revised:* 12 2020

Our concern in this work is to consider the nonlinear fractional gas dynamic equation as the following[34]:

$$D_{\tau}^{\nu}x(\rho,\tau) = x(\rho,\tau) - \frac{1}{2}\frac{\partial x^2(\rho,\tau)}{\partial \rho} - x^2(\rho,\tau), \quad 0 < \nu \leq 1, \tag{1.1}$$

subject to initial condition

$$x(\rho,0) = \varphi(\rho), \tag{1.2}$$

where $x(\rho,\tau)$ is an unknown function, and $\varphi(\rho)$ is the continuous function.

2. Preliminaries of Fractional Calculus

Some fractional calculus definitions and notation needed [22, 23, 34] in the course of this work are discussed in this section

Definition 2.1. *A real function $x(\tau), \tau > 0$, is said to be in the space $C_{\vartheta}, \vartheta \in R$ if there exists a real number $q, (q > \vartheta)$, such that $x(\tau) = \tau^q x_1(\tau)$, where $x_1(\tau) \in C[0,\infty)$, and it is said to be in the space C_{ϑ}^m if $x^{(m)} \in C_{\vartheta}, m \in N$.*

Definition 2.2. *The Riemann Liouville fractional integral operator of order ν of a function $x(\tau) \in C_{\vartheta}, \vartheta \geq -1$ is defined as*

$$I^{\nu}x(\tau) = \begin{cases} \frac{1}{\Gamma(\nu)}\int_0^{\tau}(\tau-\kappa)^{\nu-1}x(\kappa)d\kappa, & \nu > 0, \tau > 0, \\ x(\tau), & \nu = 0, \end{cases}$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Properties of the operator I^{ν}, which we will use here, are as follows:
For $x \in C_{\vartheta}, \vartheta \geq -1, \nu, \sigma \geq -1$, then

1. $I^{\nu}I^{\sigma}\varphi(\mu) = I^{\gamma+\sigma}x(\tau)$.
2. $I^{\nu}I^{\sigma}\varphi(\mu) = I^{\sigma}I^{\nu}x(\tau)$.
3. $I^{\nu}\tau^m = \frac{\Gamma(m+1)}{\Gamma(\nu+m+1}\tau^{\nu+m}$.

Definition 2.3. *The fractional derivative of $x(\tau)$ in the Caputo sense is defined as*

$$\begin{aligned} D^{\nu}x(\tau) &= I^{m-\nu}D^m x(\tau) \\ &= \frac{1}{\Gamma(m-\nu)}\int_0^{\tau}(\tau-\xi)^{m-\nu-1}x^{(m)}(\xi)d\xi, \end{aligned} \tag{2.1}$$

for $m-1 < \nu \leq m, m \in N, \mu > 0, x \in C_{-1}^m$.

The following are the basic properties of the operator D^{ν}:

1. $D^{\nu}I^{\nu}x(\tau) = x(\tau)$.
2. $D^{\nu}I^{\nu}x(\tau) = x(\tau) - \sum_{k=0}^{m-1} x^{(k)}(0)\frac{\tau^k}{k!}$.

Definition 2.4. *The Mittag-Leffler function E_ν with $\nu > 0$ is defined as*

$$E_\nu(z) = \sum_{m=0}^{\infty} \frac{z^\nu}{\Gamma(m\nu + 1)}. \tag{2.2}$$

Definition 2.5. *The natural transform (NT) of $x(\tau)$ is symbolized by $N\{x(\tau)\}$ presented by*

$$N\{x(\tau)\} = \int_0^\infty e^{-s\tau} x(w\tau) d\tau, \tag{2.3}$$

where s and w are the NT variables.

Definition 2.6. *The Natural transform of the Caputo fractional derivative is defined as*

$$N\left\{D_\tau^{(\nu)}\left[x(\rho,\tau)\right]\right\} = \frac{s^\nu}{w^\nu} N\{x(\rho,\tau)\} - \sum_{j=0}^{n-1} s^{\nu-k-1} w^{-(\nu-k)} x^{(k)}(\rho, 0). \tag{2.4}$$

Some basic properties of the natural transform are defined as below:

1. $N\{1\} = \dfrac{1}{s^\nu}$.
2. $N\{\tau^\nu\} = \dfrac{\Gamma(\nu+1)w^\nu}{s^{\nu+1}}$.

3. Natural Homotopy Perturbation Method

Let us consider the following fractional partial differential equation:

$$D_\tau^{(\nu)}\left[x(\rho,\tau)\right] + R\left[x(\rho,\tau)\right] + F\left[x(\rho,\tau)\right] = g(\rho,\tau), \tag{3.1}$$

with $n - 1 < \nu \leq n$ and subject to initial conditions

$$\frac{\partial^r x(\rho, 0)}{\partial \tau^r} = x^{(r)}(\rho, 0), \quad r = 0, 1, ... \tag{3.2}$$

where $D_\tau^{(\nu)} = \dfrac{\partial^\nu}{\partial \tau^\nu}$ denote Caputo fractional derivative, R is linear LFDOs, F is nonlinear LFDOs and g is an inhomogenuous term.

Taking the NT on both side of (3.1), we construct

$$N\left\{D_\tau^{(\nu)}\left[x(\rho,\tau)\right]\right\} + N\left\{R\left[x(\rho,\tau)\right]\right\} + N\left\{F\left[x(\rho,\tau)\right]\right\} = N\{g(\rho,\tau)\}, \tag{3.3}$$

or equivalent

$$\begin{aligned} N\{x(\rho,\tau)\} &= \frac{w^\nu}{s^\nu} \sum_{j=0}^{n-1} s^{\nu-k-1} w^{-(\nu-k)} x^{(k)}(\rho, 0) + \\ & \frac{w^\nu}{s^\nu}\left(N\{g(\rho,\tau)\} - N\left\{R\left[x(\rho,\tau)\right]\right\} - N\left\{F\left[x(\rho,\tau)\right]\right\}\right). \end{aligned} \tag{3.4}$$

Now applying inverse NT on both side of (3.4), we get

$$\begin{aligned} x(\rho,\tau) &= N^{-1}\left[\frac{w^\nu}{s^\nu} \sum_{j=0}^{n-1} s^{\nu-k-1} w^{-(\nu-k)} x^{(k)}(\rho, 0)\right] + \\ & N^{-1}\left[\frac{w^\nu}{s^\nu}\left(N\{g(\rho,\tau)\} - N\left\{R\left[x(\rho,\tau)\right]\right\} - N\left\{F\left[x(\rho,\tau)\right]\right\}\right)\right]. \end{aligned} \tag{3.5}$$

Now we apply the HPM.

$$x(\rho,\tau)=\sum_{n=0}^{\infty}p^{n}x_{n}(\rho,\tau), \tag{3.6}$$

and the nonlinear terms $F\left[x(\rho,\tau)\right]$ is decomposed as:

$$F\left[x(\rho,\tau)\right]=\sum_{n=0}^{\infty}p^{n}H_{n}(x), \tag{3.7}$$

where $H_n(x)$ is the He's polynomial and be computed using the following formula:

$$H_{n}(x_{1},x_{2},...,x_{n})=\frac{1}{n!}\frac{\partial^{n}}{\partial p^{n}}\left[F\left(\sum_{i=0}^{n}p^{i}x_{i}(\rho,\tau)\right)\right]_{p=0}, \quad n=0,1,2,.... \tag{3.8}$$

Substituting (3.6) and (3.7) into (3.5), we have:

$$\sum_{n=0}^{\infty}p^{n}x_{n}(\rho,\tau)=N^{-1}\left[\frac{w^{\nu}}{s^{\nu}}\sum_{j=0}^{n-1}s^{\nu-k-1}w^{-(\nu-k)}x^{(k)}(\rho,0)\right]+$$
$$N^{-1}\left[\frac{w^{\nu}}{s^{\nu}}\left(N\left\{g(\rho,\tau)\right\}-N\left\{R\left[\sum_{n=0}^{\infty}p^{n}x_{n}\right]\right\}-N\left\{\sum_{n=0}^{\infty}p^{n}H_{n}\right\}\right)\right]. \tag{3.9}$$

Using the coefficient of the likes powers of p in (3.9), the following approximations are obtained:

$$p^{0}:x_{0}(\rho,\tau)=N^{-1}\left[\frac{w^{\nu}}{s^{\nu}}\sum_{j=0}^{n-1}s^{\nu-k-1}w^{-(\nu-k)}x^{(k)}(\rho,0)\right]+N^{-1}\left[\frac{w^{\nu}}{s^{\nu}}\left(N\left\{g(\rho,\tau)\right\}\right)\right],$$
$$p^{n+1}:x_{n+1}(\rho,\tau)=N^{-1}\left[\frac{w^{\nu}}{s^{\nu}}\left(N\left\{R\left[x_{n}\right]\right\}-N\left\{H_{n}\right\}\right)\right], \quad n\geq 0. \tag{3.10}$$

Hence, the series solution of (3.1) is given by:

$$x(\rho,\tau)=lim_{M\longrightarrow\infty}\sum_{n=0}^{M}x_{n}(\rho,\tau), \tag{3.11}$$

4. Applications

4.1. Example

Let us consider the fractional gas dynamic equation of the form

$$D_{\tau}^{\nu}x(\rho,\tau)=x(\rho,\tau)-\frac{1}{2}\frac{\partial x^{2}(\rho,\tau)}{\partial\rho}-x^{2}(\rho,\tau), \tag{4.1}$$

subject to initial condition

$$x(\rho,0)=e^{-\rho}. \tag{4.2}$$

Taking the Natural transform on both sides of (3.5), we get:

$$N\left\{x(\rho,\tau)\right\}=\frac{1}{s^{\nu}}e^{-\rho}+\frac{w^{\nu}}{s^{\nu}}\left[N\left\{x(\rho,\tau)-\frac{1}{2}\frac{\partial x^{2}(\rho,\tau)}{\partial\rho}-x^{2}(\rho,\tau)\right\}\right]. \tag{4.3}$$

Then by taking the inverse Natural transform of (4.3), we obtain

$$x(\rho,\tau) = e^{-\rho} + N^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{x(\rho,\tau) - \frac{1}{2}\frac{\partial x^2(\rho,\tau)}{\partial \rho} - x^2(\rho,\tau)\right\}\right]\right). \tag{4.4}$$

We now assume that

$$\begin{aligned} x(\rho,\tau) &= \sum_{n=0}^{\infty} p^n x_n(\rho,\tau), \\ \frac{\partial x^2(\rho,\tau)}{\partial \rho} &= \sum_{n=0}^{\infty} p^n H_n(x), \\ x^2(\rho,\tau) &= \sum_{n=0}^{\infty} p^n G_n(x), \end{aligned} \tag{4.5}$$

where H_n and G_n are the Adomian polynomials.

Then by using (4.4), we can re-write (4.4) in the form:

$$\sum_{n=0}^{\infty} p^n x_n = e^{-\rho} + N^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{\sum_{n=0}^{\infty} p^n x_n - \frac{1}{2}\sum_{n=0}^{\infty} p^n H_n - \sum_{n=0}^{\infty} p^n G_n\right\}\right]\right). \tag{4.6}$$

By comparing both sides of (4.6), we can easily generate the recursive relation as follows:

$$\begin{aligned} p^0 : x_0(\rho,\tau) &= e^{-\rho}. \\ p^1 : x_1(\rho,\tau) &= N^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{x_0 - \frac{1}{2}H_0 - G_0\right\}\right]\right) \\ &= N^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{x_0(\rho,\tau) - \frac{1}{2}\frac{\partial x_0^2(\rho,\tau)}{\partial \rho} - x_0^2(\rho,\tau)\right\}\right]\right) \\ &= N^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{e^{-\rho}\right\}\right]\right) \\ &= e^{-\rho}N^{-1}\left(\frac{w^{\nu}}{s^{2\nu}}\right) \\ &= e^{-\rho}\frac{\tau^{\nu}}{\Gamma(1+\nu)}. \end{aligned}$$

$$\begin{aligned} p^2 : x_2(\rho,\tau) &= N^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{x_1 - \frac{1}{2}H_1 - G_1\right\}\right]\right) \\ &= N^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{x_1(\rho,\tau) - \frac{\partial (x_0 x_1)}{\partial \rho} - 2x_0 x_1\right\}\right]\right) \\ &= N^{-1}\left(\frac{w^{\nu}}{s^{\nu}}\left[N\left\{e^{-\rho}\frac{\tau^{\nu}}{\Gamma(1+\nu)}\right\}\right]\right) \\ &= e^{-\rho}N^{-1}\left(\frac{w^{2\nu}}{s^{3\nu}}\right) \\ &= e^{-\rho}\frac{\tau^{2\nu}}{\Gamma(1+2\nu)}. \end{aligned}$$

$$
\begin{aligned}
p^3 : x_3(\rho,\tau) &= N^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{x_2 - \frac{1}{2}H_2 - G_2\right\}\right]\right)\\
&= N^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{x_2 - \frac{\partial(x_0x_2)}{\partial\rho} - \frac{1}{2}\frac{\partial x_1}{\partial\rho} - 2x_0x_2 - x_1^2\right\}\right]\right)\\
&= N^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{e^{-\rho}\frac{\tau^{2\nu}}{\Gamma(1+2\nu)}\right\}\right]\right)\\
&= e^{-\rho}N^{-1}\left(\frac{w^{3\nu}}{s^{4\nu}}\right)\\
&= e^{-\rho}\frac{\tau^{3\nu}}{\Gamma(1+3\nu)}.\\
&\vdots
\end{aligned}
$$

Hence, the approximate series solution is given by

$$
\begin{aligned}
x(\rho,\tau) &= e^{-\rho}\left[1 + \frac{\tau^\nu}{\Gamma(1+\nu)} + \frac{\tau^{2\nu}}{\Gamma(1+2\nu)} + \frac{\tau^{3\nu}}{\Gamma(1+3\nu)} + \cdots\right]\\
&= e^{-\rho}\sum_{r=0}^{\infty}\frac{\tau^{r\nu}}{\Gamma(1+r\nu)}\\
&= e^{-\rho}E_\nu(\tau). \qquad (4.7)
\end{aligned}
$$

For $\nu = 1$ the above solution reduces to exact solution $x(\rho,\tau) = e^{\tau-\rho}$. From Eqs. (4.7), the approximate solution of the given problem (4.1) by using NHPM is the same results as that obtained by DTM [34] and it clearly appears that the approximate solution remains closed form to exact solution.

4.2. Example

Consider the inhomogeneous fractional gas dynamics equation of the form

$$
D_\tau^\nu x(\rho,\tau) = x(\rho,\tau) - \frac{1}{2}\frac{\partial x^2(\rho,\tau)}{\partial\rho} - x^2(\rho,\tau) - e^{\tau-\rho}, \qquad (4.8)
$$

subject to initial condition

$$
x(\rho,0) = 1 - e^{-\rho}. \qquad (4.9)
$$

Taking the Natural transform on both sides of (4.8), we get:

$$
\begin{aligned}
N\left\{x(\rho,\tau)\right\} &= \frac{1-e^{-\rho}}{s^\nu} + \frac{w^\nu e^{-\rho}}{s^\nu(s^\nu - w^\nu)} +\\
&\quad \frac{w^\nu}{s^\nu}\left[N\left\{x(\rho,\tau) - \frac{1}{2}\frac{\partial x^2(\rho,\tau)}{\partial\rho} - x^2(\rho,\tau)\right\}\right]. \qquad (4.10)
\end{aligned}
$$

Applying the inverse Natural transform of (4.10), we obtain

$$
x(\rho,\tau) = 1 - e^{-\rho}E_\nu(\tau) + N^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{x(\rho,\tau) - \frac{1}{2}\frac{\partial x^2(\rho,\tau)}{\partial\rho} - x^2(\rho,\tau)\right\}\right]\right). \qquad (4.11)
$$

We now suppose that

$$x(\rho,\tau)=\sum_{n=0}^{\infty}p^n x_n(\rho,\tau). \tag{4.12}$$

Then by using (4.12), we can re-write (4.11) in the form:

$$\begin{aligned}\sum_{n=0}^{\infty}p^n x_n &= 1-e^{-\rho}E_\nu(\tau)+\\ &N^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{\sum_{n=0}^{\infty}p^n x_n-\frac{1}{2}\sum_{n=0}^{\infty}p^n H_n-\sum_{n=0}^{\infty}p^n G_n\right\}\right]\right).\end{aligned} \tag{4.13}$$

where H_n and G_n are the Adomian polynomial which represent the nonlinear terms $\frac{\partial x^2(\rho,\tau)}{\partial\rho}$ and $x^2(\rho,\tau)$ respectively.

By comparing both sides of (4.13), we can easily generate the recursive relation as follows:

$$\begin{aligned}p^0: x_0(\rho,\tau) &= 1-e^{-\rho}E_\nu(\tau).\\ p^1: x_1(\rho,\tau) &= N^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{x_0-\frac{1}{2}H_0-G_0\right\}\right]\right)\\ &= N^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{x_0(\rho,\tau)-\frac{1}{2}\frac{\partial x_0^2(\rho,\tau)}{\partial\rho}-x_0^2(\rho,\tau)\right\}\right]\right)\\ &= 0.\\ \\ p^2: x_2(\rho,\tau) &= N^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{x_1-\frac{1}{2}H_1-G_1\right\}\right]\right)\\ &= N^{-1}\left(\frac{w^\nu}{s^\nu}\left[N\left\{x_1(\rho,\tau)-\frac{\partial(x_0x_1)}{\partial\rho}-2x_0x_1)\right\}\right]\right)\\ &= 0.\end{aligned} \tag{4.14}$$

Hence, the approximate series solution is given by

$$x(\rho,\tau)=1-e^{-\rho}E_\nu(\tau). \tag{4.15}$$

For $\nu=1$ the above solution reduces to exact solution $x(\rho,\tau)=1-e^{\tau-\rho}$. From Eqs. (4.15) , the approximate solution of the given problem (4.8) by using NHPM is the same results as that obtained by DTM [34] and it clearly appears that the approximate solution remains closed form to exact solution.

5. Conclusion

In this work, we suggest new technique called the Natural homotopy perturbation method for solving fractional gas dynamics equation. The new technique provides an elegant series solution which converge very rapidly with reduced computational size. The results obtained by the NHPM are in excellent agreement with the results of the existing methods. Thus, the proposed technique is a powerful, reliable and efficient mathematical tool for solving nonlinear PDEs.

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