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The structure of ideals, point derivations, amenability and weak amenability of extended Lipschitz algebras

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Abstract

Let (X, d) be a compact metric space and let K be a nonempty compact subset of X. Let $\alpha \in (0, 1]$ and let $\mathrm{Lip}(X, K, d^\alpha)$ denote the Banach algebra of all continuous complex-valued functions f on X for which

$$p_{(K,d^\alpha)}(f) = \sup\{\frac{|f(x) - f(y)|}{d^\alpha(x, y)} : x, y \in K, x \neq y\} < \infty$$

when it is equipped with the algebra norm $||f||_{\mathrm{Lip}(X,K,d^\alpha)} = ||f||_X + p_{(K,d^\alpha)}(f)$, where $||f||_X = \sup\{|f(x)| : x \in X\}$. In this paper we first study the structure of certain ideals of $\mathrm{Lip}(X, K, d^\alpha)$. Next we show that if K is infinite and $\mathrm{int}(K)$ contains a limit point of K then $\mathrm{Lip}(X, K, d^\alpha)$ has at least a nonzero continuous point derivation and applying this fact we prove that $\mathrm{Lip}(X, K, d^\alpha)$ is not weakly amenable and amenable.

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1. Introduction and preliminaries

Let A be a complex algebra and let φ be a multiplicative linear functional on A. A linear functional D on A is called a *point derivation* on A at φ if

$$D(fg) = \varphi(f)Dg + \varphi(g)Df,$$

for all $f, g \in A$. We say that φ is a *character* on A if $\varphi(f) \neq 0$ for some $f \in A$. We denote by $\Delta(A)$ the set of all characters on A which is called the *character space* of A. For each $\varphi \in \Delta(A)$, we denote by $\ker(\varphi)$ the set of all $f \in A$ for which $\varphi(f) = 0$. Clearly, $\ker(\varphi)$ is a proper ideal of A.

Let $(A, \|\cdot\|)$ be a commutative unital complex Banach algebra. We know that φ is continuous and $\|\varphi\| = 1$ for all $\varphi \in \Delta(A)$. Moreover, $\Delta(A)$ is nonempty and it is a compact Hausdorff space with the Gelfand topology. We know that $\ker(\varphi)$ is a maximal ideal of A for all $\varphi \in \Delta(A)$ and every maximal ideal space of A has the form $\ker(\psi)$ for some $\psi \in \Delta(A)$. We denote by $\mathfrak{D}_\varphi$ the set of all continuous point derivations of A at $\varphi \in \Delta(A)$. Clearly, $\mathfrak{D}_\varphi$ is a complex linear subspace of A^*, the dual space of A. For a subset S of $\Delta(A)$, we define $\ker(S) = A$ when $S = \emptyset$ and $\ker(S) = \bigcap_{\varphi \in S} \ker(\varphi)$ when $S \neq \emptyset$. For a nonempty subset S of $\Delta(A)$ we define

$$I_A(S) = \{f \in A : \textit{there is an open set } V \textit{ in } \Delta(A) \textit{ with } S \subseteq V \textit{ such that } \varphi(f) = 0 \textit{ for all } \varphi \in V\},$$

and $J_A(S) = \overline{I_A(S)}$, the closure of $I_A(S)$ in $(A, \|\cdot\|)$. Clearly, $I_A(S)$ is an ideal of A and so $J_A(S)$ is a closed ideal of A. For an ideal I of A, the *hull* of I is the set of all $\varphi \in \Delta(A)$ for which $\varphi(f) = 0$ for all $f \in I$. We denote by $\mathrm{hull}(I)$ the hull of I. Clearly, S is contained in $\mathrm{hull}(J_A(S))$ for each nonempty subset S of $\Delta(A)$.

Let $(A, \|\cdot\|)$ be a commutative unital complex Banach algebra. Then A is called *regular* if for every proper closed subset S of $\Delta(A)$ and each $\varphi \in \Delta(A) \setminus S$, there exists an f in A such that $\hat{f}(\varphi) = 1$ and $\hat{f}(S) = \{0\}$, where $\hat{f}$ is the Gelfand transform of f.

The following theorem is due to Šilov. For a proof see [13] or [8].

Theorem 1.1. Let $(A, \|\cdot\|)$ be a regular commutative unital complex Banach algebra and S be a nonempty closed subset of $\Delta(A)$. Then $\mathrm{hull}(J_A(S)) = S$.

Let $(A, \|\cdot\|)$ be a commutative unital complex Banach algebra and I be a proper ideal of A. We say that A is *primary* if it is contained in exactly one maximal ideal of A. If $\varphi \in \Delta(A)$ and I is a primary ideal of A such that $\mathrm{hull}(I) = \{\varphi\}$, then I is called *primary* at φ. If A is regular then a closed ideal I of A is primary at $\varphi \in \Delta(A)$ if and only if $J_A(\{\varphi\}) \subseteq I \subseteq \ker(\varphi)$.

Let A be a complex algebra and $\mathfrak{X}$ be an A-module with respect to module operations $(a, x) \to x \cdot a : A \times \mathfrak{X} \to \mathfrak{X}$ and $(a, x) \to a \cdot x : A \times \mathfrak{X} \to \mathfrak{X}$. We say that $\mathfrak{X}$ is *symmetric* or *commutative* if $a \cdot x = x \cdot a$ for all $a \in A$ and $x \in \mathfrak{X}$. A complex linear map $D : A \to \mathfrak{X}$ is called an $\mathfrak{X}$-*derivation* on A if $D(ab) = Da \cdot b + a \cdot Db$ for all $a, b \in A$. For each $x \in \mathfrak{X}$, the map $\delta_x : A \to \mathfrak{X}$ defined by

$$\delta_x(a) = a \cdot x - x \cdot a \qquad (a \in A),$$

is an $\mathfrak{X}$-derivation on A. An $\mathfrak{X}$-derivation D on A is called *inner* $\mathfrak{X}$-derivation on A if $D = \delta_x$ for some $x \in X$.

Let $(A, \|\cdot\|)$ be a complex Banach algebra and $(\mathfrak{X}, \|\cdot\|)$ be an A-module. We say that $\mathfrak{X}$ is a *Banach A-module* if there exists a constant k such that

$$\|a \cdot x\| \leq k\|a\|\|x\|, \qquad \|x \cdot a\| \leq k\|a\|\|x\|,$$

for all $a \in A$ and $x \in \mathfrak{X}$.

If $\mathfrak{X}$ is a Banach A-module then $\mathfrak{X}^*$, the dual space of $\mathfrak{X}$, is a Banach A-module with the natural module operations

$$(a \cdot \lambda)(x) = \lambda(x \cdot a),\ (\lambda \cdot a)(x) = \lambda(a \cdot x) \qquad (a \in A,\ \lambda \in \mathfrak{X}^*,\ x \in \mathfrak{X}).$$

Let A be a complex Banach algebra and $\mathfrak{X}$ be a Banach A-module. The set of all continuous $\mathfrak{X}$-derivations on A is a complex linear space, denoted by $\mathcal{Z}^1(A, \mathfrak{X})$. The set of all inner $\mathfrak{X}$-derivations on A is a complex linear subspace of $\mathcal{Z}^1(A, \mathfrak{X})$, denoted by $\mathcal{B}^1(A, \mathfrak{X})$. The quotient space $\mathcal{Z}^1(A, \mathfrak{X})/\mathcal{B}^1(A, \mathfrak{X})$ is denoted by $\mathcal{H}^1(A, \mathfrak{X})$ and called the *first cohomology group* of A with coefficients in $\mathfrak{X}$.

Definition 1.2. Let A be a complex Banach algebra. We say that A is *amenable* if $\mathcal{H}^1(A, \mathfrak{X}^*) = \{0\}$ for every Banach A-module $\mathfrak{X}$.

The notion of amenability of complex Banach algebras was first given by Johnson in [6].

Definition 1.3. Let A be a complex Banach algebra. We say that A is *weakly amenable* if $\mathcal{H}^1(A, A^*) = \{0\}$, that is, every continuous A^*-derivation on A is inner.

The notion of weak amenability was first defined for commutative complex Banach algebras by Bade, Curtis and Dales in [4] as the following:

A commutative complex Banach algebra A is called weakly amenable if $\mathcal{Z}^1(A, \mathfrak{X}) = \{0\}$ for every symmetric Banach A-module $\mathfrak{X}$.

Later Johnson extended the definition of weak amenability to any complex Banach algebra (not necessarily commutative) as introduced in Definition 1.3. Of course, these definitions are equivalent when A is commutative (See [4, Theorem 1.5] and [7, Theorem 3.2]).

Let X be a compact Hausdorff space. We denote by $C(X)$ the commutative unital complex Banach algebra consisting of all complex-valued continuous functions on X under the *uniform norm* on X which is defined by

$$\|f\|_X = \sup\{|f(x)| : x \in X\} \qquad (f \in C(X)).$$

A complex *Banach function algebra* on X is a complex subalgebra A of $C(X)$ such that A separates the points of X, contains 1_X (the constant function on X with value 1) and it is a unital Banach algebra under an algebra norm $\|\cdot\|$. Since $C(X)$ separates the points of X by Urysohn's lemma [11, Theorem 2.12], $1_X \in C(X)$ and $(C(X), \|\cdot\|_X)$ is a unital complex Banach algebra, we deduce that $(C(X), \|\cdot\|_X)$ is a complex Banach function algebra on X.

Let $(A, \|\cdot\|)$ be a complex Banach function algebra on X. For each $x \in X$, the map $e_x : A \to \mathbb{C}$, defined by $e_x(f) = f(x)$ $(f \in A)$, is an element of $\Delta(A)$ which is called the *evaluation character* on A at x. It follows that A is semisimple and $\|f\|_X \leq \|\hat{f}\|_{\Delta(A)}$ for all $f \in A$. Moreover, the map $E_X : X \to \Delta(A)$ defined by $E_X(x) = e_x$ is injective and continuous. If E_X is surjective, then we say that A is *natural*. In this case, E_X is a homeomorphism from X onto $\Delta(A)$. It is known that if $(A, \|\cdot\|)$ is a self-adjoint inverse-closed Banach function algebra on X then A is natural. Therefore, $(C(X), \|\cdot\|_X)$ is natural.

Let A be a complex Banach function algebra on a compact Hausdorff X. If A is regular, then for each proper closed subset E of X and each $x \in X \setminus E$ there exists a function f in A such that

$f(x) = 1$ and $f(E) = \{0\}$. Moreover, the converse of the above statement holds whenever A is natural.

Let (X, d) be a metric space and Y be a nonempty subset of X. A complex-valued function f on Y is called a *Lipschitz function* on (Y, d) if there exists a positive constant M such that $|f(x) - f(y)| \leq Md(x, y)$ for all $x, y \in Y$.

The following lemma is a version of Urysohn's lemma for Lipschitz functions.

Lemma 1.4. *Let (X, d) be a metric space, H be a compact subset of X and L be a closed subset of X such that $H \cap L = \emptyset$. Then there exists a real-valued Lipschitz function h on (X, d) satisfying $0 \leq h(x) \leq 1$ for all $x \in X$, $h(x) = 1$ for all $x \in H$ and $h(x) = 0$ for all $x \in L$.*

Proof . It is sufficient to define $f = 0$ when $H = \emptyset$. Let $H \neq \emptyset$ and let $x \in H$. Since $H \subseteq X \setminus L$ and $X \setminus L$ is an open set in (X, d), then there exists a positive number r_x such that

$$\{y \in X : d(y, x) < r_x\} \subseteq X \setminus L.$$

Let $0 < \delta_x < r_x$. Then $H \subseteq \bigcup_{x \in H} \{y \in X : d(y, x) < \delta_x\}$. Since H is compact in (X, d), there exist $x_1, \ldots, x_n \in H$ such that

$$H \subseteq \bigcup_{j=1}^{n} \{y \in X : d(y, x_j) < \delta_{x_j}\}.$$

Let $j \in \{1, \ldots, n\}$. We define $g_j : X \to \mathbb{C}$ by

$$g_j(x) = \begin{cases} 1 & d(x, x_j) < \delta_{x_j}, \\ \frac{r_{x_j} - d(x, x_j)}{r_{x_j} - \delta_{x_j}} & \delta_{x_j} \leq d(x, x_j) < r_{x_j}, \\ 0 & r_{x_j} \leq d(x, x_j). \end{cases}$$

Clearly, $0 \leq g_j(x) \leq 1$ for all $x \in X$. By simple calculations, we can show that

$$|g_j(x) - g_j(y)| \leq \frac{1}{r_{x_j} - \delta_{x_j}} d(x, y),$$

for all $x, y \in X$. Therefore, g_j is a bounded real-valued Lipschitz function on (X, d).

Let $h_1 = g_1$, $h_2 = (1 - g_1)g_2$, $\ldots$, $h_n = (1 - g_1) \ldots (1 - g_{n-1})g_n$. If $j \in \{1, \ldots, n\}$ then h_j is a bounded Lipschitz function on (X, d). Set $h = h_1 + \ldots + h_n$. Then h is a bounded Lipschitz function on (X, d) and $h = 1 - (1 - g_1) \ldots (1 - g_n)$. Moreover, $0 \leq h(x) \leq 1$ for all $x \in X$, $h(x) = 1$ for all $x \in H$ and $h(x) = 0$ for all $x \in L$. □

Let (X, d) be a metric space. For $\alpha \in (0, 1]$ we define the map $d^\alpha : X \times X \to \mathbb{R}$, by $d^\alpha(x, y) = (d(x, y))^\alpha$ $(x, y \in X)$. Then d^α is a metric on X and the induced topology on X by d^α coincides with the induced topology on X by d.

Let (X, d) be a compact metric space and $\alpha \in (0, 1]$. We denote by $\text{Lip}(X, d^\alpha)$ the set of all complex-valued Lipschitz function on (X, d^α). Then $\text{Lip}(X, d^\alpha)$ is a complex subalgebra of $C(X)$ and $1_X \in \text{Lip}(\mathrm{X}, \mathrm{d}^\alpha)$. Moreover, $\text{Lip}(X, d^\alpha)$ separates the points of X. For a nonempty subset K of X and a complex-valued function f on K, we set

$$p_{(K, d^\alpha)}(f) = \sup\{\frac{|f(x) - f(y)|}{d^\alpha(x, y)} : x, y \in K, x \neq y\}.$$

Clearly, $f \in \mathrm{Lip}(X, d^\alpha)$ if and only if $p_{(X,d^\alpha)}(f) < \infty$. The d^α-Lipschitz norm $\|\cdot\|_{\mathrm{Lip}(X,d^\alpha)}$ on $\mathrm{Lip}(X, d^\alpha)$ is defined by

$$\|f\|_{\mathrm{Lip}(X,d^\alpha)} = \|f\|_X + p_{(X,d^\alpha)}(f) \qquad (f \in \mathrm{Lip}(X, d^\alpha)).$$

Then $(\mathrm{Lip}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$ is a commutative unital complex Banach algebra.

Lipschitz algebras were first studied by Sherbert in [11]. The structure of ideals and point derivations of Lipschitz algebras studied by Sherbert in [12].

Let (X, d) be a compact metric space, K be a nonempty compact subset of X and $\alpha \in (0, 1]$. We denote by $\mathrm{Lip}(X, K, d^\alpha)$ the set of $f \in C(X)$ for which $p_{(K,d^\alpha)}(f) < \infty$. In fact,

$$\mathrm{Lip}(X, K, d^\alpha) = \{f \in C(X) : f|_K \in \mathrm{Lip}(K, d^\alpha)\}.$$

Clearly, $\mathrm{Lip}(X, d^\alpha) \subseteq \mathrm{Lip}(X, K, d^\alpha)$ and $\mathrm{Lip}(X, K, d^\alpha) = \mathrm{Lip}(X, d^\alpha)$ if and only if $X \setminus K$ is finite. Moreover, $\mathrm{Lip}(X, K, d^\alpha)$ is a self-adjoint inverse-closed complex subalgebra of $C(X)$. It is easy to see that $\mathrm{Lip}(X, K, d^\alpha)$ is a complex subalgebra of $C(X)$ and a unital Banach algebra under the algebra norm $\|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)}$ defined by

$$\|f\|_{\mathrm{Lip}(X,K,d^\alpha)} = \|f\|_X + p_{(K,d^\alpha)}(f) \qquad (f \in \mathrm{Lip}(X, K, d^\alpha)).$$

Therefore, $(\mathrm{Lip}(X, K, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ is a natural Banach function algebra on X. This algebra is called *extended Lipschitz algebra* of order α on (X, d) with respect to K. It is clear that $\mathrm{Lip}(X, K, d^\alpha) = C(X)$ if and only if K is finite.

Extended Lipschitz algebras were first introduced in [5]. Some properties of these algebras have been studied in [1, 2, 3]. It is shown [3, Proposition 2.1] that $\mathrm{Lip}(X, K, d^\alpha)$ is regular.

Let (X, d) be a metric space, f be a real-valued function on X and $k > 0$. The real-valued function $T_k f$ on X defined by

$$(T_k f)(x) = \begin{cases} -k & f(x) < -k, \\ f(x) & -k \leqslant f(x) \leqslant k, \\ k & f(x) > k, \end{cases} \qquad (x \in X)$$

is called the *truncation* of f at k.

The following result is useful in the sequel and its proof is straightforward.

Theorem 1.5. Let (X, d) be a compact metric space, K be a nonempty compact subset of X and $\alpha \in (0, 1]$. Suppose that f is a real-valued function in $\mathrm{Lip}(X, K, d^\alpha)$ and $k > 0$. Then $T_k f$ is on element of $\mathrm{Lip}(X, K, d^\alpha)$.

In Section 2, we determine the structure of certain ideals of extended Lipschitz algebras. In Section 3, we show that certain extended Lipschitz algebras have a nonzero continuous point derivation. In Section 4, we show that certain extended Lipschitz algebras are not weakly amenable and amenable.

2. Certain ideals of extended Lipschitz algebras

Throughout this section we always assume that (X, d) is a compact metric space, K is a nonempty compact subset of X and $\alpha \in (0, 1]$.

For an ideal I of a commutative complex algebra A, we define

$$I^2 = \{\sum_{i=1}^{n} f_i g_i : n \in \mathbb{N}, \quad f_i, g_i \in I \quad (i \in \{1, 2, \ldots, n\})\}.$$

Clearly, I^2 is an ideal of A and $I^2 \subseteq I$.

We denote the interior of K in (X, d) by $\mathrm{int}(K)$. When $\mathrm{int}(K) \neq \emptyset$, for a nonempty compact subset H of $\mathrm{int}(K)$, we determine the structure of $J_A(E_X(H))$ and show that

$$J_A(E_X(H)) = \overline{(\ker(E_X(H)))^2} = \bigcap_{x \in H} J_A(\{e_x\}),$$

where $A = \mathrm{Lip}(X, K, d^\alpha)$. We also characterize closed primary ideals of $\mathrm{Lip}(X, K, d^\alpha)$ at interior points of K.

Lemma 2.1. *Let $A = \mathrm{Lip}(X, K, d^\alpha)$ and H be a nonempty compact subset of K. Let B be the set of all $f \in A$ satisfying:*

(i) $f(H)=\{0\}$,

(ii) for each $\varepsilon > 0$ there exists an open set U in (X, d) with $H \subseteq U$ such that $\frac{|f(x)-f(y)|}{d^\alpha(x,y)} < \varepsilon$ for all $x, y \in U \cap K$ with $x \neq y$.

Then B is a closed complex linear subspace of A.

Proof . Clearly, B is a complex linear subspace of A. Let $f \in \overline{B}$, the closure of B in $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in B such that

$$\lim_{n\to\infty} \|f_n - f\|_{\mathrm{Lip}(X,K,d^\alpha)} = 0. \tag{2.1}$$

Let $x \in H$. Then $\lim\limits_{n\to\infty} f_n(x) = f(x)$ by (2.1) and $f_n(x) = 0$ for each $n \in \mathbb{N}$ by (i). Hence, $f(x) = 0$ and so f satisfies in (i).

Let $\varepsilon > 0$ be given. Then there exists a function $g \in B$ such that

$$\|f - g\|_{\mathrm{Lip}(X,K,d^\alpha)} < \frac{\varepsilon}{2}. \tag{2.2}$$

Since $g \in B$, there exists an open set U in (X, d) with $H \subseteq U$ such that for all $x, y \in U \cap K$ with $x \neq y$ we have

$$\frac{|g(x) - g(y)|}{d^\alpha(x, y)} < \frac{\varepsilon}{2}. \tag{2.3}$$

Let $x, y \in U \cap K$ with $x \neq y$. Applying (2.2) and (2.3) we have

$$\begin{aligned} \frac{|f(x)-f(y)|}{d^\alpha(x,y)} &\leqslant \frac{|(f-g)(x)-(f-g)(y)|}{d^\alpha(x,y)} + \frac{|g(x)-g(y)|}{d^\alpha(x,y)} \\ &\leqslant p_{(K,d^\alpha)}(f-g) + \frac{\varepsilon}{2} \\ &\leqslant \|f-g\|_{\mathrm{Lip}(X,K,d^\alpha)} + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, f satisfies in (ii). Therefore, $f \in B$ and so B is closed in $(A, \|\cdot\|_{\mathrm{lip}(X,K,d^\alpha)})$. $\square$

The following lemma was first given by Šhilov in [13].

Lemma 2.2. Let $(A, \|\cdot\|)$ be a regular commutative unital complex Banach algebra and L be a nonempty compact subset of $\Delta(A)$. An element $f \in A$ belongs to $J_A(L)$ if and only if there is a sequence $\{f_n\}_{n=1}^{\infty}$ in A satisfying:

(a) for each $n \in \mathbb{N}$ there exists an open set V in $\Delta(A)$ with $L \subseteq U_n$ such that $f_n|_{U_n} = f|_{U_n}$,

(b) $\lim\limits_{n\to\infty} \|f_n\| = 0$.

Theorem 2.3. *Suppose that* $\mathrm{int}(K) \neq \emptyset$ *and* $A = \mathrm{Lip}(X, K, d^{\alpha})$. *Let* H *be a nonempty compact subset of* $\mathrm{int}(K)$. *Then* $J_A(E_X(H))$ *is the set of all* $f \in A$ *satisfying:*

(i) *$f(H)=\{0\}$,*

(ii) *for each $\varepsilon > 0$ there exists an open set U in X with $H \subseteq U$ such that $\frac{|f(x)-f(y)|}{d^{\alpha}(x,y)} < \varepsilon$ for all $x, y \in U \cap K$ with $x \neq y$.*

Proof . Let B be the set of all $f \in A$ satisfying (i) and (ii). It is enough to show that

$$J_A(E_X(H)) = B. \tag{2.4}$$

Let $f \in I_A(E_X(H))$. Then there exists an open set V in $\Delta(A)$ with $E_X(H) \subseteq V$ such that $\widehat{f}(V) = \{0\}$. Set $U = E_X^{-1}(V)$. Since $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^{\alpha})})$ is a natural Banach function algebra on X, the map $E_X : X \to \Delta(A)$ is a homeomorphism from X onto $\Delta(A)$. Hence, U is an open set in (X, d) with $H \subseteq U$ and $f(U) = \{0\}$ and so $f(H) = \{0\}$. Thus f satisfies in (i).

Let $\varepsilon > 0$ be given. Suppose that $x, y \in U \cap K$ with $x \neq y$. Then

$$\frac{|f(x) - f(y)|}{d^{\alpha}(x, y)} = 0 < \varepsilon.$$

Hence, f satisfies in (ii). Therefore, $f \in B$. So

$$I_A(E_X(H)) \subseteq B. \tag{2.5}$$

On the other hand, B is closed in $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^{\alpha})})$ by Lemma 2.1. Hence, by (2.5) we have

$$J_A(E_X(H)) \subseteq B. \tag{2.6}$$

Let $f \in B$ such that $f(x) \geq 0$ for all $x \in X$. Since $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^{\alpha})})$ is a regular commutative unital complex Banach algebra and $E_X(H)$ is a nonempty compact subset of $\Delta(A)$, to prove $f \in J_A(E_X(H))$, by Lemma 2.2, it is enough to show that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in A satisfying:

(a) for each $n \in \mathbb{N}$ there is an open set U_n in X with $H \subseteq U_n$ such that $f_n|_{U_n} = f|_{U_n}$.

(b) $\lim\limits_{n\to\infty} f_n = 0$ in $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^{\alpha})})$.

Let $n \in \mathbb{N}$. We define $S_n = \{x \in X : \mathrm{dist}(x, H) < (\frac{1}{n})^{\frac{1}{\alpha}}\}$, $E_n = \{x \in X : f(x) < \frac{1}{n^3}\}$ and $\Omega_n = S_n \cap E_n$, where $\mathrm{dist}(x, H) = \inf\{d(x, y) : y \in H\}$. Then Ω_n is an open set in (X, d), $H \subseteq \Omega_n$, $\Omega_{n+1} \subseteq \Omega_n$ and $f(\Omega_n) \subseteq [0, \frac{1}{n^3})$. Set $S_0 = X$ and $V_n = K \cap (\Omega_n \cup (X \setminus S_{n-1}))$. We define the function $h_n : V_n \to \mathbb{R}$ by

$$h_n(x) = \begin{cases} f(x) & (x \in K \cap \Omega_n), \\ 0 & (x \in K \cap (X \setminus S_{n-1})). \end{cases}$$

Set $U_n = \mathrm{int}(K) \cap \Omega_n$. Then U_n is an open set in (X, d), $H \subseteq U_n$ and $h_n|_{U_n} = f|_{U_n}$.

We claim that for each $n \in \mathbb{N}$ we have $\|h_n\|_{V_n} \leq \frac{1}{n^3}$ and

$$\sup\{\frac{|h_n(x)-h_n(y)|}{d^\alpha(x,y)} : x,y \in V_n, x \neq y\} \leq \sup\{\frac{|f(x)-f(y)|}{d^\alpha(x,y)} : x,y \in K \cap \Omega_n\} + \frac{1}{n}.$$

Since $h_1 = f$ on $K \cap \Omega_1$, $V_1 = K \cap \Omega_1$ and $f(\Omega_1) \subseteq [0,1)$, our claim is justified when $n = 1$.

Let $n \in \mathbb{N}$ with $n \geq 2$. If $x, y \in V_n$, then

$$|h_n(x)-h_n(y)| = \begin{cases} |f(x)-f(y)| & (x,y \in K \cap \Omega_n), \\ f(x) & (x \in K \cap \Omega_n, y \in K \cap (X \setminus S_{n-1})), \\ f(y) & (x \in K \cap (X \setminus S_{n-1}), y \in K \cap \Omega_n), \\ 0 & (x,y \in K \cap (X \setminus S_{n-1})). \end{cases}$$

Let $x \in K \cap \Omega_n$ and $y \in K \cap (X \setminus S_{n-1})$. Since H is a nonempty compact set in (X,d), there exists $z_0 \in H$ such that $\operatorname{dist}(x,H) = d(x,z_0)$. So we have

$$\begin{aligned} d^\alpha(x,y) &\geq d^\alpha(y,z_0) - d^\alpha(x,z_0) \\ &\geq \tfrac{1}{n-1} - (\operatorname{dist}(x,H))^\alpha \\ &> \tfrac{1}{n-1} - \tfrac{1}{n} \\ &= \tfrac{1}{n(n-1)}. \end{aligned}$$

Moreover, $f(x) \leq \frac{1}{n^3}$. Therefore,

$$\begin{aligned} |h_n(x)-h_n(y)| &= \tfrac{f(x)}{d^\alpha(x,y)} d^\alpha(x,y) \\ &\leq \tfrac{n(n-1)}{n^3} d^\alpha(x,y) \\ &\leq \tfrac{1}{n} d^\alpha(x,y). \end{aligned}$$

The same inequality holds if $x \in K \cap (X \setminus S_{n-1})$ and $y \in K \cap \Omega_n$. Therefore,

$$\sup\{\frac{|h_n(x)-h_n(y)|}{d^\alpha(x,y)} : x,y \in V_n,\ x \neq y\} \leq \sup\{\frac{|f(x)-f(y)|}{d^\alpha(x,y)} : x,y \in K \cap \Omega_n,\ x \neq y\} + \frac{1}{n}.$$

Moreover, $\|h_n\|_{V_n} \leq \|f\|_{\Omega_n} \leq \frac{1}{n^3}$. Hence, our claim is justified when $n \in \mathbb{N}$ with $n \geq 2$.

Let $n \in \mathbb{N}$. By Sherbert's extension theorem [12, Proposition 1.4], there exists a function $g_n : K \to \mathbb{R}$ such that $g_n|_{V_n} = h_n$, $\|g_n\|_K = \|h_n\|_{V_n}$ and

$$\sup\{\frac{|g_n(x)-g_n(y)|}{d^\alpha(x,y)} : x,y \in K,\ x \neq y\} = \sup\{\frac{|h_n(x)-h_n(y)|}{d^\alpha(x,y)} : x,y \in V_n,\ x \neq y\}.$$

By Tietze's extension theorem [10, Theorem 20.4], there exists a function $f_n \in C(X)$ such that $f_n|_K = g_n$ and $\|f_n\|_X = \|g_n\|_K$. Therefore, $f_n \in \operatorname{Lip}(X,K,d^\alpha)$ and $f_n|_{U_n} = f|_{U_n}$. So (a) holds. Moreover, $\|f_n\|_X < \frac{1}{n^3}$ and

$$p_{(K,d^\alpha)}(f_n) \leq \sup\{\frac{|f(x)-f(y)|}{d^\alpha(x,y)} : x,y \in K \cap \Omega_n,\ \ x \neq y\} + \frac{1}{n}.$$

Let $\varepsilon > 0$ be given. Since f satisfies in (ii), there exists an open set U in (X,d) with $H \subseteq U$ such that $\frac{|f(x)-f(y)|}{d^\alpha(x,y)} < \frac{\varepsilon}{3}$ for all $x,y \in U \cap K$ with $x \neq y$. It is easy to see that $H = \bigcap_{n=1}^{\infty} \overline{\Omega_n}$. Since $X \setminus U$ is a compact set in (X, d), $X \setminus \overline{\Omega_n}$ is an open set in (X, d) for each $n \in \mathbb{N}$ and $\Omega_{n+1} \subseteq \Omega_n$ for all $n \in \mathbb{N}$, we deduce that there exists $N \in \mathbb{N}$ with $\frac{1}{N} < \frac{\varepsilon}{3}$ and $\overline{\Omega_N} \subseteq U$. So for all $n \in \mathbb{N}$ with $n \geq N$ we have

$$\begin{aligned} \|f_n\|_{\mathrm{Lip}(X,K,d^\alpha)} &= \|f_n\|_X + p_{(K,d^\alpha)}(f_n) \\ &\leq \tfrac{1}{n^3} + \sup\{\tfrac{|f(x)-f(y)|}{d^\alpha(x,y)} : x, y \in K \cap \Omega_n, x \neq y\} + \tfrac{1}{n} \\ &\leq \tfrac{1}{n} + \tfrac{\varepsilon}{3} + \tfrac{1}{n} \\ &< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

Hence, $\lim_{n\to\infty} \|f_n\|_{\mathrm{Lip}(X,K,d^\alpha)} = 0$ and so (b) holds.

It is clear that B is a self-adjoint complex subspace of $\mathrm{Lip}(X, K, d^\alpha)$. Hence, $\mathrm{Re}\, f, \mathrm{Im}\, f \in B$. On the other hand, $|g| \in B$ whenever $g \in B$. Therefore, $g^+, g^- \in B$ if $g \in B$ and g is real-valued where $g^+ = \frac{1}{2}(|g| + g)$ and $g^- = \frac{1}{2}(|g| - g)$.

Let $f \in B$. Set $f_1 = (\mathrm{Re}\, f)^+$, $f_2 = (\mathrm{Re}\, f)^-$, $f_3 = (\mathrm{Im}\, f)^+$ and $f_4 = (\mathrm{Im}\, f)^-$. Then $f_j \in B$ and $f_j \geq 0$ for all $j \in \{1, 2, 3, 4\}$. By above argument, $f_j \in J_A(E_X(H))$ for all $j \in \{1, 2, 3, 4\}$. Since, $J_A(E_X(H))$ is a complex linear subspace of A and $f = (f_1 - f_2) + i(f_3 - f_4)$, we deduce that $f \in J_A(E_X(H))$. So

$$B \subseteq J_A(E_X(H)). \tag{2.7}$$

From (2.6) and (2.7) we have $J_A(E_X(H)) = B$. Hence, the proof is complete. □

Theorem 2.4. *Suppose that* $\mathrm{int}(K) \neq \emptyset$ *and* $A = \mathrm{Lip}(X, K, d^\alpha)$. *Let* H *be a nonempty compact subset of* $\mathrm{int}(K)$. *Then*

$$J_A(E_X(H)) = \overline{(\ker(E_X(H)))^2}.$$

Proof . Let g be a real-valued function in $\ker(E_X(H))$ and let $f = g^2$. Let $n \in \mathbb{N}$. We define $g_n = T_{\frac{1}{\sqrt{n}}} g$, $f_n = T_{\frac{1}{n}} f$ and

$$U_n = \{x \in X : \quad |f(x)| < \frac{1}{n}\}.$$

Then $g_n, f_n \in A$ by Theorem 1.5, $\|g_n\|_X \leq \frac{1}{\sqrt{n}}$, $p_{(K,d^\alpha)}(g_n) \leq p_{(K,d^\alpha)}(g)$, $f_n = (g_n)^2$, $\|f_n\|_X \leq \frac{1}{n}$, U_n is an open set in (X, d) with $H \subseteq U_n$ and $f_n|_{U_n} = f|_{U_n}$. Moreover, for all $x, y \in K$ with $x \neq y$ we have

$$\begin{aligned} \tfrac{|f_n(x)-f_n(y)|}{d^\alpha(x,y)} &= \tfrac{|(g_n(x))^2-(g_n(y))^2|}{d^\alpha(x,y} \\ &\leq \tfrac{||g_n(x)|-|g_n(y)||}{d^\alpha(x,y)}(|g_n(x)| + |g_n(y)|) \\ &\leq 2\|g_n\|_X p_{(K,d^\alpha)}(g_n) \\ &\leq \tfrac{2}{\sqrt{n}} p_{(K,d^\alpha)}(g_n) \\ &\leq \tfrac{2}{\sqrt{n}} p_{(K,d^\alpha)}(g). \end{aligned}$$

Hence,

$$p_{(K,d^\alpha)}(f_n) \leq \frac{2}{\sqrt{n}} p_{(K,d^\alpha)}(g). \tag{2.8}$$

Since $\|f_n\|_X \leq \frac{1}{n}$ and (2.8) holds for all $n \in \mathbb{N}$, we deduce that

$$\lim_{n\to\infty} \|f_n\|_{\mathrm{Lip}(X,K,d^\alpha)} = 0.$$

Therefore, $f \in J_A(E_X(H))$ by regularity of A and Lemma 2.2.

Let f,g be real-valued functions in $\ker(E_X(H))$. Then $f+g, f-g \in \ker(E_X(H))$. By the above argument $(f+g)^2, (f-g)^2 \in J_A(E_X(H))$. So $fg \in J_A(E_X(H))$ since $fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$ and $J_A(E_X(H))$ is a complex linear subspace of A.

Since A is a natural Banach function algebra on X, we deduce that $\ker(E_X(H))$ is self-adjoint. This implies that $\mathrm{Re}\ f, \mathrm{Im}\ f \in \ker(E_X(H))$ if $f \in \ker(E_X(H))$.

Let $f, g \in \ker(E_X(H))$. We define $f_1 = \mathrm{Re}\ f$, $f_2 = \mathrm{Im}\ f$, $g_1 = \mathrm{Re}\ g$ and $g_2 = \mathrm{Im}\ g$. Then f_1, f_2, g_1 and g_2 are real-valued functions in $\ker(E_X(H))$ and so $f_1g_1, f_2g_2, f_1g_2, f_2g_1 \in J_A(E_X(H))$. Hence, $fg \in J_A(E_X(H))$. This implies that

$$(\ker(E_X(H)))^2 \subseteq J_A(E_X(H)). \tag{2.9}$$

Since $J_A(E_X(H))$ is closed in $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$, from (2.9) we conclude that

$$\overline{(\ker(E_X(H)))^2} \subseteq J_A(E_X(H)). \tag{2.10}$$

Let $f \in I_A(E_X(H))$. Then there exists an open set U in (X, d) with $H \subseteq U$ such that $e_x(f) = 0$ for all $x \in U$. Since H is a compact subset of X, $X \setminus U$ is a closed subset of X and $H \cap (X \setminus U) = \emptyset$, by Lemma 1.4, there exists a bounded Lipschitz function h on (X, d) such that $h(x) = 1$ for all $x \in H$ and $h(x) = 0$ for all $x \in X \setminus U$. Let $g = 1 - h$. Then $g \in A$, $g(x) = 0$ for all $x \in H$ and $g(x) = 1$ for all $x \in X \setminus U$. Moreover, $g^2 \in (\ker(E_X(H)))^2$ and $g^2(x) = 1$ for all $x \in X \setminus U$. So $f = fg^2$. Since $\ker(E_X(H))$ is an ideal of A, we deduce that $f \in (\ker(E_X(H)))^2$. Therefore,

$$I_A(E_X(H)) \subseteq (\ker(E_X(H)))^2.$$

Since $J_A(E_X(H)) = \overline{I_A(E_X(H))}$, we deduce that

$$J_A(E_X(H)) \subseteq \overline{(\ker(E_X(H)))^2}. \tag{2.11}$$

From (2.10) and (2.11) we have

$$J_A(E_X(H)) = \overline{(\ker(E_X(H)))^2},$$

and so the proof is complete. □

We now determine the set of all closed primary ideals of A at e_x as following, where $x \in \mathrm{int}(K)$.

Theorem 2.5. *Suppose that* $\mathrm{int}(K) \neq \emptyset$ *and* $A = \mathrm{Lip}(X, K, d^\alpha)$. *Let* $x \in \mathrm{int}(K)$ *and let* I *be a closed linear subspace of* A. *Then* I *is a closed primary ideal of* A *at* e_x *if and only if* $J_A(\{e_x\}) \subseteq I \subseteq \ker(e_x)$.

Proof . Let I be a closed primary ideal of A at e_x. Since A is a regular commutative complex unital Banach algebra, we have

$$J_A(\{e_x\}) \subseteq I \subseteq \ker(e_x).$$

Let I be a closed complex linear subspace of A such that

$$J_A(\{e_x\}) \subseteq I \subseteq \ker(e_x). \tag{2.12}$$

Let $g \in I$ and $f \in A$. Since $f - f(x)1_X \in \ker(e_x)$, we have $(f - f(x)1_X)g \in (\ker(e_x))^2$. Hence, $(f - f(x)1_X)g \in J_A(\{e_x\})$ by Theorem 2.4 and so $(f - f(x)1_X)g \in I$ by (2.12). This implies that $fg \in I$ since I is a complex linear subspace of A. Hence, I is an ideal of A. Since $I \subseteq \ker(e_x)$, we deduce that

$$\{e_x\} \subseteq \mathrm{hull}(I). \tag{2.13}$$

From (2.12) we have

$$\mathrm{hull}(I) \subseteq \mathrm{hull}(J_A(\{e_x\})). \tag{2.14}$$

Since $\{e_x\}$ is a closed subset of $\Delta(A)$ and $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ is a regular complex Banach algebra, by Theorem 1.1, we deduce that

$$\mathrm{hull}(J_A(\{e_x\})) = \{e_x\}. \tag{2.15}$$

From (2.13), (2.14) and (2.15) we have

$$\mathrm{hull}(I) = \{e_x\}.$$

Therefore, I is primary at e_x. $\square$

Theorem 2.6. *Suppose that* $\mathrm{int}(K) \neq \emptyset$ *and* $A = \mathrm{Lip}(X, K, d^\alpha)$. *Let* H *be a nonempty compact subset of* $\mathrm{int}(K)$. *Then*

$$J_A(E_X(H)) = \bigcap_{x \in H} J_A(\{e_x\}).$$

Proof . Let $f \in I_A(E_X(H))$ and let $x \in H$. Then there exists an open set V in $\Delta(A)$ with $E_X(H) \subseteq V$ such that $\widehat{f}(V) = \{0\}$. Since $\{e_x\} \subseteq E_X(H)$, we have $\{e_x\} \subseteq V$. Hence, $f \in I_A(\{e_x\})$ and so $f \in J_A(\{e_x\})$. Thus

$$I_A(E_X(H)) \subseteq I_A(\{e_x\}) \subseteq J_A(\{e_x\}). \tag{2.16}$$

Since (2.16) holds for all $x \in H$, we deduce that

$$I_A(E_X(H)) \subseteq \bigcap_{x \in H} J_A(\{e_x\}).$$

This implies that

$$J_A(E_X(H)) \subseteq \bigcap_{x \in H} J_A(\{e_x\}), \tag{2.17}$$

since $J_A(E_X(H)) = \overline{I_A(E_X(H))}$ and $\bigcap_{x \in H} J_A(\{e_x\})$ is closed in $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$.

Let $f \in A \setminus J_A(E_X(H))$. If $f(H) \neq \{0\}$, then $e_{x_0}(f) = f(x_0) \neq 0$ for some $x_0 \in H$. Since $e_{x_0} : A \to \mathbb{C}$ is continuous at f and $e_{x_0}(f) \neq 0$, there exists a positive number δ such that $e_{x_0}(g) \neq 0$ whenever $g \in A$ and $\|g - f\|_{\mathrm{Lip}(X,K,d^\alpha)} < \delta$. This implies that

$$\{g \in A : \|g - f\|_{\mathrm{Lip}(X,K,d^\alpha)} < \delta\} \cap I_A(\{e_{x_0}\}) = \emptyset,$$

and so $f \in A \setminus J_A(\{e_{x_0}\})$.

Let $f(H) = \{0\}$. By Theorem 2.3, there exists $\varepsilon > 0$ such that for each open set U in (X, d) with $H \subseteq U$ we have

$$\frac{|f(x) - f(y)|}{d^\alpha(x, y)} \geq \varepsilon,$$

for some $x, y \in U \cap K$ with $x \neq y$. Let $n \in \mathbb{N}$. We define

$$U_n = \{x \in X : \quad \mathrm{dist}(x, H) < (\frac{1}{n})^{\frac{1}{\alpha}}\}.$$

Then U_n is an open set in (X, d) with $H \subseteq U_n$. Thus there exist $x_n, y_n \in U_n \cap K$ with $x_n \neq y_n$ such that

$$\frac{|f(x_n) - f(y_n)|}{d^\alpha(x_n, y_n)} \geq \varepsilon.$$

Since $x_n, y_n \in U_n$, we deduce that there exist $z_n, w_n \in H$ such that $d(z_n, x_n) < (\frac{1}{n})^{\frac{1}{\alpha}}$ and $d(w_n, y_n) < (\frac{1}{n})^{\frac{1}{\alpha}}$. Since H is a compact set in (X, d) and $\{z_n\}_{n=1}^{\infty}$ is a sequence in H, there is a strictly increasing function $\gamma : \mathbb{N} \to \mathbb{N}$ and an element z of H such that

$$\lim_{n\to\infty} d(z_{\gamma(n)}, z) = 0.$$

Since H is a compact set in (X, d) and $\{w_{\gamma(n)}\}_{n=1}^{\infty}$ is a sequence in H, there exists a strictly increasing function $\eta : \mathbb{N} \to \mathbb{N}$ and an element w of H such that

$$\lim_{n\to\infty} d(w_{\eta(\gamma(n))}, w) = 0.$$

Let $n_k = \eta(\gamma(k))$ for all $k \in \mathbb{N}$. Then $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence in $\mathbb{N}$, $\{z_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{z_n\}_{n=1}^{\infty}$, $\{w_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{w_n\}_{n=1}^{\infty}$, $\lim\limits_{k\to\infty} d(z_{n_k}, z) = 0$ and $\lim\limits_{k\to\infty} d(w_{n_k}, w) = 0$. Since $d(z_{n_k}, x_{n_k}) < (\frac{1}{n_k})^{\frac{1}{\alpha}}$ and $d(w_{n_k}, y_{n_k}) < (\frac{1}{n_k})^{\frac{1}{\alpha}}$, we conclude that $\lim\limits_{k\to\infty} d^{\alpha}(x_{n_k}, z) = 0$ and $\lim\limits_{k\to\infty} d^{\alpha}(y_{n_k}, w) = 0$. So $\lim\limits_{k\to\infty} d^{\alpha}(x_{n_k}, y_{n_k}) = d^{\alpha}(z, w)$. We claim that $z = w$. If $z \neq w$, then $d^{\alpha}(z, w) \neq 0$ and so

$$\lim_{k\to\infty} \frac{|f(x_{n_k}) - f(y_{n_k})|}{d^{\alpha}(x_{n_k}, y_{n_k})} = \frac{|f(z) - f(w)|}{d^{\alpha}(z, w)}. \tag{2.18}$$

Since $\frac{|f(x_{n_k})-f(y_{n_k})|}{d^{\alpha}(x_{n_k},y_{n_k})} \geq \varepsilon$ for all $k \in \mathbb{N}$, we deduce that $\frac{|f(z)-f(w)|}{d^{\alpha}(z,w)} \geq \varepsilon$ by (2.18). But $\frac{|f(z)-f(w)|}{d^{\alpha}(z,w)} = 0$ since $z, w \in H$. Therefore, our claim is justified by this contradiction. From $z = w$, we have $\lim\limits_{k\to\infty} d(y_{n_k}, z) = 0$. Let U be an open set in (X, d) with $z \in U$. Then there exists $k \in \mathbb{N}$ such that $x_{n_k}, y_{n_K} \in U$. Therefore, there exists $x_{n_k}, y_{n_K} \in U \cap K$ such that

$$\frac{|f(x_{n_k}) - f(y_{n_k})|}{d^{\alpha}(x_{n_k}, y_{n_k})} \geq \varepsilon.$$

Hence, $f \notin J_A(\{e_z\})$ by Theorem 2.3. So $f \notin \bigcap\limits_{x\in H} J_A(\{e_x\})$. Therefore,

$$\bigcap_{x\in H} J_A(\{e_x\}) \subseteq J_A(E_X(H)). \tag{2.19}$$

From (2.17) and (2.19), we have $J_A(E_X(H)) = \bigcap\limits_{x\in H} J_A(\{e_x\})$. □

3. Point derivations of extended Lipschitz algebras

Throughout this section we assume that (X, d) is a compact metric space, K is an infinite compact subset of X, $\alpha \in (0, 1]$,

$$W(K) = \{(x, y) \in K \times K : x \neq y\},$$

and

$$W_x = \{\{(x_n, y_n)\}_{n=1}^{\infty} : (x_n, y_n) \in W(K) \quad (n \in \mathbb{N}), \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x\},$$

where x is a nonisolated point of K in (X, d).

Lemma 3.1. *Let $A = \mathrm{Lip}(X, K, d^\alpha)$, x be a nonisolated point of K in (X, d) and $\{(x_n, y_n)\}_{n=1}^\infty$ be an element of W_x. Let $n \in \mathbb{N}$ and define the map $\phi_n : A \to \mathbb{C}$ by*

$$\phi_n(f) = \frac{f(x_n) - f(y_n)}{d^\alpha(x_n, y_n)} \qquad (f \in A). \tag{3.1}$$

Then $\phi_n \in A^*$ and $\|\phi_n\| \le 1$.

Proof . It is obvious that ϕ_n is a complex linear functional on A. Since

$$|\phi_n(f)| = \frac{|f(x_n) - f(y_n)|}{d^\alpha(x_n, y_n)} \le p_{(K,d^\alpha)}(f) \le \|f\|_{\mathrm{Lip}(X,K,d^\alpha)}$$

for all $f \in A$, we conclude that $\phi_n \in A^*$ and $\|\phi_n\| \le 1$. □

Let $A = \mathrm{Lip}(X, K, d^\alpha)$ and B^* denote the closed unit ball of A^*. Since B^* is weak* compact, every net in B^* has a subnet that it converges in A^* with the weak* topology. Let x be a nonisolated point of K in (X, d). We denote by Ω_x the set of all sequences $\{\phi_n\}_{n=1}^\infty$ defined by (3.1) as $\{(x_n, y_n)\}_{n=1}^\infty$ varries over W_x. We denote by Φ_x the set of all $\Lambda \in A^*$ for which there exists a sequence $\{\phi_n\}_{n=1}^\infty$ in Ω_x and a subnet $\{\phi_{n_\gamma}\}_\gamma$ of $\{\phi_n\}_{n=1}^\infty$ such that $\lim_\gamma \phi_{n_\gamma} = \Lambda$ in A^* with the weak* topology.

Theorem 3.2. *Let $A = \mathrm{Lip}(X, K, d^\alpha)$ and x be a nonisolated point of K in (X, d). Then*

(i) *Φ_x is a nonempty subset of B^*.*

(ii) *$\Phi_x \subseteq \mathfrak{D}_{e_x}$.*

Proof . (i). Let $\{\phi_n\}_{n=1}^\infty$ be an element of Ω_x. Then $\phi_n \in B^*$ for all $n \in \mathbb{N}$ by Lemma 3.1. Since B^* is weak* compact subset of A^*, there exists a subnet $\{\phi_{n_\gamma}\}_\gamma$ of $\{\phi_n\}_{n=1}^\infty$ and an element $D \in B^*$ such that $\lim_\gamma \phi_{n_\gamma} = D$ in A^* with the weak* topology. This implies that $D \in \Phi_x$ and so Φ_x is nonempty.

Let $\Lambda \in \Phi_x$. Then there exists an element $\{\phi_n\}_{n=1}^\infty$ of W_x and a subnet $\{\phi_{n_\gamma}\}_\gamma$ of $\{\phi_n\}_{n=1}^\infty$ such that $\lim_\gamma \phi_{n_\gamma} = \Lambda$ in A^* with the weak* topology. This implies that $\lim_\gamma \phi_{n_\gamma}(f) = \Lambda(f)$ for all $f \in A$. Let $f \in A$. Then $\lim_\gamma |\phi_{n_\gamma}(f)| = |\Lambda(f)|$. By Lemma 3.1, we have $\|\phi_{n_\gamma}\| \le 1$ for each γ. Hence, $|\phi_{n_\gamma}(f)| \le \|f\|_{\mathrm{Lip}(X,K,d^\alpha)}$ for each γ. This implies that $|\Lambda(f)| \le \|f\|$. Therefore, $\|\Lambda\| \le 1$ and so $\Lambda \in B^*$. Thus (i) holds.

(ii). Let $D \in \Phi_x$. Then there exists an element $\{\phi_n\}_{n=1}^\infty$ of W_x and a subnet $\{\phi_{n_\gamma}\}_\gamma$ of $\{\phi_n\}_{n=1}^\infty$ such that $\lim_\gamma \phi_{n_\gamma} = D$ in A^* with the weak* topology. Since $\{\phi_n\}_{n=1}^\infty \in W_x$, there exists an element $\{(x_n, y_n)\}_{n=1}^\infty$ of W_x such that for all $n \in \mathbb{N}$ we have

$$\phi_n(f) = \frac{f(x_n) - f(y_n)}{d^\alpha(x_n, y_n)} \qquad (f \in A).$$

Let $f, g \in A$. Then

$$\begin{aligned} D(fg) &= \lim_\gamma \phi_{n_\gamma}(fg) \\ &= \lim_\gamma \tfrac{(fg)(x_{n_\gamma}) - (fg)(y_{n_\gamma})}{d^\alpha(x_{n_\gamma}, y_{n_\gamma})} \\ &= \lim_\gamma \tfrac{f(x_{n_\gamma})[g(x_{n_\gamma}) - g(y_{n_\gamma})] + g(y_{n_\gamma})[f(x_{n_\gamma}) - f(y_{n_\gamma})]}{d^\alpha(x_{n_\gamma}, y_{n_\gamma})} \\ &= \lim_\gamma [f(x_{n_\gamma}) \tfrac{g(x_{n_\gamma}) - g(y_{n_\gamma})}{d^\alpha(x_{n_\gamma}, y_{n_\gamma})} + g(y_{n_\gamma}) \tfrac{f(x_{n_\gamma}) - f(y_{n_\gamma})}{d^\alpha(x_{n_\gamma}, y_{n_\gamma})}] \\ &= \lim_\gamma [f(x_{n_\gamma})\phi_{n_\gamma}(g) + g(y_{n_\gamma})\phi_{n_\gamma}(f)] \\ &= f(x)Dg + g(x)Df \\ &= e_x(f)Dg + e_x(g)Df. \end{aligned}$$

This implies that D is a continuous point derivation at e_x and so $D \in \mathfrak{D}_{e_x}$. Thus $\Phi_x \subseteq \mathfrak{D}_{e_x}$ and so (ii) holds. $\square$

Theorem 3.3. *Let $A = \mathrm{Lip}(X, K, d^\alpha)$, $x \in \mathrm{int}(K)$ and x be a nonisolated point of K. Then A has a nonzero point derivation at e_x.*

Proof . Since x is a nonisolated point of K, there exists a sequence $\{x_n\}_{n=1}^\infty$ in $K \setminus \{x\}$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

We define the sequence $\{y_n\}_{n=1}^\infty$ in K with $y_n = x$ for all $n \in \mathbb{N}$. Then $\{(x_n, y_n)\}_{n=1}^\infty$ is an element of W_x. Let the sequence $\{\phi_n\}_{n=1}^\infty$ given by (3.1) in terms of the sequence $\{(x_n, y_n)\}_{n=1}^\infty$. We define the function $g_x : X \to \mathbb{C}$ by $g_x(t) = (d(x, t))^\alpha$, $t \in X$. Since d^α is a metric on X, we conclude that $g_x \in \mathrm{Lip}(X, d^\alpha)$ and so $g_x \in A$. Moreover, $\phi_n(g_x) = 1$ for all $n \in \mathbb{N}$. Since B^* is weak* compact and $\{\varphi_n\}_{n=1}^\infty$ is a sequence in B^*, we deduce that there exists a subnet $\{\varphi_{n_\gamma}\}_\gamma$ of $\{\varphi_n\}_{n=1}^\infty$ and an element D of B^* such that $\lim_\gamma \varphi_{n_\gamma} = D$ in A^* with the weak* topology. Clearly, $D \in \Phi_x$ and $Dg_x = 1$. Hence, $D \neq 0$ and by part (ii) of Theorem 3.2 we have $D \in \mathfrak{D}_{e_x}$. Therefore, $\mathfrak{D}_{e_x} \setminus \{0\}$ is nonempty and so A has a nonzero continuous point derivation at e_x. $\square$

Let $\mathfrak{X}$ be a complex Banach space, M be a nonempty subset of $\mathfrak{X}$ and N be a nonemty subset of $\mathfrak{X}^*$, the dual space of $\mathfrak{X}$. Recall that $M^\perp$ and ${}^\perp N$ denote $\{\Lambda \in \mathfrak{X}^* : \Lambda f = 0\ (f \in M)\}$ and $\{f \in \mathfrak{X} : \Lambda f = 0\ (\Lambda \in N)\}$, respectively. We know [8, Theorem 4.7] that if M is a complex linear subspace of $\mathfrak{X}$ then ${}^\perp(M^\perp)$ is a closed complex linear subspace of $\mathfrak{X}$ and ${}^\perp(M^\perp) = \overline{M}$, the closure of M in $(\mathfrak{X}, \|\cdot\|)$.

Theorem 3.4. *Let $\mathrm{int}(K) \neq \emptyset$, $A = \mathrm{Lip}(X, K, d^\alpha)$, x be a nonisolated point of K and $x \in \mathrm{int}(K)$. Suppose that $\mathrm{sp}(\Phi_x)$ denotes the weak* closure of the complex linear subspace of A spanned by Φ_x. Then $\mathfrak{D}_{e_x} = \mathrm{sp}(\Phi_x)$.*

Proof . By Theorem 3.2, Φ_x is a nonempty subset of $\mathfrak{D}_{e_x}$. Since $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ is a semisimple commutative unital Banach algebra, we conclude that $\mathfrak{D}_{e_x}$ is a weak* closed complex linear subspace of A^* by [12, Proposition 8.2]. So

$$\mathrm{sp}(\Phi_x) \subseteq \mathfrak{D}_{e_x}. \tag{3.2}$$

Now we show that

$${}^\perp(\mathrm{sp}(\Phi_x)) \subseteq {}^\perp \mathfrak{D}_{e_x}. \tag{3.3}$$

Since $\overline{(\ker(e_x))^2} \oplus \mathbb{C}.1$ is a closed complex linear subspace of $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ and [12, Proposition 8.4] implies that $\mathfrak{D}_{e_x} = (\overline{(\ker(e_x))^2} \oplus \mathbb{C}.1)^\perp$, we deduce that

$${}^\perp\mathfrak{D}_{e_x} = \overline{(\ker(e_x))^2} \oplus \mathbb{C}.1, \tag{3.4}$$

by [9, Theorem 4.7(a)]. On the other hand, by Theorem 2.4, we have

$$\overline{(\ker(e_x))^2} = J_A(\{e_x\}). \tag{3.5}$$

From (3.4) and (3.5) we get

$${}^\perp\mathfrak{D}_{e_x} = J_A(\{e_x\}) \oplus \mathbb{C}1_X. \tag{3.6}$$

Let $f \in A \setminus^\perp \mathfrak{D}_{e_x}$. Then $f \notin J_A(\{e_x\})$. We define $g = f - f(x)1_X$. Then $g \in A$, $g(x) = 0$ and $f = g + f(x)1_X$. According to (3.6) and $f \in A \setminus^\perp \mathfrak{D}_{e_x}$, we deduce that $g \notin J_A(\{e_x\})$. Since

$x \in \text{int}(K)$ and $E_X(\{x\}) = \{e_x\}$, by Theorem 2.3, there exists $\varepsilon > 0$ such that for each open set U in (X,d) with $x \in U$ we have $\frac{|g(z)-g(w)|}{d^\alpha(z,w)} \geq \varepsilon$ for some $z, w \in U \cap K$ with $z \neq w$. Let $n \in \mathbb{N}$ and set $U_n = \{y \in X : \quad d(y,x) < \frac{1}{n}\}$. Then there exist $z_n, w_n \in U_n \cap K$ with $z_n \neq w_n$ such that $\frac{|g(z_n)-g(w_n)|}{d^\alpha(z_n,w_n)} \geq \varepsilon$. Clearly, $\lim_{n\to\infty} d(z_n,x) = \lim_{n\to\infty} d(w_n,x) = 0$. So $\{(z_n,w_n)\}_{n=1}^\infty$ is an element of W_x. Let $\{\phi_n\}_{n=1}^\infty$ given by (3.1) in terms of the sequence $\{(z_n,w_n)\}_{n=1}^\infty$. Clearly, $|\phi_n(g)| \geq \varepsilon$ for all $n \in \mathbb{N}$. Since $\{\phi_n\}_{n=1}^\infty$ is an element of Ω_x, by given argument in the proof of part (i) of Theorem 3.2, there exists a subnet $\{\phi_{n_\gamma}\}_\gamma$ of $\{\phi_n\}_{n=1}^\infty$ and an element $D \in A^*$ such that $D = \lim_\gamma \phi_{n_\gamma}$ in A^* with the weak* topology. Such D is an element of Φ_x and so $D \in \text{sp}(\Phi_x)$. Since $|Dg| = \lim_\gamma |\phi_{n_\gamma}(g)|$ and $|\phi_{n_\gamma}(g)| \geq \varepsilon$ for all γ, we deduce that $|Dg| \geq \varepsilon$ and so $Dg \neq 0$. Since $D \in \Phi_x$ and $\Phi_x \subseteq \mathfrak{D}_{e_x}$ by part (ii) of Theorem 3.2, we conclude that $D \in \mathfrak{D}_{e_x}$ and so $D1_X = 0$. Now we have

$$Df = D(g + f(x)1_X) = Dg + f(x)D1_X = Dg.$$

Therefore, $Df \neq 0$ and so $f \in A \setminus^\perp (\text{sp}(\Phi_x))$. Hence,

$$A \setminus^\perp \mathfrak{D}_{e_x} \subseteq A \setminus^\perp (\text{sp}(\Phi_x)).$$

This implies (3.3) holds. From (3.3) we get

$$\left({}^\perp\mathfrak{D}_{e_x}\right)^\perp \subseteq \left({}^\perp (\text{sp}(\Phi_x))\right)^\perp. \tag{3.7}$$

Since $\mathfrak{D}_{e_x}$ and $\text{sp}(\Phi_x)$ are closed in A^* with the weak* topology, we deduce that

$$\mathfrak{D}_{e_x} \subseteq \text{sp}(\Phi_x), \tag{3.8}$$

by (3.7) and [9, Theorem 4.7(b)]. From (3.2) and (3.8), we get $\mathfrak{D}_{e_x} = \text{sp}(\Phi_x)$. □

4. Amenability and weak amenability of extended Lipschitz algebras

Let A be a commutative unital complex Banach algebra. It is known [4] that if A is weakly amenable, then every continuous point derivation of A is zero. Considering this fact, we obtain the following result.

Theorem 4.1. *Let (X,d) be a compact metric space, $\alpha \in (0,1]$ and K be an infinite compact subset of X with $\text{int}(K) \neq \emptyset$ and $\text{int}(K)$ contains a limit point of K in (X,d). Then*

(i) $\text{Lip}(X,K,d^\alpha)$ *is not weakly amenable.*
(ii) $\text{Lip}(X,K,d^\alpha)$ *is not amenable.*

Proof . (i). Let $x \in \text{int}(K)$ and x be a limit point of K. Then x is a nonisolated point of K in (X,d). Since $x \in \text{int}(K)$, we deduce that $\text{Lip}(X,K,d^\alpha)$ has a nonzero continuous derivation at e_x by Theorem 3.3. Therefore, $\text{Lip}(X,K,d^\alpha)$ is not weakly amenable.

(ii). It is obvious by (i). □

Let X be an infinite set and (X,d) be a compact metric space. Then $\text{int}(X) = X$ in (X,d) and so $\text{int}(X)$ has a limit point of X in (X,d). Since $\text{Lip}(X,X,d^\alpha) = \text{Lip}(X,d^\alpha)$, we immediately get the following result as a consequence of Theorem 4.1.

Corollary 4.2. Let X be an infinite set, (X,d) be a compact metric space and $\alpha \in (0,1]$. Then

(i) $\text{Lip}(X,d^\alpha)$ is not weakly amenable,
(ii) $\text{Lip}(X,d^\alpha)$ is not amenable.

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