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Subordination and superordination results of multivalent functions associated with the Dziok-Srivastava operator

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Abstract

Using the techniques of the differential subordination and superordination, we derive certain subordination and superordination properties of multivalent functions associated with the Dziok-Srivastava operator.

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1. Introduction

Let $A(p,k)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_{n+p} z^{n+p} \quad (p,k \in \mathbb{N} = \{1,2,3,\dots\}), \tag{1.1}$$

which are analytic in the open unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$; we write $A(p) := A(p,1)$.

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Suppose that f and g are analytic in U. We say that the function f is subordinate to g in U, or g superordinate to f in U, and we write $f(z) \prec g(z)$, if there exists an analytic function w in U with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \mathrm{U}$. If g is univalent in U, then the following equivalence relationship holds (see [13], [14] and [15]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathrm{U}) \subset g(\mathrm{U}).$$

For the functions $f_j \in A(p,k)$ given by

$$f_j(z) = z^p + \sum_{n=k}^{\infty} a_{n+p,j} z^{n+p}, \quad z \in \mathrm{U}, \quad (j = 1, 2),$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{n=k}^{\infty} a_{n+p,1} a_{n+p,2} z^{n+p} = (f_2 * f_1)(z), \quad z \in \mathrm{U}.$$

For the complex parameters $a_1, \dots, a_q$ and $b_1, \dots, b_s$, with $b_j \notin \mathbb{Z}_0^- := \{0, -1, -2, \dots\}$, $j = 1, \dots, s$, the generalized hypergeometric function ${}_qF_s$ is defined (see [26]) by the following infinite series

$${}_qF_s(a_1, \dots, a_q; b_1, \dots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdot \ldots \cdot (a_q)_n}{(b_1)_n \ldots (b_s)_n} \frac{z^n}{n!}, \quad z \in \mathrm{U},$$

$$(q \le s + 1;\ q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

where $(\theta)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_n = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} = \begin{cases} 1, & \text{if } \theta = 0 \\ \theta(\theta+1)\dots(\theta+n-1), & \text{if } \theta \in \mathbb{N}. \end{cases}$$

Corresponding to the function $h_p(a_1, \dots, a_q; b_1, \dots, b_s; z)$ defined by

$$h_p(a_1, \dots, a_q; b_1, \dots, b_s; z) = z^p\ {}_qF_s(a_1, \dots, a_q; b_1, \dots, b_s; z), \ z \in \mathrm{U},$$

Dziok and Srivastava [4] considered a linear operator

$$H_p(a_1, \dots, a_q; b_1, \dots, b_s) : A(p,k) \to A(p,k)$$

defined by the following Hadamard product:

$$H_p(a_1, \dots, a_q; b_1, \dots, b_s) f(z) = h_p(a_1, \dots, a_q; b_1, \dots, b_s; z) * f(z), \ z \in \mathrm{U}, \tag{1.2}$$

$$(q \le s+1;\ q, s \in \mathbb{N}_0).$$

If $f \in A(p,k)$ is given by (1.1), then we have

$$H_p(a_1, \dots, a_q; b_1, \dots, b_s) f(z) = f(z) = z^p + \sum_{n=k}^{\infty} \Gamma_n a_{n+p} z^{n+p}, \ z \in \mathrm{U}, \tag{1.3}$$

where

$$\Gamma_n = \frac{(a_1)_n \cdot \ldots \cdot (a_q)_n}{(b_1)_n \ldots (b_s)_n} \frac{1}{n!} \quad (n \in \mathbb{N}).$$

To simplify the notations, we write

$$H_{p,q,s}(a_1)f(z) := H_p(a_1,\ldots,a_q;b_1,\ldots,b_s)f(z).$$

From (1.2) or (1.3) it follows that

$$z\left(H_{p,q,s}\left(a_1\right)f(z)\right)' = a_1 H_{p,q,s}\left(a_1+1\right)f(z) - (a_1-p)H_{p,q,s}\left(a_1\right)f(z),\ z\in \mathrm{U}.$$

It should be remarked that the linear operator $H_{p,q,s}\left(a_1\right)$ is a generalization of many other linear operators considered earlier. In particular, for $f\in A\left(p\right)$ we have the following special cases:

(i) $H_{1,2,1}(a,b;c)f =: \left(I_c^{a,b}\right)f\ \left(a,b\in\mathbb{C};\ c\notin\mathbb{Z}_0^-\right)$, where the linear operator $I_c^{a,b}$ was investigated by Hohlov [8];

(ii) $H_{p,2,1}(n+p,1;1)f =: D^{n+p-1}f\ (n\in\mathbb{N};\ n>-p)$, where the linear operator D^{n+p-1} was studied by Goel and Sohi [7]. In the case when $p=1$, $D^n f$ is the Ruscheweyh derivative of f (see [22]);

(iii) $H_{p,2,1}(\delta+p,1;\delta+p+1)f(z) =: J_{p,\delta}(f)(z) = \dfrac{p+\delta}{z^\delta}\displaystyle\int_0^z t^{\delta-1}f(t)dt\ (\delta>-p)$, where $J_{p,\delta}$ is the generalized Bernardi–Libera–Livingston integral operator (see [3]);

(iv) $H_{p,2,1}(p+1,1;p+1-\lambda)f(z) =: \Omega_z^{(\lambda,p)}f\left(z\right) = \dfrac{\Gamma\left(p+1-\lambda\right)}{\Gamma\left(p+1\right)}z^\lambda D_z^\lambda f\left(z\right)\ (-\infty\le\lambda<p+1)$, where $D_z^\lambda f$ is the fractional integral of f of order $-\lambda$ when $-\infty\le\lambda<0$, and fractional derivative of f of order λ when $0\le\lambda<p+1$. The extended fractional differintegral operator $\Omega_z^{(\lambda,p)}$ was introduced and studied by Patel and Mishra [21], while the fractional differential operator $\Omega_z^{(\lambda,p)}$ with $0\le\lambda<1$ was investigated by Srivastava and Aouf [25]. The operator $\Omega_z^{(\lambda,1)} =: \Omega_z^\lambda$ was introduced by Owa and Srivastava [20] (see also Owa [19]);

(v) $H_{p,2,1}(a,1;c)f =: L_p(a,c)f\ (a\in\mathbb{R};\ c\in\mathbb{R}\setminus\mathbb{Z}_0^-)$, where the linear operator $L_p(a,c)$ was studied by Saitoh [23], which yields the operator $L(a,c)$ introduced by Carlson and Shaffer [1] for $p=1$;

(vi) $H_{1,2,1}(\mu,1;\lambda+1)f =: I_{\lambda,\mu}f(z)\ (\lambda>-1;\ \mu>0)$, where $I_{\lambda,\mu}$ is the Choi–Saigo–Srivastava operator [3], which is closely related to the Carlson–Shaffer [1] operator $L(\mu,\lambda+1)$;

(vii) $H_{p,2,1}(p+1,1;n+p)f =: I_{n,p}f\ (n\in\mathbb{Z};\ n>-p)$, where the operator $I_{n,p}$ was considered by Liu and Noor [10];

(viii) $H_{p,2,1}(\lambda+p,c;a)f =: I_p^\lambda(a,c)f\ (a,c\in\mathbb{R}\setminus\mathbb{Z}_0^-;\ \lambda>-p)$, where $I_p^\lambda(a,c)$ is the Cho–Kwon–Srivastava operator [2].

In recent years, many interesting subclasses of analytic functions associated with the Dziok–Srivastava operator $H_{p,q,s}\left(a_1\right)$ and its many special cases were investigated by (for example) Dziok and Srivastava ([4] and [5]), Gangadharan et al. [6], Liu and Noor [10], Liu [9], Liu and Srivastava [12], Liu and Patel [11], and many others (see also [2, 16, 17, 27]). In the present paper we shall use the method based upon the differential subordination to derive inclusion relationships and other interesting properties and characteristics of the Dziok–Srivastava operator $H_{p,q,s}\left(a_1\right)$.

2. Preliminaries lemmas

Let $P[c,k]$ denote the class of functions of the form

$$\varphi(z) = c + c_k z^k + c_{k+1} z^{k+1} + \ldots,$$

that are analytic in U; we write $P[k] := P[1,k]$.

Definition 2.1. *[15] Denote by $\mathcal{Q}$ the set of all functions f that are analytic and injective on $\mathrm{U} \setminus E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial \mathrm{U} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and such that $f'(z) \neq 0$ for $\zeta \in \mathrm{U} \setminus E(f)$.

In our present investigation, we shall require the following lemmas.

Lemma 2.2. *[14] Let h be analytic and convex (univalent) in U, with $h(0) = 1$, and let $\varphi \in P[k]$. If*

$$\varphi(z) + \frac{z\varphi'(z)}{\gamma} \prec h(z),$$

where $\gamma \neq 0$ and $\operatorname{Re} \gamma \geq 0$, then

$$\varphi(z) \prec q(z) = \frac{\gamma}{k} z^{-\frac{\gamma}{k}} \int_0^z t^{\frac{\gamma}{k}-1} h(t)\, dt \prec h(z),$$

and q is the best dominant.

Lemma 2.3. *[24] Let q be a convex (univalent) function in U, let $\sigma \in \mathbb{C}$ and $\theta \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, with*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\operatorname{Re}\frac{\sigma}{\theta}\right\}.$$

If the function φ is analytic in U and

$$\sigma\varphi(z) + \theta z\varphi'(z) \prec \sigma q(z) + \theta z q'(z),$$

then $\varphi(z) \prec q(z)$, and q is the best dominant.

Lemma 2.4. *[15] Let q be a convex (univalent) function in U and let $k \in \mathbb{C}$, with $\operatorname{Re} k > 0$. If*

$$\varphi \in P[q(0), 1] \cap \mathcal{Q},$$

and $\varphi(z) + kz\varphi'(z)$ is univalent in U, then

$$q(z) + kzq'(z) \prec \varphi(z) + kz\varphi'(z)$$

implies $q(z) \prec \varphi(z)$, and q is the best subordinant.

Lemma 2.5. *[28, Chapter 14] For any real or complex numbers a, b, c $\left(c \notin \mathbb{Z}_0^-\right)$ we have*

$$\int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a}\, dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a,b;c;z) \quad (\operatorname{Re} c > \operatorname{Re} b > 0); \tag{2.1}$$

$${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z); \tag{2.2}$$

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1\left(a,b;c;\frac{z}{1-z}\right). \tag{2.3}$$

3. Main results

Unless otherwise mentioned, we assume throughout the sequel that $a_i > 0$ for $i = 1, \ldots, q$, $\alpha > 0$, $\mu > 0$ and $-1 \leq B < A \leq 1$. Now, we will prove the following sharp subordination result:

Theorem 3.1. *Let $0 \leq j < p$, and for $f \in A(p,k)$ suppose that*

$$\frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}} \neq 0, \ z \in \mathrm{U},$$

whenever $\mu \in (0, +\infty) \setminus \mathbb{N}$. Let define the function Φ_j by

$$\Phi_j(z) = (1-\alpha)\left[\frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu} + \alpha \frac{\left(H_{p,q,s}(a_1+1) f(z)\right)^{(j)}}{z^{p-j}} \left[\frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu-1}, \tag{3.1}$$

where all the powers are the principal ones, i.e. $\log 1 = 0$.

If

$$\Phi_j(z) \prec \left[\frac{p!}{(p-j)!}\right]^{\mu} \frac{1+Az}{1+Bz}, \tag{3.2}$$

then

$$\left[\frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu} \prec \left[\frac{p!}{(p-j)!}\right]^{\mu} q(z), \tag{3.3}$$

where

$$q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1} \ {}_2F_1\left(1, 1; \frac{\mu a_1}{\alpha k} + 1; \frac{Bz}{Bz+1}\right), & \text{if } B \neq 0, \\ 1 + \frac{\mu a_1}{\mu a_1 + \alpha k} Az, & \text{if } B = 0, \end{cases}$$

and $\left[\frac{p!}{(p-j)!}\right]^{\mu} q$ is the best dominant of (3.3). *Furthermore, we have*

$$\operatorname{Re}\left[\frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu} > \left[\frac{p!}{(p-j)!}\right]^{\mu} \eta, \ z \in \mathrm{U}, \tag{3.4}$$

where η is given by

$$\eta = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1} \ {}_2F_1\left(1, 1; \frac{\mu a_1}{\alpha k} + 1; \frac{B}{B-1}\right), & \text{if } B \neq 0, \\ 1 - \frac{\mu a_1}{\mu a_1 + \alpha k} A, & \text{if } B = 0, \end{cases}$$

and the estimate (3.4) *is the best possible.*

Proof . Letting

$$\varphi(z) = \left[\frac{(p-j)!}{p!} \frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu}, \ z \in \mathrm{U}, \tag{3.5}$$

by choosing the principal branch in (3.5) we note that $\varphi \in P[k]$. Differentiating both the sides of (3.5), by using in the resulting equation the assumption (3.2) and the fact that

$$z\left(H_{p,q,s}(a_1) f(z)\right)^{(j+1)} = a_1 \left(H_{p,q,s}(a_1+1) f(z)\right)^{(j)} - (a_1 - p + j)\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}, \ \ z \in \mathrm{U}, \quad (0 \leq j < p) \tag{3.6}$$

we obtain

$$\varphi(z)+\frac{z\varphi'(z)}{\frac{\mu a_1}{\alpha}}\prec\frac{1+Az}{1+Bz}.$$

Now, by using Lemma 2.2, with $\gamma=\frac{\mu a_1}{\alpha}$, in the above differential subordination, we deduce that

$$\varphi(z)\prec q(z)=\frac{\mu a_1}{\alpha k}z^{-\frac{\mu a_1}{\alpha k}}\int_0^z t^{\frac{\mu a_1}{\alpha k}-1}\left(\frac{1+At}{1+Bt}\right)dt=$$
$$\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1,1;\frac{\mu a_1}{\alpha k}+1;\frac{Bz}{Bz+1}\right), & \text{if}\quad B\neq 0,\\ 1+\frac{\mu a_1}{\mu a_1+\alpha k}Az, & \text{if}\quad B=0,\end{cases}$$

where we used a change of variable followed by the use of the identities (2.1), (2.2) and (2.3), respectively. This completes the proof of the assertion (3.3).

Next, we will show that

$$\inf\{\operatorname{Re} q(z):|z|<1\}=q(-1). \tag{3.7}$$

Indeed, we have

$$\operatorname{Re}\frac{1+Az}{1+Bz}\geq\frac{1-Ar}{1-Br}\quad(|z|<r<1).$$

Setting

$$g(s,z)=\frac{1+Asz}{1+Bsz}\quad(0\leq s\leq 1;z\in\mathrm{U})$$

and

$$dv(s)=\frac{\mu a_1}{\alpha k}s^{\frac{\mu a_1}{\alpha k}-1}ds$$

which is a positive measure on the closed interval $[0,1]$, we get that

$$q(z)=\int_0^1 g(s,z)\,dv(s),$$

so that

$$\operatorname{Re} q(z)\geq\int_0^1\frac{1-Asr}{1-Bsr}dv(s)=q(-r)\quad(|z|\leq r<1).$$

Now, taking $r\to 1^-$ in the above inequality we obtain the assertion (3.7). The estimate (3.4) is the best possible since the function $\left[\frac{p!}{(p-j)!}\right]^{\mu}q$ is the best dominant of (3.3). □

Corollary 3.2. *Let $0\leq j<p$ and $f\in A(p,k)$. If*

$$\frac{\left(H_{p,q,s}(a_1+1)f(z)\right)^{(j)}}{z^{p-j}}\prec\frac{p!}{(p-j)!}\frac{1+A^*z}{1+Bz},$$

where

$$A^*=\begin{cases}\frac{B\ {}_2F_1\left(1,1;\frac{\mu a_1}{\alpha k}+1;\frac{B}{B-1}\right)}{B+\ {}_2F_1\left(1,1;\frac{\mu a_1}{\alpha k}+1;\frac{B}{B-1}\right)-1}, & \textit{if}\quad B\neq 0,\\ \frac{a_1+k}{a_1}, & \textit{if}\quad B=0,\end{cases}$$

then $H_{p,q,s}(a_1)f$ is p–valent in U.

Proof . Putting $\mu = \alpha = 1$ and replacing A by A^* in Theorem 3.1, we get

$$\operatorname{Re} \frac{z\left(H_{p,q,s}\left(a_1\right) f(z)\right)^{(j)}}{z^{p-j+1}} = \operatorname{Re} \frac{\left(H_{p,q,s}\left(a_1\right) f(z)\right)^{(j)}}{z^{p-j}} > 0,\ z \in \mathrm{U}.$$

Since the function $\phi(z) = z^{p-j+1}$ is $(p-j+1)$–valently starlike in U, in view of the result [18, Theorem 8] we obtain that the function $H_{p,q,s}\left(a_1\right) f$ is p–valent in U. □

Theorem 3.3. *Let $0 \le j < p$, and for $f \in A(p,k)$ let define the function F_α by*

$$F_\alpha(z) = (1-\alpha-\alpha a_1+\alpha p) H_{p,q,s}\left(a_1\right) f(z) + \alpha a_1 H_{p,q,s}\left(a_1+1\right) f(z). \tag{3.8}$$

If

$$\frac{F_\alpha^{(j)}(z)}{z^{p-j}} \prec (1-\alpha+\alpha p) \frac{p!}{(p-j)!} \frac{1+Az}{1+Bz}, \tag{3.9}$$

then

$$\frac{\left(H_{p,q,s}\left(a_1\right) f(z)\right)^{(j)}}{z^{p-j}} \prec \frac{p!}{(p-j)!} q(z), \tag{3.10}$$

where

$$q(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1,1;\frac{1-\alpha+\alpha p}{\alpha k}+1;\frac{Bz}{Bz+1}\right), & \text{if } B \neq 0, \\ 1 + \frac{1-\alpha+\alpha p}{1-\alpha+\alpha(p+k)} Az, & \text{if } B = 0, \end{cases}$$

and $\frac{p!}{(p-j)!}q$ is the best dominant of (3.10). Furthermore, we have

$$\operatorname{Re} \frac{\left(H_{p,q,s}\left(a_1\right) f(z)\right)^{(j)}}{z^{p-j}} > \frac{p!}{(p-j)!} \xi,\ z \in \mathrm{U}, \tag{3.11}$$

where ξ is given by

$$\xi = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\frac{1-\alpha+\alpha p}{\alpha k}+1;\frac{B}{B-1}\right), & \text{if } B \neq 0, \\ 1 - \frac{\mu a_1}{\mu a_1+\alpha k} A, & \text{if } B = 0, \end{cases}$$

and the estimate in (3.11) is the best possible.

Proof . Using the definition (3.8) and the identity (3.6), it follows that

$$F_\alpha^{(j)}(z) = (1-\alpha+\alpha j)\left(H_{p,q,s}\left(a_1\right) f(z)\right)^{(j)} + \alpha z \left(H_{p,q,s}\left(a_1\right) f(z)\right)^{(j+1)}, \tag{3.12}$$

for $0 \le j < p$. Putting

$$\varphi(z) = \frac{(p-j)!}{p!} \frac{\left(H_{p,q,s}\left(a_1\right) f(z)\right)^{(j)}}{z^{p-j}},\ z \in \mathrm{U}, \tag{3.13}$$

we have that $\varphi \in P[k]$. Differentiating both the sides of (3.13), using (3.9) and (3.12) in the resulting equation, by a simple calculation we get

$$\varphi(z) + \frac{\alpha}{1-\alpha+\alpha p} z\varphi'(z) \prec \frac{1+Az}{1+Bz}.$$

The remaining part of the proof is similar to that of Theorem 3.1, so we omit these details. □

Theorem 3.4. *Let $0 \le j < p$, and for $\delta > -p$ let define the operator $J_{p,\delta} : A(p,k) \to A(p,k)$ by*

$$J_{p,\delta}(f)(z) = \frac{p+\delta}{z^{\delta}} \int_0^z t^{\delta-1} f(t)\, dt,\ z \in \mathrm{U}.$$

If

$$\frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}} \prec \frac{p!}{(p-j)!} \frac{1+Az}{1+Bz}, \tag{3.14}$$

then

$$\frac{\left(H_{p,q,s}(a_1) J_{p,\delta}(f)(z)\right)^{(j)}}{z^{p-j}} \prec \frac{p!}{(p-j)!} q(z), \tag{3.15}$$

where

$$q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1} \ {}_2F_1\left(1,1; \frac{\delta+p}{k}+1; \frac{Bz}{Bz+1}\right), & \text{if } B \neq 0, \\ 1 + \frac{\delta+p}{\delta+p+k} Az, & \text{if } B = 0, \end{cases}$$

and $\frac{p!}{(p-j)!} q$ is the best dominant of (3.15)*. Furthermore, we have*

$$\operatorname{Re} \frac{\left(H_{p,q,s}(a_1) J_{p,\delta}(f)(z)\right)^{(j)}}{z^{p-j}} > \frac{p!}{(p-j)!} k,\ z \in \mathrm{U}, \tag{3.16}$$

where k is given by

$$k = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1} \ {}_2F_1\left(1,1; \frac{\delta+p}{k}+1; \frac{B}{B-1}\right), & \text{if } B \neq 0, \\ 1 - \frac{\delta+p}{\delta+p+k} A, & \text{if } B = 0, \end{cases}$$

and the estimate in (3.16) *is the best possible.*

Proof . Letting

$$\varphi(z) = \frac{(p-j)!}{p!} \frac{\left(H_{p,q,s}(a_1) J_{p,\delta}(f)(z)\right)^{(j)}}{z^{p-j}},\ z \in \mathrm{U},$$

we have that $\varphi(z) \in P[k]$. Differentiating the above definition formula, by using (3.14) and the identity

$$z\left(H_{p,q,s}(a_1) J_{p,\delta}(f)(z)\right)^{(j+1)} = (\delta+p)\left(H_{p,q,s}(a_1) f(z)\right)^{(j)} - (\delta+j)\left(H_{p,q,s}(a_1) J_{p,\delta}(f)(z)\right)^{(j)}$$

in the resulting equation, we get

$$\varphi(z) + \frac{z\varphi'(z)}{\delta+p} \prec \frac{1+Az}{1+Bz}.$$

Now, the assertion (3.15) and the estimate (3.16) follow by employing the same techniques that was used in the proof of Theorem 3.1. □

Theorem 3.5. *Let q be a univalent function in* U*, such that q satisfies*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\frac{\mu a_1}{\alpha}\right\},\ z \in \mathrm{U}. \tag{3.17}$$

Let $0 \le j < p$, *and for* $f \in A(p,k)$ *suppose that*

$$\frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}} \neq 0, \ z \in \mathrm{U},$$

whenever $\mu \in (0,+\infty) \setminus \mathbb{N}$. *Let the function* Φ_j *defined by* (3.1), *and suppose that it satisfies the following subordination:*

$$\left[\frac{(p-j)!}{p!}\right]^{\mu} \Phi_j(z) \prec q(z) + \frac{\alpha}{\mu a_1} z q'(z). \tag{3.18}$$

Then,

$$\left[\frac{(p-j)!}{p!} \frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu} \prec q(z),$$

and q *is the best dominant of the above subordination.*

Proof . If the function φ is defined by (3.5), from Theorem 3.1 we obtain

$$\left[\frac{(p-j)!}{p!}\right]^{\mu} \Phi_j(z) = \varphi(z) + \frac{\alpha}{\mu a_1} z \varphi'(z). \tag{3.19}$$

Combining (3.18) and (3.19) we find that

$$\varphi(z) + \frac{\alpha}{\mu a_1} z \varphi'(z) \prec q(z) + \frac{\alpha}{\mu a_1} z q'(z), \tag{3.20}$$

and by using Lemma 2.3 and (3.20) we easily get the assertion of Theorem 3.5. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.5 we obtain the following special case:

Corollary 3.6. *For* $-1 \le B < A \le 1$, *suppose that*

$$\operatorname{Re} \frac{1-Bz}{1+Bz} > \max\left\{0; -\frac{\mu a_1}{\alpha}\right\}, \ z \in \mathrm{U}.$$

Let $0 \le j < p$, *and for* $f \in A(p,k)$ *suppose that*

$$\frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}} \neq 0, \ z \in \mathrm{U},$$

whenever $\mu \in (0,+\infty) \setminus \mathbb{N}$. *Let the function* Φ_j *defined by* (3.1), *and suppose that it satisfies the following subordination:*

$$\left[\frac{(p-j)!}{p!}\right]^{\mu} \Phi_j(z) \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{\mu a_1} \frac{(A-B)z}{(1+Bz)^2}.$$

Then,

$$\left[\frac{(p-j)!}{p!} \frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu} \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ *is the best dominant of the above subordination.*

Theorem 3.7. *Let $0 \le j < p$, and for $f \in A(p,k)$ suppose that*

$$\frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}} \neq 0,\ z \in \mathrm{U},$$

whenever $\mu \in (0,+\infty) \setminus \mathbb{N}$. Suppose that

$$\left[\frac{(p-j)!}{p!} \frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu} \in P[1] \cap \mathcal{Q},$$

such that $\left[\frac{(p-j)!}{p!}\right]^{\mu} \Phi_j(z)$ is univalent in U, where the function Φ_j is defined by (3.1). If q is a convex (univalent) function in U, and

$$q(z) + \frac{\alpha}{\mu a_1} z q'(z) \prec \left[\frac{(p-j)!}{p!}\right]^{\mu} \Phi_j(z),$$

then

$$q(z) \prec \left[\frac{(p-j)!}{p!} \frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu},$$

and q is the best subordinant of the above subordination.

Proof . If the function φ is defined by (3.5), from (3.19) we have

$$q(z) + \frac{\alpha}{\mu a_1} z q'(z) \prec \left[\frac{(p-j)!}{p!}\right]^{\mu} \Phi_j(z) = \varphi(z) + \frac{\alpha}{\mu a_1} z \varphi'(z).$$

Now, an application of Lemma 2.4 yields the assertion of Theorem 3.7. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.7, we get the following special case:

Corollary 3.8. *Let $0 \le j < p$, and for $f \in A(p,k)$ suppose that*

$$\frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}} \neq 0,\ z \in \mathrm{U},$$

whenever $\mu \in (0,+\infty) \setminus \mathbb{N}$. Suppose that

$$\left[\frac{(p-j)!}{p!} \frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu} \in P[1] \cap \mathcal{Q},$$

such that $\left[\frac{(p-j)!}{p!}\right]^{\mu} \Phi_j(z)$ is univalent in U, where the function Φ_j is defined by (3.1), and suppose that $-1 \le B < A \le 1$. If

$$\frac{1+Az}{1+Bz} + \frac{\alpha}{\mu a_1} \frac{(A-B)z}{(1+Bz)^2} \prec \left[\frac{(p-j)!}{p!}\right]^{\mu} \Phi_j(z),$$

then

$$\frac{1+Az}{1+Bz} \prec \left[\frac{(p-j)!}{p!} \frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu}$$

and the function $\frac{1+Az}{1+Bz}$ is the best subordinant of the above subordination.

Combining the Theorem 3.5 and Theorem 3.7, we easily get the following *Sandwich-type result*:

Theorem 3.9. *Let* $0 \leq j < p$*, and for* $f \in A(p,k)$ *suppose that*

$$\frac{\left(H_{p,q,s}\left(a_1\right) f(z)\right)^{(j)}}{z^{p-j}} \neq 0,\ z \in \mathrm{U},$$

whenever $\mu \in (0,+\infty) \setminus \mathbb{N}$*. Suppose that*

$$\left[\frac{(p-j)!}{p!}\frac{\left(H_{p,q,s}\left(a_1\right) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu} \in P\left[q\left(0\right),k\right] \cap \mathcal{Q},$$

such that $\left[\frac{(p-j)!}{p!}\right]^{\mu}\Phi_j\left(z\right)$ *is univalent in* U*, where the function* Φ_j *is defined by* (3.1)*.*

Let q_1 *be a convex (univalent) function in* U*, and suppose that* q_2 *is a univalent function in* U *that* q_2 *satisfies* (3.17)*. If*

$$q_1\left(z\right) + \frac{\alpha}{\mu a_1} z q_1^{'}\left(z\right) \prec \left[\frac{(p-j)!}{p!}\right]^{\mu}\Phi_j\left(z\right) \prec q_2\left(z\right) + \frac{\alpha}{\mu a_1} z q_2^{'}\left(z\right),$$

then

$$q_1\left(z\right) \prec \left[\frac{(p-j)!}{p!}\frac{\left(H_{p,q,s}\left(a_1\right) f(z)\right)^{(j)}}{z^{p-j}}\right]^{\mu} \prec q_2\left(z\right),$$

and q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant of the above double subordination.*

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