Int. J. Nonlinear Anal. Appl. 12 (2021) 1, 1017–1036
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.21419.2257

Some estimation procedures of the PDF and CDF of the generalized inverted Weibull distribution with comparison

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(Communicated by Madjid Eshaghi Gordji)

Abstract

Different estimation procedures for the probability density and cumulative distribution functions of the generalized inverted Weibull distribution are discussed. For this purpose, the parametric and non-parametric estimation approaches as maximum likelihood, uniformly minimum variance unbiased, percentile, least squares and weighted least squares estimators are considered and compared. The expectations and mean square error of the maximum likelihood and uniformly minimum variance unbiased estimation are provided in the closed-form whereas, for non-parametric estimation methods (percentile, least squares and weighted least squares), the expectations and mean square error are computed via the simulation data. The Monte Carlo simulations are provided to assess the performances of the proposed estimation methods. Finally, the analysis of the real data set has been presented for illustrative purposes.

Keywords: Generalized inverted Weibull distribution, Maximum likelihood estimator, Uniformly minimum variance unbiased estimator, Percentile estimator, Least squares estimator, Weighted least squares estimator.
2010 MSC: 62F10

1. Introduction

The statistical distributions play an important role in describing the properties of real-world phenomena and they can be utilized in modeling real-life data in engineering, environmental, actuarial, medical sciences, biological studies, economics, hydrology, finance, and insurance. Among

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Received: September 2020 *Revised:* October 2020

the classical distributions, the inverse models and the shapes of density and failure rate functions of the inverse models are investigated by [20], whereas [12], particularly, studied the inverse Weibull distribution and offered the names complementary Weibull distribution. The inverse Weibull distribution can be applied in the numerous branches of science, such as medicine, reliability engineering, bio-engineering, degradation phenomenon of mechanical components, load-strength relationship for a component and ecology.

The inverse Weibull distribution is extended by many researchers. Gusmão et al. [15] proposed the generalized inverse Weibull (GIW) distribution that enhances the IW distribution function by power. They also introduced the log-generalized inverse Weibull distribution accompanied by the corresponding regression model. Shahbaz et al. [30] recommended the Kumaraswamy-inverse Weibull distribution based on the cumulative distribution function (CDF) of [23] distribution. The transmuted inverse Weibull distribution and the performance of the new distribution to modeling the reliability data are investigated by [21]. Jain et al. [16] introduced inverse generalized Weibull (IGW), generalized inverse generalized Weibull (GIGW) and a mixture of two GIGW distributions with the discussion of statistical properties of the distributions. Khan and King [22] proposed the generalized inverse Weibull distribution with reliability applicability of the distribution for the engineering studies, which has the upside-down hazard rate function. Okasha et al. [29] focused on the development of the inverse Weibull distribution by the Marshall-Olkin method that leads to a more flexible distribution for modeling lifetime data. Basheer [9] introduced the generalized alpha power inverse Weibull distribution, which was constructed via the alpha power transformation method. Recently, Afify et al. [1] introduced the extended odd Weibull exponential distribution, which covers the different types of density such as symmetric, asymmetric (right-skewed or left-skewed) and reversed-J shaped. Jia et al. [17] proposed the q-Weibull distribution, which can be applied to describe complex systems with maximum likelihood (ML) and least squares (LS) estimates. The q-Weibull distribution can model the different types of real-life data since it represents the unimodal, bathtub-shaped and monotone hazard rate function.

The necessity of the estimation of the probability density function (PDF) and CDF is felt due to their applications in the estimation of differential entropy, Renyi entropy, negentropy, Kullback-Leibler divergence, Fisher information, cumulative residual entropy, Bonferroni curve, Lorenz curve, hazard rate function, reverse hazard rate function, etc. According to this essential requirement, Bagheri et al. [8] estimated the PDF and the CDF of a three-parameter generalized Exponential-Poisson distribution when all parameters except the shape are considered to be known. Alizadeh et al. (2015a, 2015b, 2015c) concentrated on the estimation of the PDF and the CDF of the generalized exponential distribution, Weibull distribution and exponentiated Weibull distribution, respectively. Bagheri et al. [7] obtained the estimators of PDF and CDF of the Weibull extension model when all parameters except the shape are considered to be known. Alizadeh et al. [3] considered estimation of the PDF and CDF of the inverse Weibull distribution with three parameters and investigated the application properties of the maximum likelihood estimators via the simulation and real data.

Several studies have been provided in the concept of the PDF and CDF estimations for different distributions, included: Pareto [11], exponential Gumbel [6], generalized Logistic [31], Frechet [27], inverse Rayleigh [26], Lindley [25] and Pareto-Rayleigh [28].

Consider the random variable X from generalized inverted Weibull distribution, then the cumulative distribution and probability density functions of the generalized inverted Weibull distribution are given respectively, as

$$F(x) = 1 - \left(1 - e^{-(\frac{\lambda}{x})^{\beta}}\right)^{\alpha}, \qquad x > 0, \alpha, \beta, \lambda > 0,$$

and

$$f(x)=\frac{\alpha\beta\lambda^{\beta}}{x^{\beta+1}}e^{-(\frac{\lambda}{x})^{\beta}}\left(1-e^{-(\frac{\lambda}{x})^{\beta}}\right)^{\alpha-1},\qquad x>0,\alpha,\beta,\lambda>0.$$

The generalized inverse Weibull distribution is the same as inverse generalized Weibull due to [16].

The aim of this paper is to obtain the different estimation of the PDF and CDF of the generalized inverse Weibull distribution based on the parametric and non-parametric approaches. The performance of the estimation methods are evaluated by both simulation and real data, that both verify the superiority of the maximum likelihood method.

The contents of the paper are organized as follows. The parametric estimation methods, maximum likelihood and the uniformly minimum variance unbiased (UMVU) estimators, of the PDF and CDF and their mean square errors (MSE) are derived, respectively in Sections 2 and 3. In Section 4, the non-parametric estimation approaches as the percentile (PC), least squares and weighted least squares (WLS) of the PDF and CDF are discussed. The estimators are compared by simulation and two real data applications in Sections 5 and 6, respectively.

Its worth to mention that, throughout the paper (except for Section 6), both β and λ are supposed to be known, whereas the parameter α is let to be unknown.

2. Maximum likelihood estimator of the PDF and CDF of GIW distribution

Consider the random sample $X_1, X_2, \ldots, X_n$ from the generalized inverse Weibull distribution, the log-likelihood function is represented as

$$\begin{aligned}\ln L(\alpha,\beta,\lambda) &= n\ln\alpha+n\ln\beta+n\beta\ln\lambda-\sum_{i=1}^{n}(\frac{\lambda}{x_i})^{\beta}-(\beta+1)\sum_{i=1}^{n}\ln x_i\\ &\quad+(\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-(\frac{\lambda}{x_i})^{\beta}}\right),\end{aligned}$$

where λ and β are known. According to maximizing the log-likelihood function, the maximum likelihood estimator of the parameter α is computed as

$$\hat{\alpha}_{ML}=\frac{n}{-\sum_{i=1}^{n}\ln\left(1-e^{-(\frac{\lambda}{x_i})^{\beta}}\right)}.$$

Based on the invariant property of ML estimators, the ML estimators of the PDF and CDF of the GIW distribution are obtained as

$$\begin{aligned}\hat{f}(x) &= \frac{\hat{\alpha}_{ML}\beta\lambda^{\beta}}{x^{\beta+1}}e^{-(\frac{\lambda}{x})^{\beta}}\left(1-e^{-(\frac{\lambda}{x})^{\beta}}\right)^{\hat{\alpha}_{ML}-1},\qquad x>0,\\ \hat{F}(x) &= 1-\left(1-e^{-(\frac{\lambda}{x})^{\beta}}\right)^{\hat{\alpha}_{ML}},\qquad x>0,\end{aligned}$$

respectively. Consider the sufficient statistic $T=-\sum_{i=1}^{n}\ln\left(1-e^{-(\frac{\lambda}{x_i})^{\beta}}\right)$, it can be shown that T has Gamma distribution with the following PDF

$$f_T(t)=\frac{\alpha^n}{\Gamma(n)}t^{n-1}e^{-\alpha t},\qquad t>0,\ \alpha>0.$$

After some elementary algebra, the PDF of the ML estimator of α is obtained as follows

$$g(w)=\frac{(n\alpha)^n}{w^{n+1}\Gamma(n)}e^{-\frac{n\alpha}{w}},\quad w>0,\alpha>0,$$

where $w = \hat{\alpha}_{ML}$. It results that, $\hat{\alpha}_{ML}$ has the inverse Gamma distribution with parameters $\left(n, \frac{1}{n\alpha}\right)$. In the following, we obtain the r-th moment and MSE of the PDF and CDF estimators of the GIW distribution.

Theorem 2.1. *Based on the GIW distribution,*

i. The ML estimators of the PDF is biased and r-th moment of the $\hat{f}(x)$ is represented by

$$E\big(\hat{f}(x)^r\big) = \frac{2(n\alpha)^n}{\Gamma(n)} D_1^r \Bigg[\frac{n\alpha}{-r\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Bigg]^{\frac{r-n}{2}} K_{r-n}\Big(2\sqrt{-n\alpha r\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Big),$$

ii. The ML estimator of the CDF is biased and r-th moment of the $\hat{F}(x)$ is shown as

$$E\big(\hat{F}(x)^r\big) = 2\sum_{i=0}^{r}\binom{r}{i}(-1)^i\frac{(n\alpha)^n}{\Gamma(n)}\Big(\frac{n\alpha}{-i\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Big)^{-\frac{n}{2}} K_{-n}\Big(2\sqrt{-n\alpha i\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Big),$$

where $D_1 = \Big(\frac{\beta\lambda^\beta e^{-(\frac{\lambda}{x})^\beta}}{x^{\beta+1}}\Big)\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)^{-1}$ *and $K_\nu(.)$ denotes the modified Bessel function of the second kind of order ν and is defined as* $K_\nu(2\sqrt{\beta\varphi}) = 0.5\Big(\frac{\varphi}{\beta}\Big)^{\frac{\nu}{2}}\int_0^\infty x^{\nu-1}e^{-\frac{\beta}{x}}e^{-\varphi x}dx$.

Proof . *i. The ML estimator of α has inverse Gamma distribution with parameters $(n, \frac{1}{n\alpha})$, so*

$$\begin{aligned}
E\big(\hat{f}(x)^r\big) &= \int_0^\infty \Bigg[\frac{w\beta\lambda^\beta}{x^{\beta+1}}e^{-(\frac{\lambda}{x})^\beta}\Big(1-e^{-(\frac{\lambda}{x})^\beta}\Big)^{w-1}\Bigg]^r \frac{(n\alpha)^n}{w^{n+1}\Gamma(n)}e^{-\frac{n\alpha}{w}}dw \\
&= \frac{(n\alpha)^n}{\Gamma(n)}\Bigg(\frac{\beta\lambda^\beta e^{-(\frac{\lambda}{x})^\beta}}{x^{\beta+1}}\Bigg)^r\Big(1-e^{-(\frac{\lambda}{x})^\beta}\Big)^{-r}\int_0^\infty w^{r-n-1}\Big(1-e^{-(\frac{\lambda}{x})^\beta}\Big)^{rw}e^{-\frac{n\alpha}{w}}dw \\
&= \frac{(n\alpha)^n}{\Gamma(n)}D_1^r\int_0^\infty w^{r-n-1}e^{rw\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}e^{-\frac{n\alpha}{w}}dw \\
&= \frac{2(n\alpha)^n}{\Gamma(n)}D_1^r\Bigg[\frac{n\alpha}{-r\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Bigg]^{\frac{r-n}{2}}K_{r-n}\Big(2\sqrt{-n\alpha r\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Big),
\end{aligned}$$

where $w = \hat{\alpha}_{ML}$. For $r = 1$, the biasness of the PDF estimator is deduced.
ii. Analogously, for $\hat{F}(x)$, we have

$$\begin{aligned}
E\big(\hat{F}(x)^r\big) &= \int_0^\infty \Bigg[1-\Big(1-e^{-(\frac{\lambda}{x})^\beta}\Big)^w\Bigg]^r\frac{(n\alpha)^n}{w^{n+1}\Gamma(n)}e^{-\frac{n\alpha}{w}}dw \\
&= \int_0^\infty\sum_{i=0}^{r}\binom{r}{i}\Bigg[-\Big(1-e^{-(\frac{\lambda}{x})^\beta}\Big)^w\Bigg]^i\frac{(n\alpha)^n}{w^{n+1}\Gamma(n)}e^{-\frac{n\alpha}{w}}dw \\
&= \sum_{i=0}^{r}\binom{r}{i}(-1)^i\frac{(n\alpha)^n}{\Gamma(n)}\int_0^\infty w^{-n-1}\Big(1-e^{-(\frac{\lambda}{x})^\beta}\Big)^{iw}e^{-\frac{n\alpha}{w}}dw \\
&= \sum_{i=0}^{r}\binom{r}{i}(-1)^i\frac{(n\alpha)^n}{\Gamma(n)}\int_0^\infty w^{-n-1}e^{iw\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)-\frac{n\alpha}{w}}dw \\
&= 1+2\sum_{i=1}^{r}\binom{r}{i}(-1)^i\frac{(n\alpha)^n}{\Gamma(n)}\Bigg(\frac{n\alpha}{-i\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Bigg)^{-\frac{n}{2}}K_{-n}\Big(2\sqrt{-n\alpha i\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Big).
\end{aligned}$$

Obviously, the biasness of the $\hat{F}(x)$ is verified for $r = 1$, which complete the proof. □

Theorem 2.2. *Consider the GIW distribution,*

i. The MSE of the ML estimator of the PDF is given by

$$\begin{aligned} MSE\big(\hat{f}(x)\big) &= \frac{2(n\alpha)^n}{\Gamma(n)} D_1^2 \Bigg[\frac{n\alpha}{-2\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Bigg]^{\frac{2-n}{2}} K_{2-n}\Big(2\sqrt{-2n\alpha\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Big) \\ &-\frac{4(n\alpha)^n}{\Gamma(n)} D_1 \Bigg[\frac{n\alpha}{-\ln\Big(1-e^{-(\frac{\lambda}{x})^\beta}\Big)}\Bigg]^{\frac{1-n}{2}} K_{1-n}\Big(2\sqrt{-n\alpha\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Big) \\ &\Big(\frac{\alpha\beta\lambda^\beta}{x^{\beta+1}} e^{-(\frac{\lambda}{x})^\beta}\Big(1-e^{-(\frac{\lambda}{x})^\beta}\Big)^{\alpha-1}\Big) + \Big(\frac{\alpha\beta\lambda^\beta}{x^{\beta+1}} e^{-(\frac{\lambda}{x})^\beta}\Big(1-e^{-(\frac{\lambda}{x})^\beta}\Big)^{\alpha-1}\Big)^2. \end{aligned}$$

ii. The MSE of the ML estimator of the CDF is shown as

$$\begin{aligned} MSE\big(\hat{F}(x)\big) &= 1+\sum_{i=1}^{2}\binom{2}{i}(-1)^i\frac{(n\alpha)^n}{\Gamma(n)}\Bigg[2\Big(\frac{n\alpha}{-i\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Big)^{-\frac{n}{2}}\Bigg] \\ &K_{-n}\Big(2\sqrt{-n\alpha i\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Big) \\ &-2\Bigg[1-\frac{2(n\alpha)^n}{\Gamma(n)}\Big(\frac{n\alpha}{-\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Big)^{-\frac{n}{2}} K_{-n}\Big(2\sqrt{-n\alpha i\ln\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)}\Big)\Bigg] \\ &\Big(1-\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)^\alpha\Big)+\Big(1-\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)^\alpha\Big)^2. \end{aligned}$$

Proof . *By Theorem 2.1 and $MSE\big(\hat{f}(x)\big) = E\big(\hat{f}(x)^2\big) - 2f(x)E\big(\hat{f}(x)\big) + f(x)^2$, the proof is completed.* □

3. The uniformly minimum variance unbiased estimators of the PDF and CDF of GIW distribution

In this section, the uniformly minimum variance unbiased estimators of the PDF and CDF of the GIW distribution are derived with their moments and MSEs.
Consider the GIW random sample $X_1, X_2, \cdots, X_n$, where the parameters λ and β are considered to be known and α be unknown, then $T = -\sum_{i=1}^n \ln\big(1-e^{-(\frac{\lambda}{x_i})^\beta}\big)$ is a complete sufficient statistic for the unknown parameter α and has Gamma distribution with parameters n and $\frac{1}{\alpha}$, which denoted as Gamma$(n,\frac{1}{\alpha})$. To compute the UMVU estimators of the PDF and CDF of the GIW distribution, we need to obtain the joint PDF of X_1 and the complete sufficient statistic T, which discussed in the following Theorem.

Theorem 3.1. *The joint PDF of X_1 and T can be expressed as*

$$f(x_1,t) = \frac{\alpha^n\beta\lambda^\beta e^{-\alpha t}\Big(t+\ln\big(1-e^{-(\frac{\lambda}{x_1})^\beta}\big)\Big)^{n-2}}{\Gamma(n-1)x_1^{\beta+1}\Big(e^{(\frac{\lambda}{x_1})^\beta}-1\Big)}, \qquad t > -\ln\big(1-e^{-(\frac{\lambda}{x_1})^\beta}\big).$$

Proof . *Consider $U = -\sum_{i=2}^n \ln\big(1-e^{-(\frac{\lambda}{x_i})^\beta}\big)$, which has Gamma$\big(n-1,\frac{1}{\alpha}\big)$ distribution. Since U and X_1 are independent random variables, then the joint PDF of U and X_1 can be written as*

$$f_{U,X_1}(u,x_1) = f_U(u)f_{X_1}(x_1) = \Big(\frac{\alpha^{n-1}}{\Gamma(n-1)}u^{n-2}e^{-\alpha u}\Big)\Big(\frac{\alpha\beta\lambda^\beta}{x_1^{\beta+1}}e^{-(\frac{\lambda}{x_1})^\beta}\big(1-e^{-(\frac{\lambda}{x_1})^\beta}\big)^{\alpha-1}\Big).$$

Let $T = U - \ln\left(1 - e^{-(\frac{\lambda}{x_1})^\beta}\right)$ *and* $V = X_1$*, then*

$$\begin{aligned} f_{T,V}(t,v) &= \frac{\beta\lambda\alpha^n}{v^2\Gamma(n-1)}\Big(t + \ln\big(1 - e^{-\left(\frac{\lambda}{v}\right)^\beta}\big)\Big)^{n-2} e^{-\alpha u}\big(\frac{\lambda}{v}\big)^{\beta-1} e^{-\left(\frac{\lambda}{v}\right)^\beta} e^{(\alpha-1)(u-t)} \\ &= \frac{\alpha^n\beta\lambda^\beta e^{-\alpha t}}{\Gamma(n-1)v^{\beta+1}}\Big(t + \ln\big(1 - e^{-(\frac{\lambda}{v})^\beta}\big)\Big)^{n-2} \frac{e^{-\left(\frac{\lambda}{v}\right)^\beta}}{1 - e^{-\left(\frac{\lambda}{v}\right)^\beta}}, \qquad t > -\ln\big(1 - e^{-(\frac{\lambda}{v})^\beta}\big). \end{aligned}$$

By substituting X_1 *instead of* V*, the proof is completed.* □

Theorem 3.2. *The UMVU estimators of the PDF and CDF of the GIW distribution are given by*

$$\begin{aligned} \tilde{f}(x) &= f_{X_1|T}(x_1|t) = \frac{(n-1)\beta\lambda^\beta\Big(t + \ln\big(1 - e^{-(\frac{\lambda}{x_1})^\beta}\big)\Big)^{n-2}}{x_1^{\beta+1}t^{n-1}\Big(e^{(\frac{\lambda}{x_1})^\beta} - 1\Big)}, \qquad t > -\ln\big(1 - e^{-(\frac{\lambda}{x_1})^\beta}\big), \\ \tilde{F}(x) &= F_{X_1|T}(x_1|t) = 1 - \Big(1 + \frac{\ln\big(1 - e^{-\left(\frac{\lambda}{x_1}\right)^\beta}\big)}{t}\Big)^{n-1}, \qquad t > -\ln\big(1 - e^{-(\frac{\lambda}{x_1})^\beta}\big), \end{aligned}$$

respectively.

Proof . *First, we show that* $f_{X_1|T}(x_1|t)$ *is an unbiased estimator of the* $f(x)$*. The distribution of* X_1 *given* T *is computed as*

$$f_{X_1|T}(x_1|t) = \frac{f(x_1,t)}{f_T(t)} = \frac{(n-1)\beta\lambda^\beta\Big(t + \ln\big(1 - e^{-(\frac{\lambda}{x_1})^\beta}\big)\Big)^{n-2}}{x_1^{\beta+1}t^{n-1}\Big(e^{(\frac{\lambda}{x_1})^\beta} - 1\Big)}, \qquad t > -\ln\big(1 - e^{-(\frac{\lambda}{x_1})^\beta}\big).$$

The estimator $f_{X_1|T}(x_1|t)$ *is unbiased, which is investigated as below,*

$$E(\tilde{f}(x)) = E\left(f_{X_1|T}(x_1|t)\right) = \int_t f_{X_1|T}(x_1|t)f_T(t)dt = \int_t f_{X_1,T}(x_1,t)dt = f_{X_1}(x_1) = f(x).$$

So, $f_{X_1|T}(x_1|t)$ *is an unbiased estimator of the* $f(x)$*. Since* T *is a complete sufficient statistic, according to the Lehmann Scheffe theorem,* $f_{X_1|T}(x_1|t)$ *is the UMVU estimator of* $f(x)$*.*
Similar process is performed for the CDF,

$$F_{X_1|T}(x_1|t) = \int_0^{x_1} f_{X_1|T}(y|t)dy = \int_0^{x_1} \frac{(n-1)\beta\lambda^\beta\Big(t + \ln\big(1 - e^{-(\frac{\lambda}{y})^\beta}\big)\Big)^{n-2}}{y^{\beta+1}t^{n-1}\big(e^{(\frac{\lambda}{y})^\beta} - 1\big)}dy,$$

let $-\ln\big(1 - e^{-(\frac{\lambda}{y})^\beta}\big) = w$*, therefore*

$$\begin{aligned} F_{X_1|T}(x_1|t) &= \int_0^{-\ln(1-e^{-(\frac{\lambda}{x_1})^\beta})} \frac{(n-1)(t-w)^{n-2}}{t^{n-1}}dw \\ &= 1 - \Big(\frac{t + \ln\big(1 - e^{-(\frac{\lambda}{x_1})^\beta}\big)}{t}\Big)^{n-1}, \qquad t > -\ln\big(1 - e^{-(\frac{\lambda}{x_1})^\beta}\big). \end{aligned}$$

Analogously, we can obtain

$$E(\tilde{F}(x)) = E\left(F_{X_1|T}(x_1|t)\right) = \int_t F_{X_1|T}(x_1|t) f_T(t)dt = \int_t \int_0^{x_1} f_{X_1|T}(y|t) f_T(t)dydt$$
$$= \int_0^{x_1} \int_t f_{X_1,t}(y,t)dtdy = \int_0^{x_1} f_{X_1}(y)dy = F_{X_1}(x_1) = F(x).$$

Therefore, $F_{X_1|T}(x_1|t)$ is an unbiased estimator of $F(x)$. Since T is a complete sufficient statistic, according to Lehmann Scheffe theorem $F_{X_1|T}(x)$ is the UMVU estimator of $F(x)$. □

Proposition 3.3. *The upper incomplete gamma function for $n = 1, 2, \ldots$ is defined as below*

$$\Gamma(n,q) = \int_q^\infty t^{n-1}e^{-t}dt,$$
$$\Gamma(-n,q) = \frac{1}{n!}\Big[\frac{e^{-q}}{q^n}\sum_{k=0}^{n-1}(-1)^k(n-k-1)!q^k + (-1)^n\Gamma(0,q)\Big],$$

where $\Gamma(0,q) = -Ei(-q)$ and $Ei(.)$ is the exponential integral function which defined as $Ei(-q) = -\int_q^\infty t^{-1}e^{-t}dt$.

Theorem 3.4. *The r-th moment of the UMVU estimator of the PDF of the GIW distribution is represented as*

$$E\big(\tilde{f}(x)^r\big) = D^r \sum_{j=0}^{r(n-2)} \binom{r(n-2)}{j} a^j \alpha^{r+j} \frac{\Gamma\left(n-r-j, -a\alpha\right)}{\Gamma(n)},$$

where $D = \frac{(n-1)\beta\lambda^\beta}{x^{\beta+1}\left(e^{(\lambda/x)^\beta}-1\right)}$, $a = \ln\left(1 - e^{-(\frac{\lambda}{y})^\beta}\right)$ and $\Gamma(.,.)$ is the upper incomplete gamma function.

Proof . *As mentioned before, the complete sufficient statistic T has Gamma distribution with the parameters n and $\frac{1}{\alpha}$, therefore*

$$E\big(\tilde{f}(x)^r\big) = \int_{-a}^\infty \left[\frac{(n-1)\beta\lambda^\beta}{x^{\beta+1}t^{n-1}} \times \frac{(t+a)^{n-2}}{e^{(\frac{\lambda}{x})^\beta}-1}\right]^r \frac{\alpha^n}{\Gamma(n)} t^{n-1}e^{-\alpha t}dt$$
$$= \frac{\alpha^n D^r}{\Gamma(n)} \int_{-a}^\infty \left(1+\frac{a}{t}\right)^{r(n-2)} t^{n-r-1}e^{-\alpha t}dt$$
$$= \frac{\alpha^n D^r}{\Gamma(n)} \int_{-a}^\infty \sum_{j=0}^{r(n-2)} \binom{r(n-2)}{j} \left(\frac{a}{t}\right)^j t^{n-r-1}e^{-\alpha t}dt$$
$$= \frac{\alpha^n D^r}{\Gamma(n)} \sum_{j=0}^{r(n-2)} \binom{r(n-2)}{j} a^j \int_{-a}^\infty t^{n-r-j-1}e^{-\alpha t}dt$$
$$= D^r \sum_{j=0}^{r(n-2)} \binom{r(n-2)}{j} a^j \alpha^{r+j} \frac{\Gamma(n-r-j, -a\alpha)}{\Gamma(n)}.$$

□

Theorem 3.5. *The MSE of UMVU estimator of the PDF of GIW distribution is given by*

$$MSE(\tilde{f}(x)) = D^2 \sum_{j=0}^{2(n-2)} \binom{2(n-2)}{j} a^j \alpha^{2+j} \frac{\Gamma(n-2-j,-a\alpha)}{\Gamma(n)} - \left(\frac{\alpha\beta\lambda^\beta}{x^{\beta+1}} e^{-(\frac{\lambda}{x})^\beta} \left(1 - e^{-(\frac{\lambda}{x})^\beta}\right)^{\alpha-1} \right)^2.$$

Proof . *By Theorem 3.4 and the definition of MSE, the proof is achieved.* □

In the following, we obtain the second moment of the UMVU estimator $\tilde{F}(x)$, which can be utilized in the computation of the MSE.

Theorem 3.6. *The second moment of the UMVU estimator of the CDF of GIW distribution are demonstrate as follows*

$$E(\tilde{F}(x)^2) = \frac{\Gamma(n,-a\alpha)}{\Gamma(n)} - \frac{2}{\Gamma(n)} \sum_{j=0}^{n-1} \binom{n-1}{j} (a\alpha)^j \frac{\Gamma(n-j,-a\alpha)}{\Gamma(n)} + \sum_{j=0}^{2n-2} \binom{2n-2}{j} (a\alpha)^j \frac{\Gamma(n-j,-a\alpha)}{\Gamma(n)}.$$

Proof . *The second moment of the $\tilde{F}(x)$ is calculated as below*

$$\begin{aligned} E(\tilde{F}(x)^2) &= \int_{-a}^{\infty} \left[1 - \left(1+\frac{a}{t}\right)^{n-1}\right]^2 \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t} dt \\ &= \underbrace{\int_{-a}^{\infty} \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t} dt}_{I} - \underbrace{2\int_{-a}^{\infty} \left(1+\frac{a}{t}\right)^{n-1} \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t} dt}_{II} \\ &\quad + \underbrace{\int_{-a}^{\infty} \left(1+\frac{a}{t}\right)^{2n-2} \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t} dt}_{III}, \end{aligned}$$

where

$$I = \frac{\Gamma(n,-a\alpha)}{\Gamma(n)},$$

$$\begin{aligned} II &= 2\frac{\alpha^n}{\Gamma(n)} \int_{-a}^{\infty} \left(1+\frac{a}{t}\right)^{n-1} t^{n-1} e^{-\alpha t} dt = 2\frac{\alpha^n}{\Gamma(n)} \int_{-a}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{a}{t}\right)^j t^{n-1} e^{-\alpha t} dt \\ &= 2\frac{\alpha^n}{\Gamma(n)} \sum_{j=0}^{n-1} \binom{n-1}{j} a^j \int_{-a}^{\infty} t^{n-j-1} e^{-\alpha t} dt = \frac{2}{\Gamma(n)} \sum_{j=0}^{n-1} \binom{n-1}{j} (a\alpha)^j \frac{\Gamma(n-j,-a\alpha)}{\Gamma(n)}, \end{aligned}$$

and

$$\begin{aligned} III &= \frac{\alpha^n}{\Gamma(n)} \int_{-a}^{\infty} \sum_{j=0}^{2n-2} \binom{2n-2}{j} \left(\frac{a}{t}\right)^j t^{n-1} e^{-\alpha t} dt = \frac{\alpha^n}{\Gamma(n)} \sum_{j=0}^{2n-2} \binom{2n-2}{j} a^j \int_{-a}^{\infty} t^{n-j-1} e^{-\alpha t} dt \\ &= \sum_{j=0}^{2n-2} \binom{2n-2}{j} (a\alpha)^j \frac{\Gamma(n-j,-a\alpha)}{\Gamma(n)}. \end{aligned}$$

□

Theorem 3.7. *The MSE of $\tilde{F}(x)$ is given by*

$$MSE\big(\tilde{F}(x)\big) = \frac{\Gamma(n,-a\alpha)}{\Gamma(n)} - \frac{2}{\Gamma(n)}\sum_{j=0}^{n-1}\binom{n-1}{j}(a\alpha)^j\frac{\Gamma(n-j,-a\alpha)}{\Gamma(n)}$$
$$+\sum_{j=0}^{2n-2}\binom{2n-2}{j}(a\alpha)^j\frac{\Gamma(n-j,-a\alpha)}{\Gamma(n)} - \Big(1-\big(1-e^{-(\frac{\lambda}{x})^\beta}\big)^\alpha\Big)^2.$$

Proof . *By using Theorem 3.6 and the definition of MSE, the proof is completed.* □

4. Non-parametric estimators

In this section, we focused on some non-parametric estimation approaches as percentiles, least squares and weighted least squares estimators.

4.1. Estimators based on percentiles

Percentile estimators provided by Kao (1959, 1958), which are based on the inverting the CDF. Since the GIW distribution has a closed-form CDF, therefore the parameters of the GIW distribution can be estimated using percentiles.
Consider the random sample $X_1, X_2, \ldots, X_n$ from the GIW distribution and let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the corresponding order statistics in the ascending order. Also let $p_i = \frac{i}{(n+1)}$.
The percentile estimator of α (when λ and β are consider to be known) say $\tilde{\alpha}_{PC}$ is the value that minimize the following expression

$$\sum_{i=1}^{n}\Big[\ln(1-p_i) - \alpha\ln\big(1-e^{-(\frac{\lambda}{x_{(i)}})^\beta}\big)\Big]^2.$$

So, the percentile estimators of the parameter α can be obtained as

$$\frac{d}{d\alpha}\sum_{i=1}^{n}\Big[\ln(1-p_i) - \alpha\ln\big(1-e^{-(\frac{\lambda}{x_{(i)}})^\beta}\big)\Big]^2 = 0,$$

therefore

$$-2\sum_{i=1}^{n}\ln\Big(1-e^{-(\frac{\lambda}{x_{(i)}})^\beta}\Big)\Big[\ln(1-p_i) - \alpha\ln\big(1-e^{-(\frac{\lambda}{x_{(i)}})^\beta}\big)\Big] = 0.$$

After some calculation, the estimator of α, is given as

$$\tilde{\alpha}_{PC} = \frac{\sum_{i=1}^{n}\ln\Big(1-e^{-(\frac{\lambda}{x_{(i)}})^\beta}\Big)\ln(1-p_i)}{\sum_{i=1}^{n}\Big[\ln\big(1-e^{-(\frac{\lambda}{x_{(i)}})^\beta}\big)\Big]^2}.$$

Subsequently, the percentile estimators of the PDF and CDF are represented respectively, as

$$\tilde{f}_{PC}(x) = \frac{\tilde{\alpha}_{PC}\beta\lambda^\beta}{x^{\beta+1}}e^{-(\frac{\lambda}{x})^\beta}\Big(1-e^{-(\frac{\lambda}{x})^\beta}\Big)^{\tilde{\alpha}_{PC}-1},$$
$$\tilde{F}_{PC}(x) = 1-\Big(1-e^{-(\frac{\lambda}{x})^\beta}\Big)^{\tilde{\alpha}_{PC}}.$$

The expectations and the MSE of these estimators can be calculated by simulation.

4.2. Least squares estimators

The least squares estimator of α (when λ and β are known), say $\tilde{\alpha}_{LS}$, is the value that minimizing the following expression

$$\sum_{i=1}^{n}\Big[1-\big(1-e^{-(\frac{\lambda}{x_{(i)}})^{\beta}}\big)^{\alpha}-p_i\Big]^2.$$

Accordingly, the LS estimators of the PDF and CDF are shown respectively, as

$$\begin{aligned}\tilde{f}_{LS}(x) &= \frac{\tilde{\alpha}_{LS}\beta\lambda^{\beta}}{x^{\beta+1}}e^{-\left(\frac{\lambda}{x}\right)^{\beta}}\left(1-e^{-\left(\frac{\lambda}{x}\right)^{\beta}}\right)^{\hat{\alpha}_{LS}-1},\\ \tilde{F}_{LS}(x) &= 1-\left(1-e^{-(\frac{\lambda}{x})^{\beta}}\right)^{\hat{\alpha}_{LS}}.\end{aligned}$$

4.3. Weighted least squares estimators

The weighted least squares estimator of α (when λ and β are known), say $\tilde{\alpha}_{WLS}$ is the value that minimizing

$$\sum_{i=1}^{n}W_i\Big[1-\big(1-e^{-(\frac{\lambda}{x_{(i)}})^{\beta}}\big)^{\alpha}-p_i\Big]^2,$$

where $W_i=\left(Var\left(F(X_{(i)})\right)\right)^{-1}=\frac{(n+1)^2(n+2)}{i(n-i+1)}$. So, the WLS estimators of the PDF and CDF are illustrated respectively, by

$$\begin{aligned}\tilde{f}_{WLS}(x) &= \frac{\hat{\alpha}_{WLS}\beta\lambda^{\beta}}{x^{\beta+1}}e^{-\left(\frac{\lambda}{x}\right)^{\beta}}\left(1-e^{-(\frac{\lambda}{x})^{\beta}}\right)^{\hat{\alpha}_{WLS}-1},\\ \tilde{F}_{WLS}(x) &= 1-\left(1-e^{-(\frac{\lambda}{x})^{\beta}}\right)^{\hat{\alpha}_{WLS}}.\end{aligned}$$

Since mathematical computation of the expectation and MSE of these estimators are difficult, so simulation scheme will be applied for computing non-parametric estimators.

5. Comparison between the ML, UMVU, PC, LS and WLS estimators

Due to comparison between the five different estimation approaches (ML, UMVU, PC, LS and WLS) of the PDF and CDF of the GIW distribution, based on the efficiency, we generate data sets from the GIW distribution with different combinations of the parameters as $(\alpha,\beta,\lambda)=(0.5,0.5,0.5)$, $(1.5,0.5,2)$, $(0.5,2,1.5)$, $(4,2,3)$, $(3,2,4)$, $(2,3,4)$, $(9,5,2)$, $(2,9,5)$, $(2,5,9)$ and different sample size $n=(10,20,\ldots,60)$. In the following, we provide the algorithm of the simulation.

Step 1. For the certain value of the parametres α,β,λ with the sample size n, we generate a random sample from GIW(α,β,λ) distribution, then we compute the estimates of α with respect to different estimation methods as $(\hat{\alpha}_{ML},\tilde{\alpha}_{UMVU},\tilde{\alpha}_{PC},\tilde{\alpha}_{LS},\tilde{\alpha}_{WLS})$.

Step 2. The integrate square errors (ISE) of the generated random sample (Step 1) are computed via

$$\begin{aligned}\text{ISE}_{ML}(\hat{f}(x)) &= \frac{\sum_{i=1}^{n}\left(\hat{f}_{ML}(x_i)-f(x_i)\right)^2}{n},\\ \text{ISE}_{ML}(\hat{F}(x)) &= \frac{\sum_{i=1}^{n}\left(\hat{F}_{ML}(x_i)-F(x_i)\right)^2}{n}.\end{aligned}$$

Similarly, the ISE for the other estimation methods (UMVU, PC, LS and WLS) can be computed.

Step 3. The Step 1 and 2 are repeated for M=1000 times.

Step 4. The mean of 1000 ISEs, mean integrate square errors (MISE), are computed, that provided in Step 3.

All programs are written by **R** software. Its worth to mention that, the minimization for estimation of the PDF and CDF under PC, LS and WLS methods are performed based on the **nlm** (or **optim**) command in **R** software, for simulation data in Sections 5 and real data in Section 6.

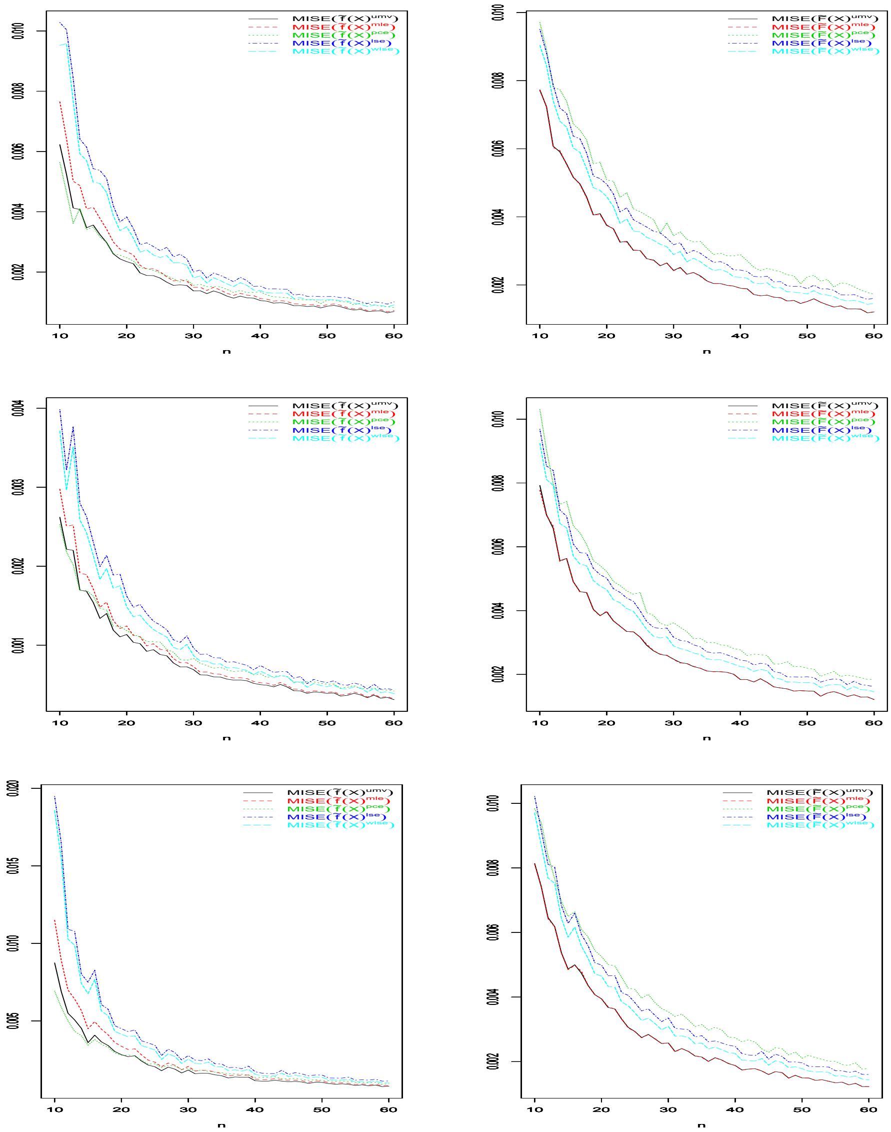

Figure 1: The comparison of MISEs of the ML, UMVU, PC, LS and WLS estimators of the PDF and CDF for $(\alpha, \beta, \lambda) = (1.5, 0.5, 2)$, (0.5,2, 1.5) and (0.5, 0.5, 0.5) based on simulation results, respectively.

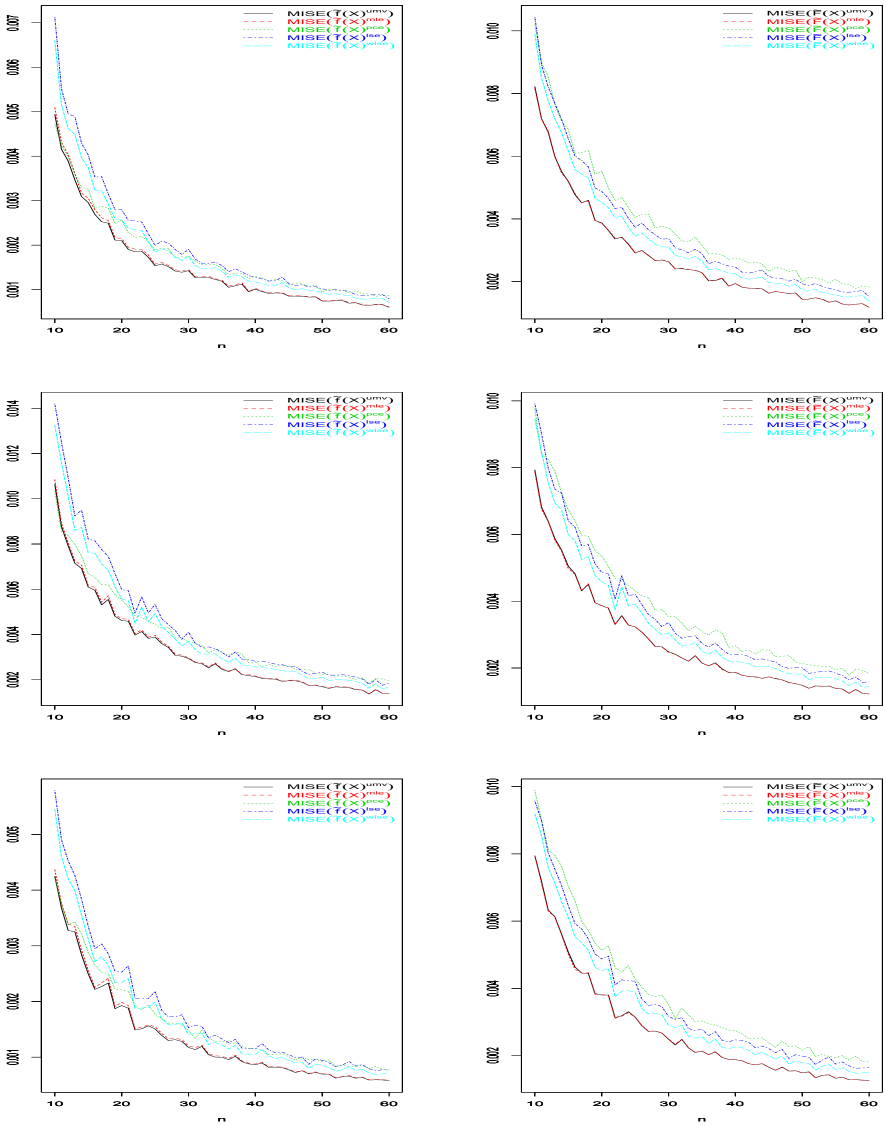

Figure 2: The comparison of MISEs of the ML, UMVU, PC, LS and WLS estimators of the PDF and CDF for $(\alpha, \beta, \lambda) = (2, 3, 4)$, (4, 2, 3) and (3, 2, 4) based on simulation results, respectively.

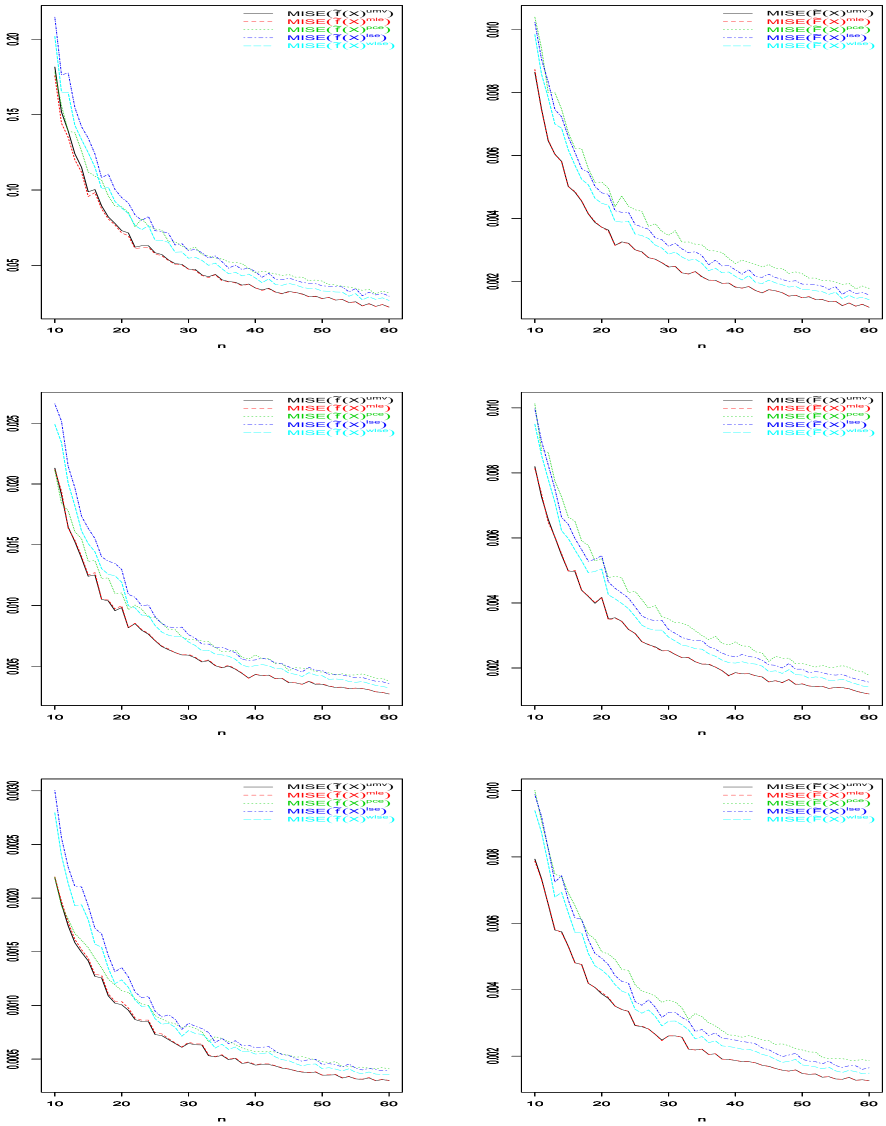

Figure 3: The comparison of MISEs of the ML, UMVU, PC, LS and WLS estimators of the PDF and CDF for $(\alpha, \beta, \lambda) = (9, 5, 2)$, (2, 9, 5) and (2, 5, 9) based on simulation results, respectively.

As can be seen from the Figures 1-3, the ML estimators of the PDF and CDF are more efficient than other estimators, the UMVU estimators of the PDF and CDF are more efficient than non-parametric estimators. As we expected, the parametric methods (ML and UMVU) are more efficient than the non-parametric ones (PC, LS and WLS). Likewise, PC estimators of the PDF and CDF are more efficient than LS and WLS estimators and also LS estimators of the PDF and CDF are more efficient than WLS estimators. In addition, it is observed that MISEs decrease with increasing sample sizes.

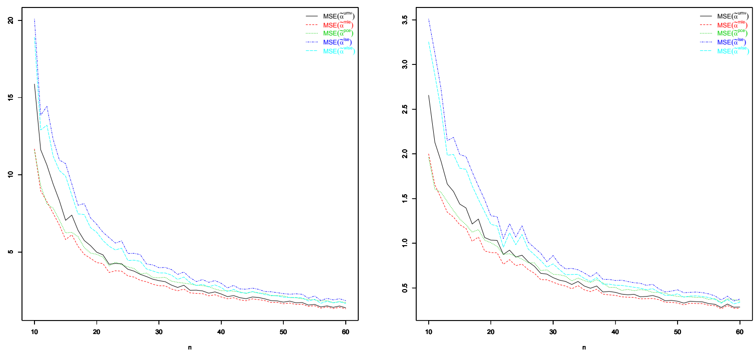

Figure 4: The MSE of the different estimators of the parameter α for $(\alpha, \beta, \lambda) = (9, 5, 2)$ (left) and $(\alpha, \beta, \lambda) = (4, 2, 3)$ (right).

Also, for more discussion in Figure 4, we compare the MSEs of $\hat{\alpha}_{ML}$, $\tilde{\alpha}_{UMVU}$, $\tilde{\alpha}_{PC}$, $\tilde{\alpha}_{LS}$ and $\tilde{\alpha}_{WLS}$ in different situations. Obviously, the ML estimator of the parameter α is more efficient than the other estimators. Also, it is observed that MSEs decrease with increasing sample sizes.

6. Data analysis

In this section, we use two real data sets and compare the different estimation approaches of the PDF and CDF of the GIW distribution.

First real data set: The first data set represents the tensile strength of 100 carbon fibers [14]. Based on tensile strength data, the Kolmogorov-Smirnov (K-S) statistic of the GIW distribution is computed about 0.0666 with p-value= 0.7659 under ML estimation, which means that GIW distribution is fitted well to tensile strength data. In Table 1, we present the estimates of α, β, λ and corresponding log-likelihood of the tensile strength data.

Table 1: Estimate of the parameters and corresponding log-likelihood for the tensile strength data.

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	Log-likelihood
ML	10266.4267	0.3108	3674.6543	-141.5167
PC	800.5651	0.4139	283.2363	-142.0424
LS	1372.2353	0.3718	593.6500	-141.7463
WLS	1697.2930	0.3716	636.8529	-141.6902

In Figures 5-7, the Q-Q plot, different estimates of PDF along with histogram and the empirical CDF and fitted CDF of the tensile strength data are depicted, respectively. Based on these plots, the ML estimator provides the best fit for the tensile strength data.

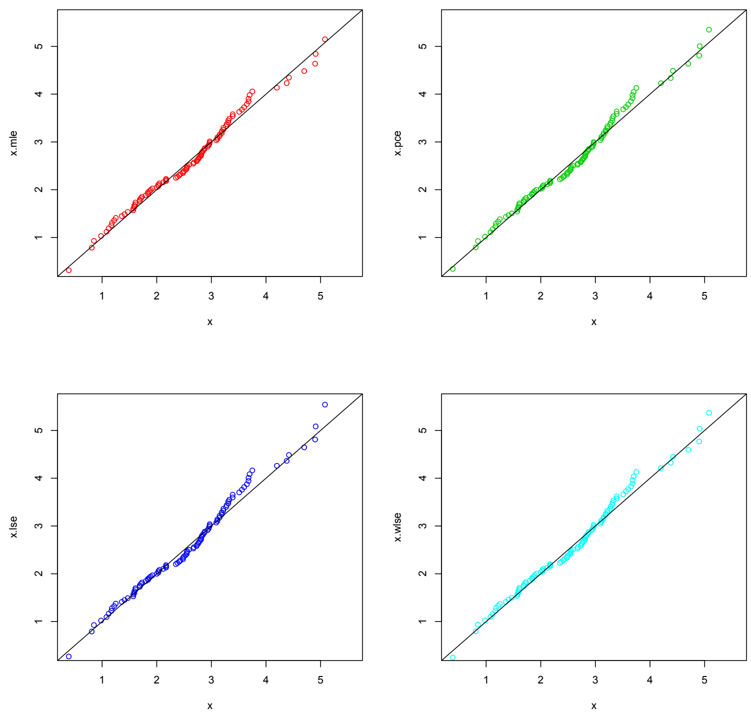

Figure 5: The Q-Q plot for the tensile strength data versus different estimators.

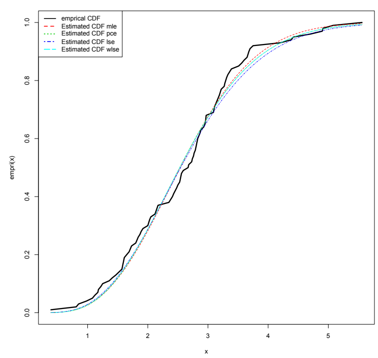

Figure 6: The empirical CDF and fitted CDF for the tensile strength data.

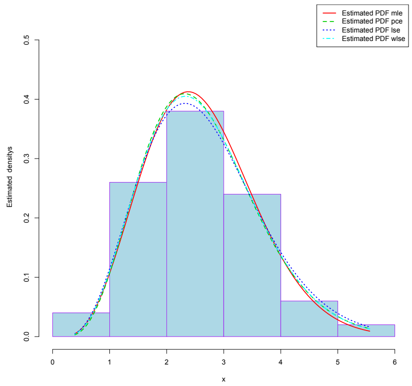

Figure 7: The different estimates of the PDF along with histogram of the tensile strength data.

Also, in this case, we consider some model selection criteria (MSCs) as "pure" maximum likelihood, Akaike information criterion (AIC), corrected AIC (AICc), Bayes information criterion (BIC, also known as Schwarz criterion), and Hannan-Quinn criterion (HQC). For more discussion about the MSCs see [10] and [13].

Table 2: The values of model selection criteria for the tensile strength data.

	ML	AIC	BIC	AICc	HQC
ML	283.0335	289.0335	296.8491	289.2835	292.1966
PC	284.0848	290.0848	297.9004	290.3348	293.2479
LS	283.4926	289.4926	297.3081	289.7426	292.6556
WLS	283.3804	289.3804	297.1959	289.6304	292.5435

In Table 2, we represent the values of model selection criteria of GIW distribution for the tensile strength data. from Table 2, all model selection criteria show that the ML estimator is better than the others.

Second real data set: The second data set [24] is the number of million revolutions before failure of twenty-three ball bearings, that collected from tests on the endurance of deep groove ball bearings.

Based on the ball bearings data, the K-S statistic of the GIW distribution is computed about 0.1114 with p-value= 0.9374 under ML estimation, which means that the GIW distribution is fitted well to ball bearings data. In Table 3, the estimates of α, β, λ and corresponding log-likelihood of the endurance of deep groove ball bearings data are demonstrated.

Table 3: Estimate of the parameters and corresponding log-likelihood for the ball bearings data.

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	Log-likelihood
ML	80.7143	0.4656	1865.2662	-112.9732
PC	17.4416	0.6129	438.2961	-113.2564
LS	4.9476	0.9462	134.6037	-113.6118
WLS	16.9615	0.6235	419.4374	-113.2289

In Table 4, the values of model selection criteria for the ball bearings data are represented that all the model selection criteria confirm the suitability of the ML estimators than the others.

Table 4: The values of model selection criteria for the ball bearings data.

	ML	AIC	BIC	AICc	HQC
ML	225.9464	231.9464	235.3529	233.2095	232.8031
PC	226.5128	232.5128	235.9193	233.7760	233.3695
LS	227.2237	233.2237	236.6302	234.4868	234.0804
WLS	226.4579	232.4579	235.8644	233.7211	233.3146

In Figures 8-10, the Q-Q plot, different estimates of PDF along with histogram and the empirical CDF and fitted CDF of the ball bearings data are illustrated, respectively, that indicate the superiority of ML estimators to fit the ball bearings data.

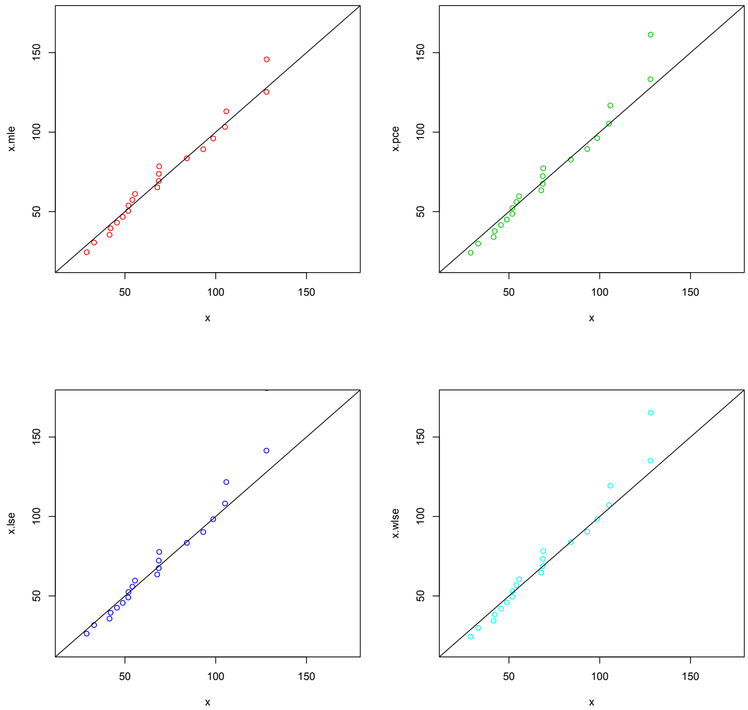

Figure 8: The Q-Q plot of endurance of deep groove ball bearings data versus different estimators.

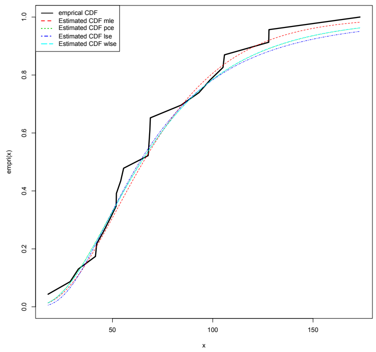

Figure 9: The empirical CDF and fitted CDF of the ball bearings data.

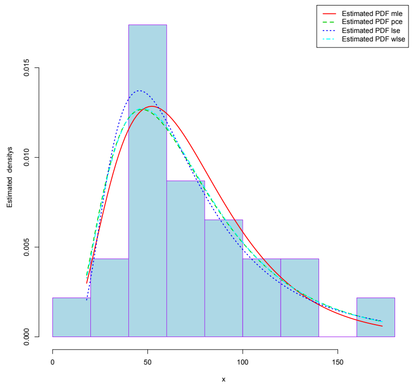

Figure 10: The different estimates of PDF along with histogram of the ball bearings data.

Concluding remarks

The different estimation approaches included the parametric (ML and UMVU) and non-parametric (PS, LS and WLS) methods of the PDF and CDF of the GIW distribution are compared with the certain values of location and scale parameters. Explicit expressions are given for the MISEs of the UMVU and ML estimators. The performances of the five estimation methods are evaluated by simulation and two real data sets. The results show suitable performance of the ML than other estimators based on the MISEs, log-likelihood, Q-Q plots, density plots and five model selection criteria. The best estimators of the PDF can be utilized to estimate functional forms of the PDF such as the differential entropy, negentropy, Rényi entropy, Kulback-Liebler divergence and Fisher information. Similarly, the best estimators of the CDF can be used to estimate cumulative residual entropy, quantile function, Bonferroni and Lorenz curves. Consequently, the best estimators of both PDF and CDF can be used to estimate functional forms of the PDF and CDF such as probability weighted moments, hazard rate function and mean deviation from the mean.

Acknowledgments

The authors would like to thank the referees, the Associate Editor and the Editor for careful reading and for their comments which greatly improved the paper.

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