Int. J. Nonlinear Anal. Appl. 12 (2021) No. 1, 1057-1071
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2019.16025.1842

Gerghaty type results via simulation and $\mathcal{C}$-class functions with application

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper we study the notion of Gerghaty type contractive mapping via simulation function along with $\mathcal{C}$-class functions and prove the existence of several fixed point results in ordinary and partially ordered metric spaces. An example is given to show the validity of our results given herein. Moreover, existence of solution of two-point boundary value second order nonlinear differential equation is obtain.

Keywords: Simulation functions, $\mathcal{C}$-class function, partially ordered metric space.
2010 MSC: Primary 54H25; Secondary 47H10.

1. Introduction

The Banach contraction principle [4] is one of the fundamental result in metric fixed point theory. Because of its importance in nonlinear analysis, number of authors have improved, generalized and extended this basic result either by defining a new contractive mapping in the context of a complete metric space or by investigating the existing contractive mappings in various abstract spaces (see, e.g., [1, 5, 6, 7, 20, 28] and references therein).

In particularly, Geraghty [8] consider an auxiliary function and generalized the Banach contraction in the frame work of complete metric space. Later on, Amini-Harandi and Emami [28] obtained similar results in the setting of partially ordered metric spaces. Using the concept of Samet [11], Cho *et al.* [6] generalized Geraghty contraction to α-Geraghty contraction and prove a fixed point theorem for such contraction. On the other hand, Khojasteh *et al.* [15] introduced the notion of $\mathcal{Z}$-contraction by using a function called simulation function and proved a version of Banach contraction principle.

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Received: September 2018 *Revised:* November 2019

2. Preliminaries

In this section we present some basic notions and results from the literature:
We denote by $\mathcal{F}$ the class of all functions $\beta : [0,\infty) \to [0,1)$ satisfying $\beta(t_n) \to 1$, implies $t_n \to 0$ as $n \to \infty$.

Definition 2.1. *[8] Let (X, d) be a metric space. A map $T : X \to X$ is called Geraghty contraction if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$

Theorem 2.2. *[8] Let (X, d) be a complete metric space. Mapping $T : X \to X$ is Geraghty contraction. Then T has a fixed point $x \in X$, and $\{T^n x_1\}$ converges to x.*

In 2015, Khojasteh *et al.* [15] introduced simulation function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$, satisfying the following assertions:

(ζ_1) $\zeta(0, 0) = 0$;

(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ_3) If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then

$$\lim_{n\to\infty} sup\, \zeta(t_n, s_n) < 0.$$

Definition 2.3. *[15] Let (X, d) be a metric space, $T : X \to X$ a mapping and ζ a simulation function. Then T is called a $\mathcal{Z}$-contraction with respect to ζ, if it satisfies*

$$\zeta(d(Tu, Tv), d(u, v)) \geq 0 \quad \textit{for all } u, v \in X. \tag{2.1}$$

Theorem 2.4. *[15] Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to ζ. Then T has a unique fixed point $u \in X$ and for every $x_0 \in X$, the Picard sequence $\{x_n\}$ where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to this fixed point of T.*

Example 2.5. *[15] Let $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$, $i = 1, 2, 3$ be defined by*

(i) $\zeta_1(t, s) = \lambda s - t$, where $\lambda \in (0, 1)$;

(ii) $\zeta_2(t, s) = s\varphi(s) - t$, where $\varphi : [0,\infty) \to [0, 1)$ is a mapping such that $\lim_{t\to r^+} sup\, \psi(t) < 1$ for all $r > 0$;

(iii) $\zeta_3 = s - \psi(s) - t$, where $\psi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\psi(t) = 0$ if and only if $t = 0$.

Then ζ_i for $i = 1, 2, 3$ are simulation functions.

Roldán-López-de-Hierro *et al.* [24] modified the notion of a simulation function by replacing (ζ_3) by (ζ_3'),

(ζ_3') : if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then

$$\lim_{n\to\infty} sup \zeta(t_n, s_n) < 0.$$

The function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying $(\zeta_1 - \zeta 2)$ and (ζ_3') is called simulation function in the sense of Roldán-López-de-Hierro.

Definition 2.6. *[2] A mapping $\mathcal{G} : [0,+\infty)^2 \to \mathbb{R}$ is called a $\mathcal{C}$-class function if it is continuous and satisfies the following conditions:*

(1) $\mathcal{G}(s,t) \leq s$;

(2) $\mathcal{G}(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$*, for all* $s,t \in [0,+\infty)$.

Definition 2.7. *[18] A mapping $\mathcal{G} : [0,+\infty)^2 \to \mathbb{R}$ has the property $\mathcal{C}_{\mathcal{G}}$, if there exists and $\mathcal{C}_{\mathcal{G}} \geq 0$ such that*

(1) $\mathcal{G}(s,t) > \mathcal{C}_{\mathcal{G}}$ *implies* $s > t$;

(2) $\mathcal{G}(s,t) \leq \mathcal{C}_{\mathcal{G}}$*, for all* $t \in [0,+\infty)$.

Some examples of $\mathcal{C}$-class functions that have property $\mathcal{C}_{\mathcal{G}}$ are as follows:

(a) $\mathcal{G}(s,t) = s - t,\ \mathcal{C}_{\mathcal{G}} = r, r \in [0,+\infty)$;

(b) $\mathcal{G}(s,t) = s - \frac{(2+t)t}{(1+t)}, \mathcal{C}_{\mathcal{G}} = 0$;

(c) $\mathcal{G}(s,t) = \frac{s}{1+kt}, k \geq 1, \mathcal{C}_{\mathcal{G}} = \frac{r}{1+k}, r \geq 2$.

For more examples of $\mathcal{C}$-class functions that have property $\mathcal{C}_{\mathcal{G}}$ see [3, 7, 18].

Definition 2.8. *[18] A $\mathcal{C}_{\mathcal{G}}$ simulation function is a mapping $\mathcal{G} : [0,+\infty)^2 \to \mathcal{R}$ satisfying the following conditions:*

(1) $\zeta(t,s) < \mathcal{G}(s,t)$ *for all* $t,s > 0$*, where* $\mathcal{G} : [0,+\infty)^2 \to \mathbb{R}$ *is a* $\mathcal{C}$*-class function;*

(2) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0,+\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$*, and* $t_n < s_n$*, then* $\lim_{n\to\infty} \sup \zeta(t_n, s_n) < \mathcal{C}_{\mathcal{G}}$.

Some examples of simulation functions and $\mathcal{C}_{\mathcal{G}}$-simulation functions are:

(1) $\zeta(t,s) = \frac{s}{s+1} - t$ for all $t,s > 0$,

(2) $\zeta(t,s) = s - \phi(s) - t$ for all $t,s > 0$, where $\phi : [0,+\infty) \to [0,+\infty)$ is a lower semi continuous function and $\phi(t) = 0$ if and only if $t = 0$.

For more examples of simulation functions and $\mathcal{C}_{\mathcal{G}}$-simulation functions see [3, 24, 15, 18, 19, 27].

Definition 2.9. *[11] Let $T : A \to B$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be α-admissible if $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$.*

Definition 2.10. *[14] An α-admissible map T is said to be triangular α-admissible if $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$ implies $\alpha(x,y) \geq 1$.*

Definition 2.11. *[9] Let (X,d) be a metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called α-continuous, if for given $x \in X$ and sequence $\{x_n\}$ with $x_n \to x$ as $n \to \infty$,*

$$\alpha(x_n, x_{n+1}) \geq 1,\ \forall\ n \in \mathbb{N} \Rightarrow Tx_n \to Tx.$$

Cho *et al.* [6] generalized the concept of Geraghty contraction to α-Geraghty contraction and prove the fixed point theorem for such contraction.

Definition 2.12. *[6] Let (X, d) be a metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called α-Geraghty contraction if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$\alpha(x, y)d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$

Theorem 2.13. *[6] Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function. Define a map $T : X \to X$ satisfying the following conditions:*

1. *T is continuous α-Geraghty contraction;*
2. *T be a triangular α-admissible;*
3. *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$.*

Then T has a fixed point $x \in X$, and $\{T^n x_1\}$ converges to x.

Lemma 2.14. *[14] Let $T : X \to X$ be a triangular α-admissible map. Assume that there exists $x_1 \subset X$ such that $\alpha(x_1, Tx_1) \geq 1$.Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Lemma 2.15. *[22] Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.2}$$

If $\{x_n\}$ is not a Cauchy sequence in X, then there exists $\varepsilon > 0$ and two sequences $x_{m(k)}$ and $x_{n(k)}$ of positive integers such that $x_{n(k)} > x_{m(k)} > k$ and the following sequences tend to ε^+ when $k \to \infty$:

$$d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{n(k)+1}), d(x_{m(k)-1}, x_{n(k)}),$$

$$d(x_{m(k)-1}, x_{n(k)+1}), d(x_{m(k)+1}, x_{n(k)+1}).$$

Motivated by the above results, we introduce the notion of Gerghaty type $\mathcal{Z}_{(\alpha,\mathcal{G})}$-contraction and prove some fixed point results in metric and partially ordered metric spaces. An example to prove the validity and application to nonlinear differential equation for the usability of our results is presented.

3. Fixed point results in usual metric space

We begin with the following notion:

Definition 3.1. *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to X$ is called a $\mathcal{Z}_{(\alpha,\mathcal{G})}$-Geraghty contraction if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$*

$$\zeta(\alpha(x, y)d(Tx, Ty), \beta(M(x, y))M(x, y)) \geq \mathcal{C}_{\mathcal{G}} \tag{3.1}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Remark 3.2. *Since the functions belonging to $\mathcal{F}$ are strictly smaller than 1,* (3.1) *implies that*

$$d(Tx, Ty) < M(x, y)$$

for any $x, y \in X$ with $x \neq y$ and for $\alpha(x, y) \geq 1$, $\zeta(t, s) < \mathcal{G}(s, t) = s - t$.

Theorem 3.3. *Let (X,d) be a complete metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to X$ be two functions. Suppose that the following conditions are satisfied:*

(1) *T is $Z_{(\alpha,\mathcal{G})}$-Geraghty contraction;*

(2) *T is triangular α-admissible;*

(3) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*

(4) *T is α-continuous.*

Then T has a fixed point $x^* \in X$ and T is a Picard operator that is, $T^n x_1$ converges to x^*.

Proof . *Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T and hence the proof is completed. Thus, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By Lemma 2.14, we have*

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{3.2}$$

for all $n \in \mathbb{N}$. Then

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\leq \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}). \end{aligned} \tag{3.3}$$

Since T is a $\mathcal{Z}_{(\alpha,\mathcal{G})}$-Geraghty contraction, we have

$$\begin{aligned} \mathcal{C}_{\mathcal{G}} &\leq \zeta(\alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}), \beta(M(x_n, x_{n+1})) M(x_n, x_{n+1})) \\ &< \mathcal{G}(\beta(M(x_n, x_{n+1})) M(x_n, x_{n+1}), \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1})). \end{aligned}$$

Using $(\mathcal{G}_1)$, we obtain

$$\alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) < \beta(M(x_n, x_{n+1})) M(x_n, x_{n+1}). \tag{3.4}$$

From (3.3) and (3.4), we have

$$d(x_{n+1}, x_{n+2}) < \beta(M(x_n, x_{n+1})) M(x_n, x_{n+1}), \tag{3.5}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} \\ &= \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

If $M(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2})$, then by definition of β, we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &< \beta(d(x_n, x_{n+1}) d(x_n, x_{n+1})) \\ &< d(x_{n+1}, x_{n+2}), \end{aligned}$$

a contradiction. Thus we conclude that $M(x_n, x_{n+1}) = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and so the sequence $\{d(x_n, x_{n+1})\}$ of real numbers is decreasing and bounded below by zero. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

We claim that $r = 0$. Suppose on contrary that $r > 0$, then by (3.5) we have

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \beta(d(x_n, x_{n+1}) < 1,$$

which yields that $\lim_{n\to\infty} \beta(d(x_n, x_{n+1}) = 1$. Since $\beta \in \mathcal{F}$, we get that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.6}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose on contrary that it is not. Thus there exists $\epsilon > 0$ such that for all $k > 0$, $m(k) > n(k) > k$ with the (smallest number satisfying the condition below) $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)-1}, x_{n(k)}) < \epsilon$. Then we have

$$\begin{aligned} \epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\ &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) \\ &\leq d(x_{m(k)}, x_{m(k)-1}) + \epsilon. \end{aligned}$$

Letting $k \to \infty$ in the above inequality, we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{3.7}$$

By using (3.6) and (3.7), we obtain

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon.$$

By Lemma 2.14, $\alpha(x_{m(k)-1}, x_{n(k)-1}) \geq 1$, thus

$$\begin{aligned} d(x_{m(k)}, x_{n(k)}) &= d(Tx_{m(k)-1}, Tx_{n(k)-1}) \\ &\leq \alpha(x_{m(k)-1}, x_{n(k)-1}) d(Tx_{m(k)-1}, Tx_{n(k)-1}). \end{aligned} \tag{3.8}$$

Since T is a $\mathcal{Z}_{(\alpha,\mathcal{G})}$-Geraghty contraction, we have

$$\zeta(\alpha(x_{m(k)-1}, x_{n(k)-1}) d(Tx_{m(k)-1}, Tx_{n(k)-1}), \beta(M(x_{m(k)-1}, x_{n(k)-1})) M(x_{m(k)-1}, x_{n(k)-1})) \geq \mathcal{C}_{\mathcal{G}}.$$

This implies

$$\begin{aligned} &\mathcal{C}_{\mathcal{G}} \\ \leq\ &\zeta(\alpha(x_{m(k)-1}, x_{n(k)-1}) d(Tx_{m(k)-1}, Tx_{n(k)-1}), \beta(M(x_{m(k)-1}, x_{n(k)-1})) M(x_{m(k)-1}, x_{n(k)-1})) \\ <\ &\mathcal{G}(\beta(M(x_{m(k)-1}, x_{n(k)-1})) M(x_{m(k)-1}, x_{n(k)-1}), \alpha(x_{m(k)-1}, x_{n(k)-1}) d(Tx_{m(k)-1}, Tx_{n(k)-1})). \end{aligned}$$

Using $(\mathcal{G}_1)$, we obtain

$$\alpha(x_{m(k)-1}, x_{n(k)-1}) d(Tx_{m(k)-1}, Tx_{n(k)-1})) < \beta(M(x_{m(k)-1}, x_{n(k)-1})) M(x_{m(k)-1}, x_{n(k)-1}), \tag{3.9}$$

where

$$\begin{aligned} &M((x_{m(k)-1}, x_{n(k)-1})) \\ =\ &\max\{d(x_{m(k)-1}, x_{n(k)-1}), d((x_{m(k)-1}, Tx_{m(k)-1}), d(x_{n(k)-1}, Tx_{n(k)-1})\} \\ =\ &\max\{d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)})\}. \end{aligned}$$

If $M(x_{m(k)-1}, x_{n(k)-1}) = d(x_{m(k)-1}, x_{m(k)})$, we have

$$\begin{aligned} \alpha(x_{m(k)-1}, x_{n(k)-1}) d(x_{m(k)}, x_{n(k)}) &< \beta(d(x_{m(k)-1}, x_{m(k)})) d(x_{m(k)-1}, x_{m(k)}) \\ &< d(x_{m(k)-1}, x_{m(k)}), \end{aligned}$$

a contradiction. Similarly, we have contradiction when $M(x_{m(k)-1}, x_{n(k)-1}) = d(x_{n(k)-1}, x_{n(k)})$. Thus we conclude that $M(x_{m(k)-1}, x_{n(k)-1}) = d(x_{m(k)-1}, x_{n(k)-1})$. So

$$\frac{d(x_{m(k)}, x_{n(k)})}{d(x_{m(k)-1}, x_{n(k)-1})} \leq \beta(d(x_{m(k)-1}, x_{n(k)-1}) < 1.$$

Letting $k \to \infty$ in above inequality, we derive that

$$\lim_{k\to\infty} \beta(d(x_{m(k)-1}, x_{n(k)-1})) = 1.$$

This implies

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = 0.$$

Hence $\epsilon = o$, which is a contradiction. Thus we conclude that $\{x_n\}$ is a Cauchy sequence. It follows from completeness of X that there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Since T is α-continuous and $\alpha(x_n, x_{n-1}) \geq 1$, we get $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*$ and so $x^* = Tx^*$. This completes the proof. □

Theorem 3.4. *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ be two mappings. Suppose that the following conditions are satisfied:*

(1) *T is $\mathcal{Z}_{(\alpha,\mathcal{G})}$-Geraghty contraction;*

(2) *T is triangular α-admissible;*

(3) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*

(4) *if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x^* \in X$ as $n \to \infty$, then there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all k.*

Then T has a fixed point $x^* \in X$ and T is a Picard operator, that is, $T^n x_1$ converges to x^* .
Proof . *Following the arguments those given in Theorem 3.3, we conclude that the sequence x_n defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges to $x^* \in X$. By condition (4) we deduce that there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all k. Also*

$$\begin{aligned} d(x_{n(k)+1}, Tx^*) &= d(Tx_{n(k)}, Tx^*) \\ &\leq \alpha(x_{n(k)}, x^*) d(Tx_{n(k)}, Tx^*). \end{aligned} \tag{3.10}$$

Since T is $\mathcal{Z}_{(\alpha,\mathcal{G})}$-Geraghty contraction, we have

$$\begin{aligned} &\mathcal{C}_{\mathcal{G}} \leq \zeta(\alpha(x_{n(k)}, x^*) d(Tx_{n(k)}, Tx_*), \beta(M(x_{n(k)}, x^*)) M(x_{n(k)}, x^*)) \\ &\quad < \mathcal{G}(\beta(M(x_{n(k)}, x^*)) M(x_{n(k)}, x^*)), \alpha(x_{n(k)}, x^*) d(Tx_{n(k)}, Tx^*)). \end{aligned}$$

By definition of $\mathcal{G}$, we get that

$$\alpha(x_{n(k)}, x^*)d(Tx_{n(k)}, Tx^*) < \beta(M(x_{n(k)}, x^*))M(x_{n(k)}, x^*)). \tag{3.11}$$

From (3.10) and (3.11), we have

$$d(x_{n(k)+1}, Tx^*) < \beta(M(x_{n(k)}, x^*))M(x_{n(k)}, x^*)), \tag{3.12}$$

where

$$\begin{aligned} M(x_{n(k)}, x^*) &= \max\{d(x_{n(k)}, x^*), d(x_{n(k)}, Tx_{n(k)}), d(x^*, Tx^*)\} \\ &= \max\{d(x_{n(k)}, x^*), d(x_{n(k)}, x_{n(k)+1}), d(x^*, Tx^*)\}. \end{aligned}$$

Letting $k \to \infty$ in the above inequality, we get that

$$\lim_{k\to\infty} M(x_{n(k)}, x^*) = d(x^*, Tx^*). \tag{3.13}$$

Suppose $d(x^*, Tx^*) > 0$. By definition of β and (3.12)*, we have*

$$d(x_{n(k)+1}, Tx^*) < M(x_{n(k)}, x^*).$$

Letting $k \to \infty$ in the above inequality and using (3.13), we obtain that

$$d(x^*, Tx^*) < d(x^*, Tx^*),$$

a contradiction. Thus $d(x^*, Tx^*) = 0$, that is, $x^* = Tx^*$. □

For the uniqueness of fixed point, we consider the following hypothesis:

($\mathcal{U}$) For all $x, y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. Here, $Fix(T)$ denotes the set of fixed points of T.

Theorem 3.5. *Adding condition ($\mathcal{U}$) to the hypothesis of Theorem 3.3 (resp. Theorem 3.4), we obtain that x^* is the unique fixed point of T.*

Proof . *From Theorem 3.3 (resp. Theorem 3.4), we have a fixed point, namely $x^* \in X$ of T. For uniqueness, suppose there is another fixed point of T, say, $y^* \in X$. Then, by assumption ($\mathcal{U}$), there exists $z \in X$ such that*

$$\alpha(x^*, z) \geq 1 \ \ \text{and} \ \ \alpha(y^*, z) \geq 1.$$

Since T is a α-admissible, we have

$$\alpha(x^*, T^n z) \geq 1 \ \ \text{and} \ \ \alpha(y^*, T^n z) \geq 1,$$

for all n. Hence we have

$$\begin{aligned} d(x^*, T^n z) &= d(Tx^*, TT^{n-1}z) && (3.14) \\ &\leq \alpha(x^*, T^{n-1}z)d(Tx^*, TT^{n-1}z). && (3.15) \end{aligned}$$

Since T is a $\mathcal{Z}_{(\alpha,\mathcal{G})}$-Geraghty contraction, we have

$$\begin{aligned} &\mathcal{C}_{\mathcal{G}} \leq \zeta(\alpha(x^*, T^{n-1}z)d(Tx^*, TT^{n-1}z), \beta(M(x^*, T^{n-1}z))M(x^*, T^{n-1}z) \\ &\ < \mathcal{G}(\beta(M(x^*, T^{n-1}z))M(x^*, T^{n-1}z), \alpha(x^*, T^{n-1}z)d(Tx^{*)}, TT^{n-1}z)). \end{aligned}$$

By definition of $\mathcal{G}$, we have

$$\alpha(x^*, T^{n-1}z)d(Tx^*, TT^{n-1}z) < \beta(M(x^*, T^{n-1}z))M(x^*, T^{n-1}z),$$

where

$$\begin{aligned} M(x^*, T^{n-1}z) &= \max\{d(x^*, T^{n-1}z), d(x^*, Tx^*), d(T^{n-1}z, TT^{n-1}z)\} \\ &= d(x^*, T^{n-1}z). \end{aligned}$$

Hence we have

$$\alpha(x^*, T^{n-1}z)d(Tx^*, TT^{n-1}z) \; < \; \beta(d(x^*, T^{n-1}z))d(x^*, T^{n-1}z). \tag{3.16}$$

Inequality (3.14) *together with* (3.16) *gives*

$$d(x^*, T^n z) \; < \; \beta(d(x^*, T^{n-1}z))d(x^*, T^{n-1}z). \tag{3.17}$$

By definition of β, (3.17) *gives*

$$d(x^*, T^n z) < d(x^*, T^{n-1}z),$$

for all $n \in \mathbb{N}$. Thus the sequence $d(x^*, T^n z)$ is non increasing, and so there exists $u \geq 0$ such that $\lim_{n\to\infty} d(x^*, T^n z) = u$. From (3.17)*, we have*

$$\frac{d(x^*, T^n z)}{d(x^*, T^{n-1}z)} \leq \beta(d(x^*, T^{n-1}z)),$$

and so $\lim_{n\to\infty} \beta(d(x^*, T^{n-1}z)) = 1$. Consequently, we have $\lim_{n\to\infty}(d(x^*, T^{n-1}z)) = 0$, and hence $\lim_{n\to\infty} T^n z = x^*$. Similarly, we can find that $\lim_{n\to\infty} T^n z = y^*$. By uniqueness of limit, we obtain $x^* = y^*$. □

Example 3.6. *Let $X = [0, \infty)$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Let $\zeta(t, s) = \frac{8}{9}s - t, \mathcal{G}(s, t) = s - t$ for all $s, t \in [0, \infty)$, $\mathcal{C}(\mathcal{G}) = 0$ and $\beta(t) = \frac{1}{1+t}$ for all $t \geq 0$. Then it is clear that $\beta \in \mathcal{F}$. We define $T : X \to X$ by*

$$Tx = \begin{cases} \frac{1}{3}x & if \; 0 \leq x \leq 1, \\ 3x & otherwise, \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & if \; 0 \leq x, y \leq 1, \\ 0 & otherwise. \end{cases}$$

Clearly, T is α-continuous and condition (3) *of Theorem 3.3 is satisfied with $x_1 = 1$. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. Then $x, y \in [0, 1]$, so $Tx, Ty \in [0, 1]$ and thus $\alpha(Tx, Ty) = 1$. Hence T is α-admissible. Further, if $z = Ty$, then $\alpha(y, z) \geq 1$, this implies $\alpha(x, z) \geq 1$. So T is triangular α-admissible, hence condition* (2) *of Theorem 3.3 is satisfied. Finally, if $0 \leq x, y \leq 1$, then $\alpha(x, y) = 1$, and we have*

$$\begin{aligned} \zeta(\alpha(x, y)d(Tx, Ty), \beta(M(x, y))M(x, y)) &= \frac{8}{9}\beta(M(x, y))M(x, y) - \alpha(x, y)d(Tx, Ty) \\ &= \frac{8(M(x, y))}{9(1 + M(x, y))} - d(Tx, Ty), \end{aligned}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\},$$

for all $x, y \in [0,1]$.
Case-I: *If* $M(x,y) = d(x,y)$, *then*

$$\begin{aligned}
\zeta(\alpha(x,y)d(Tx,Ty), \beta(M(x,y))M(x,y)) &= \frac{8d(x,y)}{9(1+d(x,y))} - d(\frac{x}{3}, \frac{y}{3}) \\
&= \frac{8|x-y|}{9(1+|x-y|)} - |\frac{x}{3} - \frac{y}{3}| \\
&= \frac{8|x-y|}{9(1+|x-y|)} - \frac{|x-y|}{3} \\
&= \frac{8|x-y| - 3(1+|x-y|)|x-y|}{9(1+|x-y|)} \\
&= \frac{|x-y|(5-3|x-y|)}{9(1+|x-y|)} \\
&\geq 0.
\end{aligned}$$

Case-II: *If* $M(x,y) = d(x,Tx)$, *then*

$$\begin{aligned}
\zeta(\alpha(x,y)d(Tx,Ty), \beta(M(x,y))M(x,y)) &= \frac{8d(x,Tx)}{9(1+d(x,Tx))} - d(\frac{x}{3}, \frac{y}{3}) \\
&= \frac{8|x-\frac{x}{3}|}{9(1+|x-\frac{x}{3}|)} - |\frac{x}{3} - \frac{y}{3}| \\
&= \frac{16|x|}{9(3+2|x|)} - \frac{|x-y|}{3} \\
&\geq \frac{16|x|}{9(3+2|x|)} - \frac{2|x|}{9} \\
&= \frac{|x|(10-4|x|)}{9(3+2|x|)} \\
&\geq 0.
\end{aligned}$$

Similarly, if $M(x,y) = d(y,Ty)$, *we have*

$$\zeta(\alpha(x,y)d(Tx,Ty), \beta(d(y,Ty))d(y,Ty)) \geq 0.$$

Hence for $0 \leq x, y \leq 1$, T *is a generalized* $\mathcal{Z}_{(\alpha,\mathcal{G})}$*-Geraghty contraction. In either case* $\alpha(x,y) = 0$ *and* T *is a* $\mathcal{Z}_{(\alpha,\mathcal{G})}$*-Geraghty contraction. Thus all the hypothesis of Theorem 3.3 are satisfied and* T *has a fixed point* $x^* = 0$.

4. Fixed point results in partially ordered metric space

Let $(X, d, \preceq)$ be a partially ordered metric space. Many authors has proved the existence of fixed point results in the frame work of partially order metric spaces (see for example [23, 1, 20]). In this section, we obtain some new fixed point results in partially order metric spaces, as an application of our results given in above section.

Definition 4.1. *[10] Let $(X, d, \preceq)$ be a partially ordered metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called ordered continuous, for a given $x \in X$ and sequence $\{x_n\}$*

$$x_n \to x \text{ as } n \to \infty \text{ and } x_n \preceq x_{n+1} \; \forall \; n \in \mathbb{N} \text{ we have } Tx_n \to Tx.$$

Definition 4.2. *Let $(X, d, \preceq)$ be a partially ordered metric space and let $x \preceq y$ for all $x, y \in X$. A map $T : X \to X$ is called $\mathcal{Z}_\mathcal{G}$-Geraghty contraction if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$*

$$\zeta(d(Tx, Ty), \beta(M(x, y))M(x, y)) \geq \mathcal{C}_\mathcal{G},$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Theorem 4.3. *Let $(X, d, \preceq)$ be a complete partially ordered metric space with $x \preceq y$ for all $x, y \in X$. Assume that the following conditions hold true*

(1) *T is $\mathcal{Z}_\mathcal{G}$-Geraghty contraction;*

(2) *T is increasing;*

(3) *there exists $x_1 \in X$ such that $x_1 \preceq Tx_1$;*

(4) *T is ordered continuous.*

Then T has a fixed point $x^* \in X$ and T is a Picard operator that is, $T^n x_1$ converges to x^*.
Proof . *Define $\alpha : X \times X \to [0, +\infty)$ by*

$$\alpha(x, y) = \begin{cases} 1, & if \; x \preceq y, \\ 0, & otherwise. \end{cases}$$

Since T is a $Z_\mathcal{G}$-Geraghty contraction, we have

$$\begin{aligned} \mathcal{C}_\mathcal{G} &\leq \zeta(d(Tx, Ty), \beta(M(x, y))M(x, y)) \\ &< \mathcal{G}(\beta(M(x, y))M(x, y), d(Tx, Ty)). \end{aligned}$$

By definition of $\mathcal{G}$, we get

$$d(Tx, Ty) < \beta(M(x, y))M(x, y),$$

so,

$$\alpha(x, y)d(Tx, Ty) \leq d(Tx, Ty) < \beta(M(x, y))M(x, y).$$

Hence T is $\mathcal{Z}_{(\alpha,\mathcal{G})}$-Geraghty contraction. Since T is increasing, $\alpha(x, y) = 1$ implies $\alpha(Tx, Ty) = 1$ for all $x, y \in X$. Further if $z = Ty$, then $\alpha(y, z) = 1$, this implies $\alpha(x, z) = 1$. Thus, T is triangular α-admissible. Condition (2) *implies that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) = 1$, and so condition* (3) *of Theorem 3.3 is satisfied. Thus by Theorem 3.3, T has a fixed point in X.* □

Continuity of the mapping can be omitted in Theorem 4.3 and fixed point result can be obtain with an extra condition given in the following theorem:

Theorem 4.4. *Let $(X, d, \preceq)$ be a complete partially ordered metric space with $x \preceq y$ for all $x, y \in X$. Let $T : X \to X$ be a mapping satisfying*

(1) *T is $\mathcal{Z}_{\mathcal{G}}$-Geraghty contraction type mapping;*

(2) *T is increasing;*

(3) *there exists $x_1 \in X$ such that $x_1 \preceq Tx_1$;*

(4) *if x_n is a sequence in X such that $x_n \preceq Tx_{n+1}$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.*

Then T has a fixed point $x^* \in X$ and T is a Picard operator that is, $T^n x_1$ converges to x^*.

Proof . *Define $\alpha : X \times X \to [0, +\infty)$ by*

$$\alpha(x, y) = \begin{cases} 1 & if\ x \preceq y, \\ 0 & otherwise. \end{cases}$$

Since T is a $\mathcal{Z}_{\mathcal{G}}$-Geraghty contraction, we have

$$\begin{aligned} \mathcal{C}_{\mathcal{G}} &\leq \zeta(d(Tx, Ty), \beta(M(x, y))M(x, y)) \\ &< \mathcal{G}(\beta(M(x, y))M(x, y), d(Tx, Ty)). \end{aligned}$$

By the definition of $\mathcal{G}$, we obtain

$$d(Tx, Ty) < \beta(M(x, y))M(x, y).$$

This implies

$$\alpha(x, y)d(Tx, Ty) \leq d(Tx, Ty) < \beta(M(x, y))M(x, y).$$

Hence T is generalized $\mathcal{Z}_{(\alpha,\mathcal{G})}$-Geraghty contraction. Since T is increasing, $\alpha(x, y) = 1$ implies $\alpha(Tx, Ty) = 1$ for all $x, y \in X$. Further if $z = Ty$, then $\alpha(y, z) = 1$, this implies $\alpha(x, z) = 1$. Thus, T is triangular α-admissible. Condition (2) implies that there exists $x_1 \in X$ such that $\alpha(x, Tx) = 1$ and so condition (3) of Theorem 3.4 is satisfied. Condition (4) implies that the condition (4) of Theorem 3.4 is satisfied. Thus, all the conditions of Theorem 3.4 are satisfied. Hence T has a fixed point in X. □

Remark 4.5. *Uniqueness of fixed point follows from Theorem 4.3 (respectively Theorem 4.4) with the condition*
$\mathcal{U}$: *For all $x, y \in Fix(T)$ with $x \preceq y$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.*

5. Application to Differential Equations

Denote by $C([0, 1])$ the set of all continuous functions defined on $[0, 1]$ and let $d : C([0, 1]) \times C([0, 1]) \to \mathbb{R}$ be defined by

$$d(x, y) = ||x - y||_\infty = \max_{t \in [0,1]} |x(t) - y(t)|. \tag{5.1}$$

It is well known that $(C([0, 1]), d)$ is a complete metric space. Let us consider the two-point boundary value problem of the second-order differential equation:

$$\begin{aligned} &-\frac{d^2x}{dt^2} = f(t, x(t)), \quad t \in [0, 1]; \\ &x(0) = x(1) = 1, \end{aligned} \tag{5.2}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.The Green function associated to (5.2) is defined by

$$G(t,s) = \begin{cases} t(1-s) & \text{if } 0 \le t \le s \le 1, \\ s(1-t) & \text{if } 0 \le s \le t \le 1. \end{cases}$$

Assume that the following conditions hold:

(i) there exist a function $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$|f(t,a) - f(t,b)| \le \max\{|a-b|, |a-Ta|, |b-Tb|\}$$

for all $t \in [0,1]$ and $a, b \in \mathbb{R}$ with $\xi(a,b) > 0$, where $T : C[0,1] \to C[0,1]$ is defined by

$$Tx(t) = \int_0^1 G(t,s) f(s, x(s)) ds;$$

(ii) there exists $x_0 \in C[0,1]$ such that $\xi(x_0(t), Tx_0(t)) \ge 0$ for all $t \in [0,1]$;

(iii) for each $t \in [0,1]$ and $x, y \in C[0,1]$, $\xi(x(t), y(t)) > 0$ implies $\xi(Tx(t), Ty(t)) > 0$;

(iv) for each $t \in [0,1]$, if $\{x_n\}$ is e sequence in $C[0,1]$ such that $x_n \to x$ in $C[0,1]$ and $\xi(x_n(t), x_{n+1}(t)) > 0$ for all $n \in \mathbb{N}$, then $\xi(x_n(t), x(t)) > 0$ for all $n \in \mathbb{N}$.

We now prove that existence of a solution of the mentioned second-order differential equation.

Theorem 5.1. *Under assumptions $(i)-(iv)$, (5.2) has a solution in $C^2([0,1])$.*

Proof . *It is well known that $x \in C^2([0,1])$ is a solution of (5.2) is equivalent to $x \in C([0,1])$ is a solution of the integral equation (see [11])*

$$x(t) = \int_0^1 G(t,s) f(s, x(s)) ds, \ t \in [0,1]. \tag{5.3}$$

Let $T : C[0,1] \to C[0,1]$ be a mapping defined by

$$Tx(t) = \int_0^1 G(t,s) f(s, x(s)) ds. \tag{5.4}$$

Suppose that $x, y \in C([0,1])$ such that $\xi(x(t), y(t)) \ge 0$ for all $t \in [0,1]$. By applying (i), we obtain that

$$\begin{aligned}
& |Tu(x) - Tv(x)| \\
= & \int_0^1 G(t,s) f(s, x(s)) ds - \int_0^1 G(t,s) f(s, y(s)) ds \\
= & \int_0^1 G(t,s) [f(s, x(s)) - f(s, y(s))] ds \\
\le & \left(\int_0^1 G(t,s) ds \right) \left(\int_0^1 |f(s, x(s)) - f(s, y(s))| ds \right) \\
\le & \left(\int_0^1 G(t,s) ds \right) \left(\int_0^1 (\max\{|x(s) - y(s)|, |x(s) - Tx(s)|, |y(s) - Ty(s)|\} ds \right) \\
\le & \sup_{t \in [0,1]} \left(\int_0^1 G(t,s) ds \right) \left(\int_0^1 (\max \left\{ \sup_{s \in [0,1]} |x(s) - y(s)|, \right.\right. \\
& \left.\left. \sup_{s \in [0,1]} |x(s) - Tx(s)|, \sup_{s \in [0,1]} |y(s) - Ty(s)| \right\} ds \right)
\end{aligned}$$

$$\leq \sup_{t\in[0,1]}\left(\int_0^1 G(t,s)ds\right)(\max\{||x-y||_\infty, ||x-Tx||_\infty, ||y-Ty||_\infty\})\int_0^1 ds$$
$$\leq \sup_{t\in[0,1]}\left(\int_0^1 G(t,s)ds\right)(M(x,y)).$$

Since $\int_0^1 G(t,s)ds = -(t^2/2)+(t/2)$, *for all* $t\in[0,1]$, *we have* $\sup_{t\in[0,1]}(\int_0^1 G(t,s)ds) = 1/8$. *It follows that*

$$||Tx-Ty||_\infty \leq \frac{1}{8}M(x,y). \tag{5.5}$$

Let $\zeta(t,s) = \frac{1}{4}s - t, \mathcal{G}(s,t) = s-t$ *for all* $s,t\in[0,\infty)$, $\mathcal{C}(\mathcal{G}) = 0$ *and* $\beta(t) = \frac{1}{2}$ *for all* $t\geq 0$. *Then it is clear that* $\beta\in\mathcal{F}$. *Also define*

$$\alpha(x,y) = \begin{cases} 1 & \text{if } \xi(x(t),y(t))>0, t\in[0,1],\\ 0 & \text{otherwise.}\end{cases}$$

Now

$$\begin{aligned}\zeta(\alpha(x,y)d(Tx,Ty),\beta(M(x,y))M(x,y)) &= \frac{1}{4}\beta(M(x,y))M(x,y)-\alpha(x,y)d(Tx,Ty)\\ &= \frac{1}{8}M(x,y)-d(Tx,Ty),\end{aligned} \tag{5.6}$$

Then from (5.5)

$$\zeta(\alpha(x,y)d(Tx,Ty),\beta(M(x,y))M(x,y))\geq 0.$$

Therefore the mapping T *is a* $\mathcal{Z}_{(\alpha,\mathcal{G})}$*-Geraghty contraction.*

From (ii) there exists $x_0\in C[0,1]$ *such that* $\alpha(x_0,Tx_0)\geq 1$. *Next by using (iii), we get the following assertions holding for all* $x,y\in C[0,1]$

$$\begin{aligned}\alpha(x,y)\geq 1 &\Rightarrow \xi(x(t),y(t))>0 \text{ for all } t\in[0,1]\\ &\Rightarrow \xi(Tx(t),Ty(t))>0 \text{ for all } t\in[0,1]\\ &\Rightarrow \alpha(Tx,Ty)\geq 1,\end{aligned}$$

hence T *is* α*-admissible.*

Applying Theorem (3.3), *we obtain that* T *has a fixed point in* $C([0,1])$; *say* x. *Hence,* x *is a solution of* (5.2). □

Acknowledgments

This paper was funded by the University of Sargodha, Sargodha funded research project No. UOS/ORIC/2016/54. The first author, therefore, acknowledges with thanks UOS for financial support.

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