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Anti-N-order polynomial Daugavet property on Banach spaces

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Abstract

We generalize the notion of the anti-Daugavet property (a-DP) to the anti-N-order Polynomial Daugavet property (a-NPDP) for Banach spaces by identifying a good spectrum of a polynomial and prove that locally uniformly alternatively convex or smooth Banach spaces have the a-mDP for rank-1 polynomials. We then prove that locally uniformly convex Banach spaces have the a-NPDP for compact polynomials if and only if their norms are eigenvalues, and uniformly convex Banach spaces have the a-NPDP for continuous polynomials if and only if their norms belong to the approximate spectra.

Keywords: Banach spaces, local and uniform convexity, polynomials, Daugavet Equation, N-order Polynomial Daugavet property, anti-N-order Polynomial Daugavet property
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1. Introduction

The aim of this paper is to study the anti-N-order Polynomial Daugavet property associated with polynomials of degree N on a Banach space E. For each $m \in \mathbb{N}$, let $\mathcal{L}(^mE; E)$ be the Banach space of all continuous m-linear maps $T : E \times E \times \ldots \times E \to E$ endowed with the supremum norm on $\mathbf{B}_E$, the closed unit ball of E. A map $p : E \to E$ is an m-homogeneous polynomial if there exists $T \in \mathcal{L}(^mE; E)$ such that $p(x) = T(x, \ldots, x)$ for all $x \in E$, see [15, 28]. The notation $\mathcal{P}(^mE; E)$ will represent the vector space of all continuous m-homogeneous polynomials, which is a Banach space under the supremum norm $||p|| = \sup\{||p(x)|| : x \in \mathbf{B}_E\}$; in this notation, $\mathcal{P}(^1E; E) := \mathcal{L}(E)$ is the Banach space of linear operators and the space $\mathcal{P}(^0E; E)$ is identified with the space of constant

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functions, *i.e.*, $\mathcal{P}(^0E;E) := E$. The notation $\mathcal{P}(^mE)$ will represent the Banach space of all scalar valued m-homogeneous polynomials and associated with every $p \in \mathcal{P}(^mE;E)$ is the linear adjoint $p^* \in \mathcal{L}(E^*;\mathcal{P}(^mE))$ defined by

$$(p^*(\phi))(x) = \phi(p(x)) \text{ for every } x \in E \text{ and } \phi \in E^*. \tag{1.1}$$

It is clear that $||p^*|| = ||p||$, see [6]. A map $P : E \to E$ is a polynomial of degree $N \in \mathbb{N}$ if it is the sum of j-homogeneous polynomials $p_j \in \mathcal{P}(^jE;E)$ for $j = 0, 1, \ldots, N$;

$$P = p_0 + p_1 + \ldots + p_N$$

where p_0 is a constant map on E, see [28, Def. 2.8]. The notation $\mathcal{P}_N(E;E)$ will represent the vector space of all continuous polynomials of degree less than or equal to N; it is a Banach space under the supremum norm $||P|| = \sup\{||P(x)|| : x \in \mathbf{B}_E\}$ and for $1 \leqslant m \leqslant N$, $\mathcal{P}(^mE;E)$ can be viewed as subspaces of $\mathcal{P}_N(E;E)$.

We shall say that a map $P \in \mathcal{P}_N(E;E)$ is compact if P maps the closed unit ball $\mathbf{B}_E$ into a relatively compact subset of E; it is weakly compact if it maps the closed unit ball $\mathbf{B}_E$ into a relatively weakly compact subset in E, see [6], [15, Chap. 1-2].

The norm equality,

$$||I + P|| = 1 + ||P||, \tag{1.2}$$

for any $P \in \mathcal{P}_N(E;E)$ where I is the identity operator on E is known as the *Daugavet equation* (in short, DE). Moreover, E has the *N-order Polynomial Daugavet property* (in short, NPDP) if the (DE) holds for all weakly compact polynomials $P \in \mathcal{P}_N(E;E)$ or equivalently the (DE) holds for all rank-one polynomials of degree N; E has the *m-order Daugavet property* (in short, mDP) if the same assertion holds for all $p \in \mathcal{P}(^mE;E)$, noting that E has the Daugavet property (DP) if $m = 1$, see [11, 12, 27]. The N-order Polynomial Daugavet property (NPDP) comprises the m-order Daugavet property (mDP) for $1 \leqslant m \leqslant N$; the use of the notation mDP in the sequel is for the purpose of emphasis. We also note that a polynomial P satisfies the *alternative Daugavet equation* (in short, (ADE)) if there exists $\omega \in S_{\mathbb{k}}$ such that ωP satisfies the (DE) or, equivalently, if

$$\max_{|\omega|=1} ||I + \omega P|| = 1 + ||P||. \tag{1.3}$$

Remark 1.1. *A polynomial $P \in \mathcal{P}_N(E;E)$ satisfies the (DE) (resp. (ADE)) if and only if αP does so for every $\alpha \in \mathbb{R}^+$, see [2, Lem. 2.1]. It is known that the (DE) implies the (ADE) but in general they are different since the minus identity, $-I$, always satisfies the (ADE) but not the (DE). On the other hand, when $P \in \mathcal{P}(^mE;E)$ and E is a complex Banach space, then the (DE) and (ADE) are equivalent; if E is a real Banach space the equivalence holds only for polynomials of even homogeneity degree, see [11, Prop. 3.2, Cor. 3.3]. Therefore, as a consequence, the (DE) and (ADE) are necessarily equivalent for polynomials that act on complex Banach spaces.*

Lemma 1.2. *Let E be a Banach space and $P \in \mathcal{P}_N(E;E)$ be a continuous polynomial that is uniformly continuous on the closed unit ball of E and that satisfies the (DE) (or (ADE)). Then $||P|| = \sup\{||Px|| : x \in S_E\}$.*

Proof . Suppose that $||I + P|| = 1 + ||P||$, and take a sequence $\{x_n\}$ in $\mathbf{B}_E$ such that $||x_n + Px_n|| \to 1 + ||P||$. Then $||x_n|| \to 1$ necessarily. Now let $y_n = \frac{x_n}{||x_n||}$; then by the uniform continuity of P on the unit ball $\mathbf{B}_E$, we get $||y_n + Py_n|| \to 1 + ||P||$. Thus, we have $||Py_n|| \to ||P||$ necessarily. The conclusion now follows from the fact that $\sup\{||Px|| : x \in S_E\} \leqslant \sup\{||Px|| : x \in \mathbf{B}_E\} = ||P||$. □

Remark 1.3. *Let us note that for a polynomial* $p \in \mathcal{P}(^m E; E)$, $||p|| = \sup\{||p(x)|| : x \in S_E\}$. *This follows from the expression* $p(x) = T(x, \dots, x)$, *where* $T \in \mathcal{L}(^m E; E)$ *is a multilinear operator associated to* p.

Remark 1.4. *The norm of a uniformly continuous polynomial* $P \in \mathcal{P}_N(E; E)$ *that satisfies the (DE) (respectively, (ADE)) can be computed using its numerical range (respectively, numerical radius), see [11, Prop. 1.3]. Let* $\Pi(E) = \{(x, f) : x \in S_E,\ f \in S_{E^*}, f(x) = 1\}$. *The numerical range of* $P \in \mathcal{P}_N(E; E)$ *is the set of scalars* $V(P) = \{f(Px) : (x, f) \in \Pi(E)\}$ *and its numerical radius is defined by* $v(P) = \sup\{|\lambda| : \lambda \in V(P)\}$. *If* $P \in \mathcal{P}_N(E; E)$ *is a continuous polynomial that is uniformly continuous on the closed unit ball of* E *and that satisfies the (DE) (respectively, (ADE)) then we have* $||P|| = \sup ReV(P)$ *(respectively,* $||P|| = v(P)$*).*

The study of the NPDP started with the linear case by Daugavet who proved that all compact linear operators on $E = C[0, 1]$ satisfy the (DE), see [13]. Shortly afterwards, this result was extended to different classes of operators and Banach spaces besides their applications, see for example [1, 2, 7, 10, 18, 20, 21, 24, 23, 26, 30, 32] in the linear setting and [11, 27, 12] in the nonlinear setting. The outcome of such studies and their applications are interesting in their own right; for example, the characterizations of the DP in [23, 32] gave rise to interesting results, namely, if the pair $(\mathcal{M}, E)$ where $\mathcal{M}$ is a subspace of E has the DP for rank-one operators then it has the DP for weakly compact operators whereas if $\mathcal{M}$ has a separable annihilator $\mathcal{M}^\perp$ then $\mathcal{M}$ has the DP; a Banach space E has the DP if its topological dual E^* has it (the converse result is false, for instance, $E = C[0, 1]$) and such a space contains a copy of ℓ_1 but not necessarily L_1; an operator satisfies the (DE) if and only if its adjoint likewise satisfies it, see [2]. We note that these results point to the dependence of the DP on the structures of a class of operators as well as the space. In fact, previous studies such as [11, 13, 20, 24, 27] demonstrate the existence of Banach spaces where classes of operators satisfying the (DE) are very rich. However, in-depth studies in [2, 23, 22] show that there are Banach spaces where classes of operators satisfying the (DE) are very small; this is the case of a class of bounded linear operators T from a Banach space E into itself for which $||T||$ belongs to the spectrum of T.

Definition 1.5. *A Banach space* E *is said to have the* anti-Daugavet property *(in short,* $E \in$ *a-DP) for a class* $\mathcal{M}$ *of operators if, for* $T \in \mathcal{M}$, *the equivalence* $||I + T|| = 1 + ||T|| \Leftrightarrow ||T|| \in \sigma_0(T)$ *holds. If* $\mathcal{M} = \mathcal{L}(E)$, *we simply say* E *has the anti-Daugavet property, see [22, 23].*

The characterization of the anti-Daugavet property is through the spectral information of operators and geometric properties of spaces, see [2, 22, 23, 25]. However, the well-known fact in nonlinear operator theory that there is no suitable notion of the spectrum for nonlinear operators, see [3, Chap. 6-9] complicates the generalization of the notion of a-DP to nonlinear operators. The development of the spectral theory of $P \in \mathcal{P}_N(E; E)$ requires an in-depth knowledge of solvability properties of the equation $\lambda z - P(z) = 0$ for some $\lambda \in \mathbb{k}$; namely, injectivity of $\lambda I - P$ and whether it maps some bounded neighborhood of 0 onto a neighborhood of 0, continuity of $(\lambda I - P)^{-1}$, boundedness and nontriviality of the nullset $\mathcal{N}(\lambda I - P)$. Indeed, the study of nonlinear eigenvalue problems modeled on the underlying structural properties of nonlinear operators yielded several variants of spectra for nonlinear operators, each having different notions of eigenvalues, see [3, Chap. 6-9]. However, none of the known nonlinear spectra adheres to the minimal requirements, specifically most of their associated eigenvalue notions are incompatible with the general notion of the classical eigenvalues [29, Def.s 1, 2]. Further, these spectra may be disjoint from point spectra, see [3, Example 6.6] and [31], are not discrete and unbounded even if their underlying operators are bounded and compact, see [29, Example 3.5], [9, Theorem 3.5]. These present unprecedented challenges of varying magnitude in the generalization of anti-Daugavet property to polynomials; these are issues that suffice to be addressed in this study.

The discussion in the preceding paragraph demonstrates that it suffices a careful consideration of the characteristics that constitute a "good spectrum". The fundamental idea that any eigenvalue of a linear operator $L \in \mathcal{L}(E)$ associated with the eigenvector $z \in E$ is also an eigenvalue of $L \in \mathcal{L}(E)$ associated with the eigenvector $y = ||z||^{-1}z$ is key to determining the characteristics of a "good spectrum". This idea was pursued in [16] and resulted into the eigenvalue equation $p(z) - \lambda J_m(z) = 0$ where $p \in \mathcal{P}(^mE; E)$ and $J_m(z) = ||z||^{m-1}z$, see [16, Prob. 3.1]. Indeed, $p(z) - \lambda J_m(z) = 0$ if and only if $p(y) - \lambda y = 0$ for some $\lambda \in \Bbbk$ where $y = ||z||^{-1}z$. We shall therefore consider a "good spectrum" of an m-homogeneous polynomial or more generally a polynomial $P \in \mathcal{P}_N(E; E)$, to be the set of eigenvalues associated with eigenvectors of norm one; denote this set in the sequel by $\sigma_0(P) = \{\lambda \in \Bbbk : \text{there exists } x \in E, ||x|| = 1,\ Px = \lambda x\}$.

Definition 1.6. *A scalar $\lambda \in \Bbbk$ is an approximate eigenvalue of $P \in \mathcal{P}_N(E; E)$ if there is a sequence $\{x_n\} \subset E$ with $||x_n|| = 1$ for all n such that $||Px_n - \lambda x_n|| \to 0$. Denote the set of approximate eigenvalues of $P \in \mathcal{P}_N(E; E)$ by $\sigma_a(P)$.*

Definition 1.7. *A scalar $\lambda \in \Bbbk$ is an eigenvalue of $P \in \mathcal{P}_N(E; E)$ if there is a nonzero element $x \in E$ such that $P(x) = \lambda x$. Denote the set of eigenvalues of $P \in \mathcal{P}_N(E; E)$ by $\sigma_{ev}(P)$.*

Remark 1.8. *Generally, $\sigma_0(P) \subseteq \sigma_{ev}(P) \cap \sigma_a(P)$ and $\sigma_0(P)$ can be empty, see for example [16, Theorem 4.2].*

Lemma 1.9. *Let E be a Banach space. Then every continuous polynomial $P \in \mathcal{P}_N(E; E)$ whose norm $||P|| \in \sigma_a(P)$ satisfies the (DE).*

Proof . By [2, Lemma 2.1], assume without loss of generality that $||P|| = 1$. So, if $1 \in \sigma_a(P)$ then there exists some $\{x_n\} \subset S_E$ such that $||x_n - Px_n|| \to 0$. Thus,

$$||I + P|| \geqslant ||(I + P)x_n|| \geqslant 2 - ||x_n - Px_n||$$

so that $||I + P|| \geqslant 2$ as $n \to \infty$. □

Remark 1.10. *We note that for a compact polynomial $P \in \mathcal{P}_N(E; E)$, the approximate eigenvalues are precisely the "good spectrum". In this case, if $1 \in \sigma_0(P)$ then $2 \in \sigma_0(I + P)$ and the proof of the Lemma 1.9 is immediate since $||I + P|| \geqslant 1 + \max\{|\lambda| : \lambda \in \sigma_0(P)\} \geqslant 2$.*

Definition 1.11. *A Banach space E is said to have the anti-N-order Polynomial Daugavet property (in short, E has the a-NPDP) for a class $\Omega \subset \mathcal{P}_N(E; E)$ if for each $P \in \Omega$, $||I+P|| = 1+||P||$ if and only if $||P|| \in \sigma_a(P)$. If $\Omega = \mathcal{P}_N(E; E)$, we simply say E has the anti-N-order Polynomial Daugavet property; if $\Omega = \mathcal{P}(^mE; E)$ then we say E has the anti-m-order Daugavet property (in short, E has the a-mDP).*

This paper is organized as follows. We establish a suitable spectrum for use in our study in Section 1 and investigate a-NPDP of alternatively convex or smooth spaces in Section 2. We then narrow down this study to the a-NPDP of uniformly convex or more generally locally uniformly convex Banach spaces for a class of continuous polynomials of degree N in Section 3.

2. Necessary geometric conditions on a Banach space

This section deals with the study of the a-NPDP for alternatively convex Banach spaces; it encompasses the highlights of difficulties that one faces in an attempt to generalize the known a-DP techniques to nonlinear operators. The necessary and sufficient conditions for the a-DP of uniformly convex or uniformly smooth Banach spaces, and for locally uniformly convex Banach spaces for a class of compact operators were first established in [2] (also, see [25] for more generalized operators). A Banach space E is locally uniformly convex if for every sequence $\{x_n\}$ in $\mathbf{B}_E$ and $x \in S_E$ such that $\lim_{n\to\infty} ||(x_n + x)/2|| = 1$ we have $\lim_{n\to\infty} ||x_n - x|| = 0$, see [2]; it is strictly convex (rotund) if given $x, y \in S_E$ and $x \neq y$, we have $||(x + y)/2|| < 1$; it is uniformly convex (uniformly rotund) if for any sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbf{B}_E$ such that $\lim_{n\to\infty} ||x_n + y_n|| = 2$, it follows that $\lim_{n\to\infty} ||x_n - y_n|| = 0$, see [8, Chap. I-III], [25]. A point $x_0 \in S_E$ is smooth if there is a unique functional $\phi_0 \in S_{E^*}$ such that $\phi_0(x_0) = 1$; we will say a Banach space E is smooth if every point $x \in S_E$ is smooth.

The geometric condition for the a-DP in finite dimensional spaces based on the hybrid of rotundity and smoothness called alternatively convex or smooth spaces was first established in [23] and then extended to infinite dimensional Banach spaces in [22]. A Banach space E is alternatively convex or smooth space (in short, acs) if for all $x, y \in S_E$ and $x^* \in S_{E^*}$ the following implication holds:

$$x^*(x) = 1,\ \ ||x + y|| = 2 \ \Rightarrow\ x^*(y) = 1. \tag{2.1}$$

A Banach space E is locally uniformly alternatively convex or smooth (in short, luacs) if for all $x_n, y \in S_E$ and $x^* \in S_{E^*}$ the implication

$$x^*(x_n) \to 1,\ \ ||x_n + y|| \to 2 \ \Rightarrow\ x^*(y) = 1 \tag{2.2}$$

holds whereas it is uniformly alternatively convex or smooth (in short, uacs) if for all $x_n, y_n \in S_E$ and $x_n^* \in S_{E^*}$ the following implication holds:

$$x_n^*(x_n) \to 1,\ \ ||x_n + y_n|| \to 2 \ \Rightarrow\ x_n^*(y_n) \to 1. \tag{2.3}$$

We emphasize that the rotundity or smoothness of E implies acs and if the dual E^* is acs then so does E itself; the converse is necessarily true only for reflexive spaces, see [19, Prop. 2.15]. Besides, uacs and luacs Banach spaces are acs and all these three notions coincide in finite dimensions.

The main ingredient in the proof of [22, Theorem 3.1] is the norm attainment of compact linear maps on reflexive Banach spaces; this result extends to weakly continuous m-homogeneous polynomials on bounded sets, see [15, Theorem 2.30]. In general, $P \in \mathcal{P}_N(E; E)$ is norm attaining if there is an $x \in S_E$ such that $||P|| = ||P(x)||$; in this case, the supremum norm is precisely a maximum norm. However, for a non-reflexive Banach space E, there is a continuous linear functional on E which does not attain its norm, see [14, Theorem 2, p.7]. The consequence of this is that a rank-one polynomial of degree N generated by such a functional on E is not norm attaining on E.

The following theorem characterizes the anti-m-order Daugavet property for a class of rank-one polynomials on finite dimensional Banach spaces.

Theorem 2.1. *Let E be a finite dimensional Banach space. Then the following assertions are equivalent.*

(a) E has the a-mDP for rank-one polynomials.

(b) E is an acs space.

Proof . To prove the implication $(a) \Rightarrow (b)$, assume that E is not acs. Thus, there exist $u, v \in S_E$ and $f \in S_{E^*}$ such that $||u+v|| = 2$ and $f(u) = 1$ but $f(v) \neq 1$ (see Equation 2.1). Now consider a rank-one polynomial $p \in \mathcal{P}(^mE; E)$ of norm one defined by $p(x) = f(x)^m v$ for all $x \in E$. Then it is evident that the only candidates for a vector x such that $p(x)$ is a multiple of x are the multiples of v. On the other hand,

$$\begin{aligned} ||I+p|| &= \sup_{x \in \mathbf{B}_E} ||x + px|| \geqslant ||(I+p)u|| \\ &= ||u+v|| = 2 = 1 + ||p||. \end{aligned}$$

Thus, $p \in \mathcal{P}(^mE; E)$ satisfies the (DE). If $f(v)^m \neq 1$, we are done; then p shows that E fails the a-mDP. Otherwise, we argue as follows. If $f(v)^m = 1$ but $f(v) \neq 1$, we must have $f(v) = \omega$, where $\omega \neq 1$, an m^{th} root of unity. (If E is real then $f(v) = -1$ and m must be even.) Now define a new polynomial $q \in \mathcal{P}(^mE; E)$ by $q(x) = f(x)^m \frac{(u+v)}{2}$ for all $x \in E$. Then clearly, $||q|| = 1$ and Lemma 2.1 in [2] implies that $||u + (u+v)/2|| = 2$. Consequently, we have

$$||1+q|| = \sup\{||x + q(x)|| : x \in \mathbf{B}_E\} \geqslant ||u + q(u)|| = ||u + (u+v)/2|| = 2,$$

that is, the polynomial q satisfies the (DE). Moreover, the multiples of $x_0 := u + v$ are the only candidates such that $q(x)$ is a multiple of x; so, by the conditions $f(u) = 1$ and $f(v) = \omega$, we have $f(x_0/2)^m = ((1+\omega)/2)^m \neq 1$ $(\omega \neq 1)$ so that $1 \notin \sigma_0(q)$. Therefore, we conclude q is a polynomial that witnesses that E fails the a-mDP.

To prove the implication $(b) \Rightarrow (a)$, let $p \in S_{\mathcal{P}(^mE;E)}$ be a rank-one polynomial that satisfies the (DE). We shall assume below that $m \geqslant 2$ since the case $m = 1$ is well known, and then prove that $||p|| \in \sigma_0(p)$. Let p be of the form $p(x) = f(x)^m v$ where $f \in S_{E^*}$ and $v \in S_E$. Now pick a norm-attaining point $x_0 \in S_E$ of $I + p$: $||x_0|| = 1$, $||x_0 + px_0|| = 2$. Hence, $||p(x_0)|| = |f(x_0)|^m = 1$ and so $|f(x_0)| = 1$. Let $\alpha \in S_{\mathbb{k}}$ be such that $\alpha f(x_0) = 1$. Then by acs property of E, $\alpha f(p(x_0)) = 1$, that is, $\alpha f(x_0)^m f(v) = 1$; thus, $|f(v)| = 1$. Let $r = f(v)^{\frac{-m}{m-1}}$, $|r| = 1$. Then $p(rv) = rv$. Thus, rv is a normalized eigenvector of a rank-one polynomial p. Therefore, we must have that $1 \in \sigma_0(p)$. □

Remark 2.2. *To extend Theorem 2.1 to infinite dimensions, we first note that it is uncertain whether polynomials of degree N that satisfy the Daugavet equation on acs Banach spaces are norm attaining, see [23, Lemma 4.2] for linear operators. However, the following theorem whose proof is similar to the second part of Theorem 2.1 is an analogue of [23, Lemma 4.2].*

Theorem 2.3. *Suppose E is acs and $p \in \mathcal{P}(^mE; E)$ is a rank-one polynomial of norm one satisfying the (DE). Suppose in addition that $||x + px|| = 2$ for some $x \in S_E$. Then $1 \in \sigma_0(p)$.*

Theorem 2.4. *Let E be a Banach space. Consider the following statements.*

(a) E has the a-mDP for compact polynomials.
(b) E has the a-mDP for rank-one polynomials .
(c) E is an acs space.
(d) E is an luacs space.

Then the following implications hold $(a) \Rightarrow (b) \Leftrightarrow (c)$ and $(d) \Rightarrow (b)$.

Proof . The proof of the implication $(a) \Rightarrow (b)$ is immediate from the fact that rank-one polynomials on E are compact and since the proof of Theorem 2.1 remains valid in infinite dimensions, then the proof of the equivalence relation $(b) \Leftrightarrow (c)$ is also immediate. For the implication $(d) \Rightarrow (b)$, let

$p \in \mathcal{P}(^mE; E)$ be a rank-one polynomial of norm one defined by $p(x) = f(x)^m v$ for all $x \in E$ such that $f \in S_{E^*}$, $v \in S_E$ and $||I + p|| = 2$. We shall assume that $m \geqslant 2$ since the case $m = 1$ is well known, and then prove that $1 \in \sigma_0(p)$. Now

$$||I + p|| = \sup_{||x|| \leqslant 1} ||x + f(x)^m v|| = 2.$$

Thus, take a sequence $\{x_n\}$ in the closed unit ball $\mathbf{B}_E$ (up to a subsequence where necessary) such that $||x_n + f(x_n)^m v|| \to 2$ and let $\alpha = \lim_{n\to\infty} f(x_n)$, a unit scalar. Consider $x'_n = x_n/\alpha$ and $y = \alpha^{m-1}v$; then using the luacs condition (see Equation 2.2), we get $f(y) = \alpha^{m-1}f(v) = \beta$ with $\beta \in S_{\Bbbk}$: so $|f(v)| = 1$. Let $r = f(v)^{\frac{-m}{m-1}}$; then $|r| = 1$ and clearly $p(rv) = rv$.Therefore, rv is a normalized eigenvector of a rank-one polynomial $p \in S_{\mathcal{P}(^mE;E)}$ and so we must have $1 \in \sigma_0(p)$. □

3. Main Results

This section comprises the results of study of the anti-N-order polynomial Daugavet property for certain Banach spaces. We will limit this study to continuous operators on locally uniformly convex Banach spaces to enable us model the proofs on the presentations of linear analogues [2, 25]. We will further consider a class of compact polynomials of degree N that in turn comprises all continuous polynomials of degree N on finite dimensional spaces (for instance finite type polynomials of degree N), see [5]; approximable polynomials of degree N defined as uniform limit in the topology of uniform convergence on bounded sets of a sequence of finite type polynomials of degree N, see [5]; weakly continuous polynomials of degree N on bounded sets defined as polynomials that map bounded weak convergent nets into convergent nets, see [4]. The weak continuity of polynomials on balls implies uniform weak continuity; compactness and weak continuity of linear operators produce the same subspace whereas not every compact polynomial is weakly continuous since continuous polynomials are not generally weak-to-weak continuous, see [4]. Further, we will generalize these results to continuous polynomials of degree N on uniformly convex Banach spaces.

Theorem 3.1. *A compact polynomial $P \in \mathcal{P}_N(E; E)$ on a locally uniformly convex Banach space E satisfies the (DE) if and only if its norm satisfies $||P|| \in \sigma_0(P)$.*

Proof . By [2, Lemma 2.1], it is sufficient to consider the case $||P|| = 1$. Further, the proof for the converse assertion follows from Lemma 1.9. For the forward, it suffices to prove that $1 \in \sigma_0(P)$. Suppose $P \in \mathcal{P}_N(E; E)$ satisfies the (DE):

$$||I + P|| = \sup_{||x|| \leqslant 1} ||x + Px|| = 2. \tag{3.1}$$

Thus, get a sequence $\{x_n\}$ of vectors in the ball $\mathbf{B}_E$ such that

$$\lim_{n\to\infty} ||x_n + Px_n|| = 2. \tag{3.2}$$

Consequently, $||x_n|| \to 1$ and $||Px_n|| \to 1$. Now by the compactness of $P \in \mathcal{P}_N(E; E)$, assume without loss of generality that $Px_n \to y$ so that $||y|| = 1$. But since $||x_n + Px_n|| \to 2$ and $Px_n \to y$, we also have $||x_n + y|| \to 2$. Hence, $x_n \to y$ and $Px_n \to Py$. By construction, we have $Px_n \to y$; thus $Py = y$ so that $1 \in \sigma_0(P)$. □

Theorem 3.2. *Let E be a uniformly convex Banach space and $P \in \mathcal{P}_N(E; E)$ be a continuous polynomial that is uniformly continuous on the unit closed ball of E. Then P satisfies the (DE) if and only if its norm satisfies $||P|| \in \sigma_a(P)$.*

Proof . By [2, Lemma 2.1], it is sufficient to assume $||P|| = 1$. Further, the proof for the converse assertion follows from Lemma 1.9. For the forward, assume $P \in \mathcal{P}_N(E;E)$ satisfies the (DE):

$$||I + P|| = \sup_{||x||\leqslant 1} ||x + Px|| = 2. \tag{3.3}$$

Thus, by Lemma 1.2, get a sequence $\{x_n\}$ in S_E such that $||x_n + Px_n|| = 2$. Then by uniform convexity of E, we have $\lim_{n\to\infty} ||x_n - Px_n|| = 0$; so, it is evident that $1 \in \sigma_a(P)$. □

Corollary 3.3. *Let E be either L_r for $1 < r < \infty$ or $\mathcal{H}$, a Hilbert space. Then a continuous polynomial $P \in \mathcal{P}_N(E;E)$ that is uniformly continuous on the closed unit ball of E satisfies the (DE) if and only if its norm satisfies $||P|| \in \sigma_a(P)$.*

Remark 3.4. *We note that, as a consequence of Theorem 3.1, every compact m-homogeneous polynomial operator that satisfies the (DE) on uniformly convex space, in particular, a Hilbert space, has at least an eigenvalue and therefore must have nontrivial invariant subspaces, see [17, Remark 1.4]. This result was not known.*

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