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Initial value problem for a fractional neutral differential equation with infinite delay

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Abstract

In this paper, we study an initial value problem for a class of nonlinear fractional neutral functional differential equations with infinite delay involving a Caputo fractional derivative. Existence, uniqueness, and continuous dependence results are established by using a variety of tools of fractional calculus including Banach's contraction principle and Schaefer's fixed point theorem.

Keywords: fractional functional differential equations, Caputo fractional derivative, existence and continuous dependence, fixed point theorem.
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1. Introduction

Studies on the theory of fractional differential equations (FDEs) became an important branch of the theory of differential equation, which steadily attracted many authors and has been grown as a new field of applied mathematics and many applications, such as mechanics, physics, chemistry, biology, and engineering. For more details, see the monographs of Kilbas et al. [18] , Miller and Ross [20], Podlubny [21] and Samko et al. [22], and the papers of Delboso and Rodino [11], Diethelm et al. [12], El-Sayed et al. [14], Lakshmikantham [19], Usta [24, 25], Yu and Gao [27], Zhou [29], Zhou and Jiao [30], Zhou et al. [31], and the references therein.

On the other hand, fractional neutral functional differential equations (FNFDEs) arise in many areas of applied mathematics. For this reason, they have largely been studied during the last few decades. The literature related to ordinary FNFDEs with finite delay and infinite delay is very extensive, thus, we refer the reader to [2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 23], that contains a comprehensive

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description of such equations. Similarly, for more on partial neutral functional differential equations and related issues, we refer to Abbas et al. [1], Hale [15], Hermndez [16] and Yan [26]. The study of FNFDEs with infinite delay is a widespread and interesting topic in the literature and it is the main motivation of our paper. Among these interesting works, for example, Agarwal et al. [7] studied the Caputo-type problem

$$ {}^cD^{\alpha}[y(t) - g(t, y_t)] = f(t, y_t), \quad t \in [t_0, \infty), \tag{1.1}$$

$$ y_{t_0} = \psi \in \mathcal{C}, \tag{1.2}$$

where $0 < \alpha < 1, {}^c D^{\alpha}$ is a Caputo fractional derivative, $f, g : [t_0, \infty) \times \mathcal{C} \longrightarrow \mathbb{R}^n$ are suitable functions, and $\mathcal{C} = C([-\tau, 0], \mathbb{R}^n)$ be the space of continuous functions on $[-\tau, 0]$. The authors used the Krasnoselskii's fixed point theorem to obtain existence of mild solutions to (1.1)-(1.2) with finite delay. In [3], the authors considered the problem with infinite delay

$$ {}^cD^{\alpha}[y(t) - g(t, y_t)] = f(t, y_t), \quad t \in [0, b], \tag{1.3}$$

$$ y_0 = \psi \in \mathcal{B}, \qquad t \in (-\infty, 0], \tag{1.4}$$

where $0 < \alpha < 1$, $\mathcal{B}$ is called a phase space, $f, g : [0, b] \times \mathcal{B} \longrightarrow \mathbb{R}$ $(b > 0)$ are appropriate functions and ${}^cD^{\alpha}$ is a Caputo fractional derivative. They employed Banach's fixed point theorem and Schauder's fixed point theorem to investigate the existence and continuous dependence results.

Motivated by the above works, in this paper we obtain sufficient conditions for the existence, uniqueness, and continuous dependence of solutions for initial value problem for FNFDEs (1.3)-(1.4) with infinite delay involving a Caputo fractional derivative.

This paper is organized as follows. Section 2 introduces some definitions, Lemmas, preliminary facts and list the hypotheses about the properties of fractional calculus. In Section 3, we prove the existence and uniqueness of solutions to problem (1.3)–(1.4). The continuous dependence of solution to such equations is discussed in Section 4 on space $C([0, b])$. Finally, the conclusion is given in Section 5.

2. preliminaries

In this section, we give some notations, definitions, lemmas and preliminary facts that related to the fractional calculus and the phase space.

Let $[0, b]$ a compact real interval and $C([0, b], \mathbb{R})$ be the Banach space of all continuous real functions $p : [0, b] \to \mathbb{R}$ with the norm $\|p\|_{\infty} = \sup\{|p(t)| : t \in [0, b]\}$. $C^n([0, b], \mathbb{R})$ denotes the set of mappings having n times continuously differentiable on $[0, b]$, $AC^n([0, b], \mathbb{R})$ $(n \in \mathbb{N}_0)$ is the space of functions p such that $p \in C^n([0, b], \mathbb{R})$ and $p^{(n-1)} \in AC([0, b], \mathbb{R})$, and denote by $L^1[0, b]$ the space of all real functions p such that $|p(t)|$ is Lebesgue integrable on $[0, b]$ in which $|p(t)| = \int_0^b |p(t)|\, dt < \infty$.

In what following, for any function y defined on $(-\infty, b]$ and any $t \in [0, b]$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(s) = y(t + s)$, for $-\infty < s \leq 0$, and let the functional space Λ defined by

$$ \Lambda = \{y : (-\infty, b] \to \mathbb{R};\ y\,|_{(-\infty,0]} \in \mathcal{B},\ y\,|_{[0,b]} \text{ is a continuous on } [0, b]\}, $$

where $y\,|_{[0,b]}$is the restriction of y to $[0, b]$.

Definition 2.1. *([18]) The left sided Riemann-Liouville fractional integral of order $\alpha > 0$ with the lower limit zero for a function $p : [0, b] \to \mathbb{R}$ is given by*

$$ I_0^{\alpha} p(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} p(s) ds, \ t > 0, $$

provided that the right side is pointwise defined on $[0, b]$, where $\Gamma(.)$ is the gamma function.

Definition 2.2. *([18]) Let $n-1<\alpha<n$ and $p\in L^1([0,b],\mathbb{R})$. The left sided Riemann-Liouville fractional derivative of order α with the lower limit zero for a function p is defined by*

$$D_0^\alpha p(t)=\frac{d^n}{dt^n}I_0^{n-\alpha}p(t),\ t>0,$$

where $n=[\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α. Moreover, $D_0^\alpha I_0^\alpha p(t)=p(t)$.

Definition 2.3. *([28]). Let $n-1<\alpha<n$ and $p\in AC^n([0,b],\mathbb{R})$. The left sided Caputo fractional derivative of order α with the lower limit zero for a function p is determined as*

$$^cD_0^\alpha p(t)=D_0^\alpha\left(p(t)-\sum_{k=0}^{n-1}\frac{p^{(k)}(0)}{k!}t^k\right),$$

where $n=[\alpha]+1$ for $\alpha\notin\mathbb{N}_0$, $n=\alpha$ for $\alpha\in\mathbb{N}_0$. In particular, when $0<\alpha<1$, we have $^cD_0^\alpha p(t)=D_0^\alpha\left(p(t)-p(0)\right)$. Moreover, if $^cD_0^\alpha p(t)\in AC[0,b]$, then

$$I_0^\alpha\ {}^cD_0^\alpha p(t)=p(t)-p(0).$$

Note that, $^cD_0^\alpha A=0$, where A is a constant function.

Lemma 2.4. *([18]) Assume that $\alpha,\beta\geq 0$ and $p(t)\in L^1[0,b]$. Then $I_0^\alpha I_0^\beta p(t)=I_0^{\alpha+\beta}p(t)$ and $^cD_0^\alpha I^\alpha p(t)=p(t)$, for any $t\in[0,b]$.*

Definition 2.5. *A function $y\in\Lambda$ is said to be a solution of (1.3)–(1.4) if y satisfies the equation $^cD_0^\alpha[y(t)-g(t,y_t)]=f(t,y_t)$, $t\in[0,b]$, with initial condition $y_0=\psi$, $y\mid_{[0,b]}\in C[0,b]\cap L^1[0,b]$ and $\frac{\partial g}{\partial t}$ is exists.*

Lemma 2.6. *([28]) (Banach contraction principle). Let K be a non-empty closed subset of a Banach space Λ, then each contraction mapping $T:K\longrightarrow K$ has a unique fixed point.*

Lemma 2.7. *([28]) (Schaefer's fixed point theorem). Let Λ be a Banach space and let $T:\Lambda\longrightarrow\Lambda$ be a completely continuous mapping of Λ such that the set $\{x\in\Lambda : x=\lambda Tx,\ for\ some\ 0<\lambda<1\}$ is bounded. Then T has at least one fixed point.*

In this paper, we suppose that the space $(\mathcal{B},\|.\|_\mathcal{B})$ is a seminormed space of functions mapping $(-\infty,0]$ into $\mathbb{R}$ and satisfying the following essential axioms that were presented by Hale and Kato [15] and discussed in detail by Hino et al. [17]:

(H1) If $y:(-\infty,b]\to\mathbb{R}$, such that $y\in C([0,b],\mathbb{R})$ and $y_0\in\mathcal{B}$, then for every $t\in[0,b]$ the following statements hold:

(i) $y_t\in\mathcal{B}$;

(ii) $|y(t)|\leq H\,\|y_t\|_\mathcal{B}$ for some $H>0$ which is equivalent to $\|\psi(0)\|\leq H\,\|\psi\|_\mathcal{B}$ for every $\psi\in\mathcal{B}$;

(iii) $\|y_t\|_\mathcal{B}\leq M(t)\sup\limits_{0\leq s\leq t}|y(s)|+N(t)\,\|y_0\|_\mathcal{B}$, where $M(.),N(.):\mathbb{R}^+\to\mathbb{R}^+$ with M continuous and N locally bounded, such that M and N are independent of $y(.)$. Denote by $M_b=\sup\{M(t):t\in[0,b]\}$ and $N_b=\sup\{N(t):t\in[0,b]\}$.

(H2) For the function $y(.)$ in (H1), the function $t\to y_t$ is continuous from $[0,b]$ into $\mathcal{B}$.

(H3) The space $\mathcal{B}$ is complete.

3. Existence and Uniqueness results

In this section, we prove the existence and uniqueness results to problem (1.3)–(1.4) by using Lemmas 2.6 and 2.7.

In the inception, we need the following lemma to establish our results:

Lemma 3.1. *[3] Let $0 < \alpha < 1$ and $f, g \in C[0,b]$ then the function $y \in C[0,b] \cap L^1[0,b]$ solves the following linear FNFDE*

$$^cD^\alpha[y(t) - g(t)] = f(t), \quad t \in [0,b],$$
$$y_0 = \psi \in \mathcal{B}, \qquad t \in (-\infty, 0],$$

if and only if y satisfies

$$y(t) = \begin{cases} \psi(0) - g(0) + g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, & t \in [0,b], \\ \psi(t), & t \in (-\infty, 0]. \end{cases}$$

Next, we prove the uniqueness result by employing Lemma 2.6.

Theorem 3.2. *Assume that $f, g : [0,b] \times \mathcal{B} \to \mathbb{R}$ are continuous functions and the following conditions are satiesfied:*

(A1) There exists a function $\delta_1 \in L^1([0,b], \mathbb{R})$ such that, for any $y_1, y_2 \in \mathcal{B}$

$$|f(t, y_1) - f(t, y_2)| \leq \delta_1(t) \left\| y_1 - y_2 \right\|_{\mathcal{B}}, \quad t \in [0,b];$$

(A2) There exists a function $\delta_2 \in C([0,b], \mathbb{R})$ such that, for any $y_1, y_2 \in \mathcal{B}$

$$|g(t, y_1) - g(t, y_2)| \leq \delta_2(t) \left\| y_1 - y_2 \right\|_{\mathcal{B}}, \quad t \in [0,b];$$

(A3) $M_b \left(\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \mu^* \right) < 1$, where $\mu^* = \int_0^s \delta_1(\tau) d\tau < \infty$.

Then there exists a unique solution to (1.3)-(1.4) on $(-\infty, b]$.

Proof . In view of Lemma 3.1, the function y is a solution to (1.3)-(1.4) if y satisfies

$$y(t) = \begin{cases} \psi(0) - g(0,\psi) + g(t, y_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds, & t \in [0,b], \\ \psi(t), & t \in (-\infty, 0]. \end{cases}$$

Transform the FNFDE (1.3)-(1.4) to a fixed point problem, i.e. $y = \Phi y$, where Φ is an operator $\Phi : \Lambda \to \Lambda$ defined by

$$(\Phi y)(t) = \begin{cases} \psi(0) - g(0,\psi) + g(t, y_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds, & t \in [0,b], \\ \psi(t), & t \in (-\infty, 0]. \end{cases}$$

For any function $\psi : (-\infty, 0] \to \mathbb{R}$ in $\mathcal{B}$, we define the function $\overline{\psi} : (-\infty, b] \to \mathbb{R}$ by

$$\overline{\psi}(t) = \begin{cases} \psi(0), & t \in [0,b], \\ \psi(t), & t \in (-\infty, 0]. \end{cases}$$

Then, we get $\overline{\psi}_0 = \psi$. For each function $v \in C([0,b],\mathbb{R})$ with $v(0) = 0$, let $\overline{v} : (-\infty, b] \to \mathbb{R}$ be the extension of v such that

$$\overline{v}(t) = \begin{cases} v(t), & t \in [0,b], \\ 0, & t \in (-\infty, 0]. \end{cases}$$

If $y(.)$ satisfies the integral equation

$$y(t) = \psi(0) - g(0,\psi) + g(t, y_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s)ds, \quad t \in [0,b],$$

then, we can decompose $y(.)$ as $y(t) = \overline{\psi}(t) + \overline{v}(t)$, $t \in (-\infty, b]$, which implies $y_t = \overline{\psi}_t + \overline{v}_t$, for every $t \in [0,b]$ and the function $v(.)$ satisfies

$$v(t) = -g(0,\psi) + g(t, \overline{\psi}_t + \overline{v}_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{\psi}_s + \overline{v}_s)ds, \quad t \in [0,b]. \tag{3.1}$$

with $\overline{v}_0 = 0$. Set $\Lambda_0 = \{v \in \Lambda,\ z_0 = 0\}$. For $v \in \Lambda_0$ and let $\|.\|_{\Lambda_0}$ be seminorm in Λ_0 defined by

$$\|v\|_{\Lambda_0} = \|v_0\|_{\mathcal{B}} + \|v\|_\infty = \sup\{|v(t)| : t \in [0,b]\}.$$

Thus, $(\Lambda_0, \|v\|_{\Lambda_0})$ is the Banach space. Define an operator $T : \Lambda_0 \to \Lambda_0$ by

$$(Tv)(t) = \begin{cases} -g(0,\psi) + g(t, \overline{\psi}_t + \overline{v}_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{\psi}_s + \overline{v}_s)ds, & t \in [0,b], \\ 0, & t \in (-\infty, 0], \end{cases} \tag{3.2}$$

Then, $(Tv)_0 = 0$. It is clear that the operator Φ has a unique fixed point equivalent to T which has a unique fixed point too, and so we turn to proving that T has a unique fixed point. Now, we show that $T : \Lambda_0 \to \Lambda_0$ is a contraction map. In fact, by Eq.(3.2), (A1), (A2), Definition 2.1 and (H1)(ii), then for $v, v^* \in \Lambda_0$ and $t \in [0,b]$, we have

$$\begin{aligned}
& |(Tv)(t) - (Tv^*)(t)| \\
\le\ & \left| g(t, \overline{\psi}_t + \overline{v}_t) - g(t, \overline{\psi}_t + \overline{v^*}_t) \right| \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s, \overline{\psi}_s + \overline{v}_s) - f(s, \overline{\psi}_s + \overline{v^*}_s) \right| ds \\
\le\ & \delta_2(t) \|\overline{v}_t - \overline{v^*}_t\|_{\mathcal{B}} + \|\overline{v}_t - \overline{v^*}_t\|_{\mathcal{B}} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \delta_1(s) ds \\
\le\ & \left(\sup_{t\in[0,b]} |\delta_2(t)| + I_0^{\alpha-1} I_0^1 \delta_1(t) \right) \|\overline{v}_t - \overline{v^*}_t\|_{\mathcal{B}} \\
=\ & \left(\|\delta_2\|_\infty + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \int_0^s \delta_1(\tau) d\tau ds \right) \|\overline{v}_t - \overline{v^*}_t\|_{\mathcal{B}} \\
\le\ & \left(\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \mu^* \right) \|\overline{v}_t - \overline{v^*}_t\|_{\mathcal{B}} \\
\le\ & \left(\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \mu^* \right) M(t) \sup_{0\le\tau\le t} |\overline{v}(\tau) - \overline{v^*}(\tau)| + N(t) \|\overline{v}_0 - \overline{v^*}_0\|_{\mathcal{B}} \\
\le\ & \left(\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \mu^* \right) M_b \|v - v^*\|_{\Lambda_0}.
\end{aligned}$$

Consequently,

$$\|Tv - Tv^*\|_{\Lambda_0} \leq \left(\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu^* \right) M_b \|v - v^*\|_{\Lambda_0}.$$

By the condition (A3), we conclude that T is a contraction mapping. As consequence of Lemma 2.6, then v is the fixed point of T, which is the unique solution to the equation (3.1) on $[0, b]$. Setting $y = \overline{\psi} + \overline{v}$, then y is the unique solution to the problem (1.3)-(1.4) on $(-\infty, b]$. The proof is completed. □

Before given our second result, we list the following hypotheses:

(A4) $f : [0, b] \times \mathcal{B} \to \mathbb{R}$ is a continuous and there exists a function $\eta \in L^1([0, b], \mathbb{R})$, such that, for each $t \in [0, b]$ and $y \in \mathcal{B}$

$$|f(t, y)| \leq \eta(t) \|y\|_{\mathcal{B}}, \quad \int_0^t \eta(s)ds := \mu < \infty.$$

(A5) $g : [0, b] \times \mathcal{B} \to \mathbb{R}$ is continuous and completely continuous and for any bounded set in Λ_0, the set $\{t \to g(t, y_t) : y \in \mathcal{B}\}$ is equicontinuous in $C([0, b], \mathbb{R})$ and there exists a function $\sigma \in C([0, b], \mathbb{R})$, such that, for all $t \in [0, b]$ and $y \in \mathcal{B}$

$$|g(t, y)| \leq \sigma(t) \|y\|_{\mathcal{B}}.$$

We give an existence result based on the Schaefer's fixed point theorem.

Lemma 3.3. *The operator $T : \Lambda_0 \to \Lambda_0$ is completely continuous.*

Proof . Consider the operator $T : \Lambda_0 \to \Lambda_0$ defined by (3.2) and let $\mathbb{B}_r = \{v \in \Lambda_0 : \|v\|_{\Lambda_0} \leq r\} \subset \Lambda_0$. To this end, we give the proof in several steps.

Step 1. We show that T is continuous in Λ_0.

For any $v_n, v \in \Lambda_0$, $n = 1, 2, ...$ with $\|v_n - v\|_{\Lambda_0} \to 0$ as $n \to \infty$, we obtain $v_n \to v$ as $n \to \infty$.

So, by the continuity of f and g, we have $f(s, \overline{\psi}_s + (\overline{v_n})_s \to f(s, \overline{\psi}_s + \overline{v}_s)$ and $g(t, \overline{\psi}_t + (\overline{v_n})_t) \to g(t, \overline{\psi}_t + \overline{v}_t)$, as $n \to \infty$. Consequentely, we obtain

$$\sup_{s\in[0,b]} \left| f(s, \overline{\psi}_s + (\overline{v_n})_s - f(s, \overline{\psi}_s + \overline{v}_s) \right| \to 0, \text{ as } n \to \infty, \tag{3.3}$$

and

$$\sup_{t\in[0,b]} \left| g(t, \overline{\psi}_t + (\overline{v_n})_t) \to g(t, \overline{\psi}_t + \overline{v}_t) \right| \to 0, \text{ as } n \to \infty. \tag{3.4}$$

On the other hand, By Eq.(3.2), then for every $t \in [0, b]$ and $v_n, v \in \Lambda_0$, we have

$$\begin{aligned} & |(Tv_n)(t) - (Tv)(t)| \\ \leq & \left| g(t, \overline{\psi}_t + (\overline{v_n})_t) - g(t, \overline{\psi}_t + \overline{v}_t) \right| \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s, \overline{\psi}_s + (\overline{v_n})_s) - f(s, \overline{\psi}_s + \overline{v}_s) \right| ds. \end{aligned}$$

The last inequality with Eq.(3.3) and Eq.(3.4) gives $\|Tv_n - Tv\|_{\Lambda_0} \to 0$, as $n \to \infty$, where $\int_0^t (t-s)^{\alpha-1} ds < \infty$. Therefore, T is continuous.

Step 2. We show that $T(\mathbb{B}_r)$ is uniformly bounded.

Indeed, it is enough to show that for any $r > 0$ there exists a positive constant $\ell > 0$ such that $\|Tv\|_{\Lambda_0} \leq \ell$. By Eq.(3.2), (A4), (A5) and Definition 2.1, then for every $v \in \mathbb{B}_r$ and $t \in [0, b]$, we have

$$\begin{aligned}
& |(Tv)(t)| \\
\leq\ & |g(0,\psi)| + \left|g(t,\overline{\psi}_t + \overline{v}_t)\right| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,\overline{\psi}_s + \overline{v}_s)\right| ds \\
\leq\ & \sigma(0)\|\psi\|_{\mathcal{B}} + \sigma(t)\left\|\overline{\psi}_t + \overline{v}_t\right\|_{\mathcal{B}} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\left\|\overline{\psi}_s + \overline{v}_s\right\|_{\mathcal{B}} ds, \\
\leq\ & \|\sigma\|_\infty \left(\|\psi\|_{\mathcal{B}} + \left\|\overline{\psi}_t + \overline{v}_t\right\|_{\mathcal{B}}\right) + \left\|\overline{\psi}_t + \overline{v}_t\right\|_{\mathcal{B}} I_0^{\alpha-1} I_0^1 \eta(t) \\
=\ & \|\sigma\|_\infty \left(\|\psi\|_{\mathcal{B}} + \left\|\overline{\psi}_t + \overline{v}_t\right\|_{\mathcal{B}}\right) + \left\|\overline{\psi}_t + \overline{v}_t\right\|_{\mathcal{B}} \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\int_0^s \eta(\tau) d\tau ds \\
\leq\ & \|\sigma\|_\infty \left(\|\psi\|_{\mathcal{B}} + \left\|\overline{\psi}_t + \overline{v}_t\right\|_{\mathcal{B}}\right) + \left\|\overline{\psi}_t + \overline{v}_t\right\|_{\mathcal{B}} \frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu. \qquad (3.5)
\end{aligned}$$

Since

$$\begin{aligned}
\left\|\overline{\psi}_t + \overline{v}_t\right\|_{\mathcal{B}} \leq\ & \left\|\overline{\psi}_t\right\|_{\mathcal{B}} + \|\overline{v}_t\|_{\mathcal{B}} \\
\leq\ & M(t)\sup_{0\leq\tau\leq t}\left|\overline{\psi}(\tau)\right| + N(t)\left\|\overline{\psi}_0\right\|_{\mathcal{B}} + M(t)\sup_{0\leq\tau\leq t}|\overline{v}(\tau)| + N(t)\|\overline{v}_0\|_{\mathcal{B}} \\
\leq\ & M_b|\psi(0)| + N_b\|\psi\|_{\mathcal{B}} + M_b\sup_{0\leq\tau\leq t}|v(\tau)| \\
\leq\ & M_b H\|\psi\|_{\mathcal{B}} + N_b\|\psi\|_{\mathcal{B}} + M_b\|v\|_{\mathbf{\Lambda_0}} \\
\leq\ & (M_b H + N_b)\|\psi\|_{\mathcal{B}} + M_b r \qquad (3.6) \\
:\ & = r_0, \qquad (3.7)
\end{aligned}$$

the inequality Eq.(3.5) becomes

$$|(Tv)(t)| \leq \|\sigma\|_\infty (\|\psi\|_{\mathcal{B}} + r_0) + \frac{r_0 b^{\alpha-1}}{\Gamma(\alpha)}\mu := \ell.$$

Therefore, $\|Tv\|_{\Lambda_0} \leq \ell$, for every $v \in \mathbb{B}_r$. This means that $T(\mathbb{B}_r)$ is uniformly bounded.

Step 3. We will prove that $T(\mathbb{B}_r)$ is equicontinuous. By Eq.(3.2) and our hypotheses, then for each $v \in \mathbb{B}_r$, and $t_1, t_2 \in [0, b]$, with $0 \leq t_1 < t_2 \leq b$, we have

$$\begin{aligned}
& |(Tv)(t_2) - (Tv)(t_1)| \\
\leq\ & \left|g(t_2,\overline{\psi}_{t_2} + \overline{v}_{t_2}) - g(t_1,\overline{\psi}_{t_1} + \overline{v}_{t_1})\right| \\
& + \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1} f(s,\overline{\psi}_s + \overline{v}_s) - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1} f(s,\overline{\psi}_s + \overline{v}_s)\right| \\
=\ & \left|g(t_2,\overline{\psi}_{t_2} + \overline{v}_{t_2}) - g(t_1,\overline{\psi}_{t_1} + \overline{v}_{t_1})\right| + \left|I_0^{\alpha-1} I_0^1 f(t_2,\overline{\psi}_{t_2} + \overline{v}_{t_2}) - I_0^{\alpha-1} I_0^1 f(t_1,\overline{\psi}_{t_1} + \overline{v}_{t_1})\right| \\
=\ & \left|g(t_2,\overline{\psi}_{t_2} + \overline{v}_{t_2}) - g(t_1,\overline{\psi}_{t_1} + \overline{v}_{t_1})\right| + \left|\begin{array}{c} \frac{1}{\Gamma(\alpha-1)}\int_0^{t_2}(t_2-s)^{\alpha-2}\int_0^s f(\tau,\overline{\psi}_\tau + \overline{v}_\tau) d\tau ds \\ -\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}\int_0^s f(\tau,\overline{\psi}_\tau + \overline{v}_\tau) d\tau ds \end{array}\right|.
\end{aligned}$$

The complete continuity of g imply that

$$\left|g(t_2,\overline{\psi}_{t_2} + \overline{v}_{t_2}) - g(t_1,\overline{\psi}_{t_1} + \overline{v}_{t_1})\right| \to 0, \text{ as } t_1 \to t_2.$$

Hence by (A4), Definition 2.1 and Eq.(3.7), we get

$$\begin{aligned}
&|(Tv)(t_2)-(Tv)(t_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}\left|(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}\right|\int_0^s\left|f(\tau,\overline{\psi}_\tau+\overline{v}_\tau)\right|d\tau ds \\
&+\frac{1}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}\int_0^s\left|f(\tau,\overline{\psi}_\tau+\overline{v}_\tau)\right|d\tau ds \\
\leq\ & \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}\left|(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}\right|\left\|\overline{\psi}_s+\overline{v}_s\right\|_{\mathcal{B}}\int_0^s\eta(\tau)d\tau ds \\
&+\frac{1}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}\left\|\overline{\psi}_s+\overline{v}_s\right\|_{\mathcal{B}}\int_0^s\eta(\tau)d\tau ds \\
\leq\ & \frac{\mu r_0}{(\alpha-1)\Gamma(\alpha-1)}(t_1^{\alpha-1}-t_2^{\alpha-1})+2(t_2-t_1)^{\alpha-1}) \\
\leq\ & \frac{2\mu r_0}{\Gamma(\alpha)}(t_2-t_1)^{\alpha-1}.
\end{aligned}$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero. Therefore, $T(\mathbb{B}_r)$ is equicontinuous. The equicontinuity for the cases $t_1<t_2\leq 0$, and $t_1\leq 0\leq t_2$ evident.

From Steps 1 to 3 with Arzela-Ascoli theorem, we can infer that the operator $T:\Lambda_0\to\Lambda_0$ is completely continuous. The proof of Lemma is complete. □

Theorem 3.4. *Assume that (A4) and (A5) hold. If*

$$\left(\|\sigma\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu\right)M_b<1. \tag{3.8}$$

Then there exists at least a solution to (1.3)-(1.4) on $(-\infty,b]$.

Proof . In view of Lemma 3.3, the operator $T:\Lambda_0\to\Lambda_0$ is completely continuous. Now, we need to show that the set $\Omega=\{v\in\Lambda_0: v=\vartheta Tv, \text{ for some } 0<\vartheta<1\}$ is bounded.

Let $v\in\Omega$, then $v=\vartheta Tv$, for some $0<\vartheta<1$. So, for each $t\in[0,b]$, we have

$$\begin{aligned}
\frac{1}{\vartheta}|v(t)| &\leq |g(0,\psi)|+\left|g(t,\overline{\psi}_t+\overline{v}_t)\right|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|f(s,\overline{\psi}_s+\overline{v}_s)\right|ds \\
&\leq \|\sigma\|_\infty\left(\|\psi\|_{\mathcal{B}}+\left\|\overline{\psi}_t+\overline{v}_t\right\|_{\mathcal{B}}\right)+\left\|\overline{\psi}_t+\overline{v}_t\right\|_{\mathcal{B}}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\eta(s)ds \\
&\leq \|\sigma\|_\infty\|\psi\|_{\mathcal{B}}+\left(\|\sigma\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu\right)\left((M_bH+N_b)\|\psi\|_{\mathcal{B}}+M_b\|v\|_{\Lambda_0}\right).
\end{aligned}$$

For every $t\in[0,b]$ and for some $\vartheta\in(0,1)$, we obtain

$$\begin{aligned}
\|v\|_{\Lambda_0} &\leq \vartheta\|\sigma\|_\infty\|\psi\|_{\mathcal{B}}+\vartheta\left(\|\sigma\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu\right)\left((M_bH+N_b)\|\psi\|_{\mathcal{B}}+M_b\|v\|_{\Lambda_0}\right) \\
&\leq \|\sigma\|_\infty\|\psi\|_{\mathcal{B}}+(M_bH+N_b)\|\psi\|_{\mathcal{B}}\left(\|\sigma\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu\right)+\left(\|\sigma\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu\right)M_b\|v\|_{\Lambda_0}.
\end{aligned}$$

By the inequality (3.8), we get

$$\|v\|_{\Lambda_0}\leq\frac{\|\sigma\|_\infty\|\psi\|_{\mathcal{B}}+(M_bH+N_b)\|\psi\|_{\mathcal{B}}\left(\|\sigma\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu\right)}{1-\left(\|\sigma\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu\right)M_b}:=L.$$

This shows that the set Ω is bounded. As consequence of Lemma 2.7, there exists at least a fixed point v of T in Λ_0 on $[0,b]$. Thus, $y=\overline{\psi}+\overline{v}$ is the solution to (1.3)-(1.4) on $(-\infty,b]$, and the proof is completed. □

4. Dependence on the parameter λ

This section is devoted to discussion of the dependence of solution on the parameter λ for the problem (1.3)-(1.4), provided that the functions $g(t,y_t)$ and $f(t,y_t)$ are Lipschitz with respect to y_t.

Firstly, we know that Theorem 3.2 remains valid if we consider the following problem

$$^{c}D_0^{\alpha}[y(t)-g(t,y_t,\lambda)]=f(t,y_t,\lambda),\quad t\in[0,b], \tag{4.1}$$

$$y_0=\psi\in\mathcal{B}, \tag{4.2}$$

where λ is a real parameter and $f,g:[0,b]\times\mathcal{B}\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfies

$$|f(t,y_1,\lambda)-f(t,y_2,\lambda)|\le\delta_1(t)\left\|y_1-y_2\right\|_{\mathcal{B}},\quad t\in[0,b],\ y_1,y_2\in\mathcal{B}; \tag{4.3}$$

and

$$|g(t,y_1,\lambda)-g(t,y_2,\lambda)|\le\delta_2(t)\left\|y_1-y_2\right\|_{\mathcal{B}},\quad t\in[0,b],\ y_1,y_2\in\mathcal{B}; \tag{4.4}$$

with

$$M_b\left(\left\|\delta_2\right\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu^*\right)<1. \tag{4.5}$$

Here, δ_1,δ_2 and μ^* are defined as in Theorem 3.2.

On the other hand, by Theorem 3.2, if Eq.(4.3), Eq.(4.4) and Eq.(4.5) hold. Then the problem (4.1)-(4.2) has a unique solution $y(t)=y(t,\lambda)$ for each $\lambda\in\mathbb{R}$. Now, we show that the solution of (4.1)-(4.2) depends continuously on the parameter λ if for each $t\in[0,b]$ and for any $y\in\mathcal{B}$,

$$|f(t,y,\lambda_1)-f(t,y,\lambda_2)|\le\delta_1(t)\left|\lambda_1-\lambda_2\right|\ \text{ and }\ |g(t,y,\lambda_1)-g(t,y,\lambda_2)|\le\delta_2(t)\left|\lambda_1-\lambda_2\right|, \tag{4.6}$$

for all $\lambda_1,\lambda_2\in\mathbb{R}$.

Theorem 4.1. *Let $f,g:[0,b]\times\mathcal{B}\times\mathbb{R}\to\mathbb{R}$ are continuous functions. If Eq.(4.3), Eq.(4.4) and Eq.(4.5) hold. Then there exists a constant κ such that*

$$\left\|y(.,\lambda_1)-y(.,\lambda_2)\right\|_C\le\kappa\left|\lambda_1-\lambda_2\right|,\quad\forall\lambda_1,\lambda_2\in\mathbb{R}.$$

Proof . Suppose that $\lambda_i\in\mathbb{R}$ $(i=1,2)$ are arbitrary constants and $y_1(t)=y(t,\lambda_1)$, $y_2(t)=y(t,\lambda_2)$ are corresponding solutions of the problem (4.1)-(4.2). Let $t\in[0,b]$, then there are $v_1,v_2\in C([0,b])$ such that $y(t,\lambda_1)=\psi(0)+v_1(t,\lambda_1)$, $y(t,\lambda_2)=\psi(0)+v_2(t,\lambda_2)$ and satisfying

$$v_1(t,\lambda_1)=-g(0,\psi,\lambda_1)+g(t,\overline{\psi}_t+(\overline{v_1})_t,\lambda_1)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,\widetilde{\psi}_s+(\overline{v_1})_s,\lambda_1)ds,$$

$$v_2(t,\lambda_2)=-g(0,\psi,\lambda_2)+g(t,\widetilde{\psi}_t+(\overline{v_2})_t,\lambda_2)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,\widetilde{\psi}_s+(\overline{v_2})_s,\lambda_2)ds.$$

Hence, we have

$$\begin{aligned}
& |y(t,\lambda_1) - y(t,\lambda_2)| \\
= & |v_1(t,\lambda_1) - v_2(t,\lambda_2)| \\
\leq & |g(0,\psi,\lambda_1) - g(0,\psi,\lambda_2)| + \left|g(t,\overline{\psi}_t + (\overline{v_1})_t,\lambda_1) - g(t,\overline{\psi}_t + (\overline{v_2})_t,\lambda_2)\right| \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,\overline{\psi}_s + (\overline{v_1})_s,\lambda_1) - f(s,\overline{\psi}_s + (\overline{v_2})_s,\lambda_2)\right| ds \\
\leq & |g(0,\psi,\lambda_1) - g(0,\psi,\lambda_2)| + \left|g(t,\overline{\psi}_t + (\overline{v_1})_t,\lambda_1) - g(t,\overline{\psi}_t + (\overline{v_2})_t,\lambda_1)\right| \\
& + \left|g(t,\overline{\psi}_t + (\overline{v_2})_t,\lambda_1) - g(t,\overline{\psi}_t + (\overline{v_2})_t,\lambda_2)\right| \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,\overline{\psi}_s + (\overline{v_1})_s,\lambda_1) - f(s,\overline{\psi}_s + (\overline{v_2})_s,\lambda_1)\right| ds \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,\overline{\psi}_s + (\overline{v_2})_s,\lambda_1) - f(s,\overline{\psi}_s + (\overline{v_2})_s,\lambda_2)\right| ds \\
\leq & \delta_2(0)\,|\lambda_1 - \lambda_2| + \delta_2(t)\,\|(\overline{v_1})_t - (\overline{v_2})_t\|_{\mathcal{B}} + \delta_2(t)\,|\lambda_1 - \lambda_2| \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\delta_1(s)\,\|(\overline{v_1})_s - (\overline{v_2})_s\|_{\mathcal{B}}\, ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\delta_1(s)\,|\lambda_1 - \lambda_2|\, ds \\
\leq & \left(2\,\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu^*\right)|\lambda_1 - \lambda_2| + \left(\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu^*\right)\|(\overline{v_1})_t - (\overline{v_2})_t\|_{\mathcal{B}}\,.
\end{aligned}$$

Since

$$\begin{aligned}
\|(\overline{v_1})_t - (\overline{v_1})_t\|_{\mathcal{B}} &\leq M(t)\sup_{0\leq\tau\leq t}|\overline{v_1}(\tau) - \overline{v_2}(\tau)| + N(t)\,\|(\overline{v_1})_0 - (\overline{v_2})_0\|_{\mathcal{B}} \\
&\leq M_b\sup_{0\leq\tau\leq t}|v_1(\tau) - v_2(\tau)| \\
&= M_b\,\|v_1 - v_2\|_C
\end{aligned}$$

we have

$$\begin{aligned}
& \|y(.,\lambda_1) - y(.,\lambda_2)\|_C \\
\leq & \left(2\,\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu^*\right)|\lambda_1 - \lambda_2| + \left(\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu^*\right)M_b\,\|v_1 - v_2\|_C\,. \\
= & \left(2\,\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu^*\right)|\lambda_1 - \lambda_2| + \left(\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu^*\right)M_b\,\|y(.,\lambda_1) - y(.,\lambda_2)\|_C\,,
\end{aligned}$$

Eq.(4.5) gives

$$\|y(.,\lambda_1) - y(.,\lambda_2)\|_C \leq \frac{\left(2\,\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu^*\right)}{1 - \left(\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu^*\right)M_b}\,|\lambda_1 - \lambda_2|\,.$$

Take $\kappa = \frac{\left(2\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu^*\right)}{1-\left(\|\delta_2\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\mu^*\right)M_b}$, we obtain

$$\|y(.,\lambda_1) - y(.,\lambda_2)\|_C \leq \kappa\,|\lambda_1 - \lambda_2|\,.$$

Therefore, the solution of (1.3)-(1.4) depends continuously on the parameter λ. The proof is completed. □

5. Conclusion

We can conclude that the main results of this article have been successfully achieved, that is, through of Banach's contraction principle and Schaefer's fixed point theorem, extremely important results within the mathematical analysis. We obtained the existence, uniqueness, and continuous dependence of solutions of the initial value problem for a nonlinear FNFDEs introduced by the Caputo fractional derivative. This paper contributes to the growth of the FDEs, especially, involving a infinite delay. There are some articles that carried out a brief study on existence and uniqueness of solutions of FDEs, one of the aims of this paper is to contribute so that it can have a greater range of studies within the mathematical analysis of FNFDEs.

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