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Dislocated quasi rectangular b-metric spaces and fixed point theorems of cyclic and weakly cyclic contractions

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Abstract

In this paper we have established fixed point theorems for dq-rectangular b-cyclic-Banach and dq-rectangular b-cyclic-Kannan contraction mappings in dislocated quasi rectangular b-metric spaces. We have also presented examples to support some of our results.

Keywords: Dislocated quasi rectangular b-metric space, fixed point, contractions, Banach contraction, Kannan Contraction, cyclic contraction, weak cyclic contraction.
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1. Introduction

Now a days many generalizations of metric spaces and fixed point theorems for different types contraction mappings in these spaces can be found in the literature of fixed point theory. Initially, metric space was generalized by Wilson[12] by introducing the concept of quasi-metric space. Bakhtin[2] introduced the b-metric space which is generalizes the metric spaces and established basic fixed point theorems in it. Hitzler et al.[10] put forth concept of dislocated metric spaces. R. George et al.[11] introduced notion of rectangular b-metric spaces as a generalization of both metric spaces and b-metric spaces. They also proved analogue of Banach contraction principle and Kannan type contraction in rectangular b-metric spaces. In the literature, many generalizations of metric spaces are found namely dislocated b-metric space, quasi b-metric space, dislocated quasi b-metric space etc. P.G. Golhare and C.T.Aage[13] introduced the new generalization of metric spaces namely dislocated

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quasi rectangular b-metric space. We establish extensions of some well known results of fixed points theorems of cyclic and weakly cyclic contraction mappings in dislocated quasi rectangular b-metric spaces.

Bakhtin[2] defined the b-metric space as follows

Definition 1.1. *([2]) Let X be a non-empty set and mapping $d : X \times X \to [0, \infty)$ satisfies:*

(i) $d(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$,
(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,
(iii) *there exists a real number* $k \geq 1$ *such that* $d(x, y) \leq k[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$.

Then d is called b-metric on X and (X, d) is called a b-metric space with coefficient k.

Shah and Huassain[9] extended b-metric space to quasi-b-metric spaces and proved some fixed point theorems in it. Alghamdi, Husasain and Salimi[7] defined the term b-metric-like spaces or dislocated b-metric spaces to generalize metric-like spaces. Some of generalizations of metric spaces are mentioned below.

Definition 1.2. *([9]) Let X be a non-empty set. Let $d : X \times X \to [0, \infty)$ be a mapping and $k \geq 1$ be a constant such that:*

(i) $d(x, y) = 0 = d(y, x)$ *if and only if* $x = y$ *for all* $x, y \in X$,
(ii) $d(x, y) \leq k[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$.

Then pair (X, d) is called quasi-b-metric space.

Definition 1.3. *([7]) Let X be a non-empty set. Let $d : X \times X \to [0, \infty)$ be a mapping and $k \geq 1$ be a constant such that:*

(i) $d(x, y) = 0$ *then* $x = y$ *for all* $x, y \in X$,
(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,
(iii) $d(x, y) \leq k[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$.

Then pair (X, d) is called dislocated b-metric space.

Chakkrid and Cholatis[4] defined the concept of dislocated quasi-b-metric space as follows

Definition 1.4. *([4]) Let X be a non-empty set. Let the mapping $d : X \times X \to [0, \infty)$ and constant $k \geq 1$ satisfy following conditions:*

(i) $d(x, y) = 0 = d(y, x)$ *then* $x = y$ *for all* $x, y \in X$,
(ii) $d(x, y) \leq k[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$.

Then the pair (X, d) is called dislocated quasi-$b-$metric space or in short dqb-metric space.

The constant k is called coefficient of space (X, d). It is clear that b-metric spaces, quasi-b-metric spaces and b-metric-like spaces are dqb-metric spaces but converse is not true.

Example 1.5. *([8]) Let $X = R^+$ and for $p > 1, d : X \times X \to [0, \infty)$ be defined as,*

$$d(x, y) = |x - y|^p + |x|^p, \forall x, y \in X.$$

Then (X, d) is dqb-metric space with $k = 2^p > 1$. But (X, d) is not b-metric space and also not dislocated quasi metric space.

Example 1.6. *([4]) Let $X = R$ and suppose,*

$$d(x, y) = |2x - y|^2 + |2x + y|^2,$$

then (X, d) is dqb-metric space with coefficient $k = 2$ but (X, d) is not a quasi-b-metric space. Also (X, d) is not dislocated quasi metric space.

Definition 1.7. *([1]) Let X be a non-empty set and mapping $d : X \times X \to [0, \infty)$ satisfies:*

(i) $d(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$,
(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,
(iii) $d(x, y) \leq [d(x, u) + d(u, v) + d(v, y)]$ *for all* $x, y \in X$ *and all distinct points* $u, v \in X \setminus \{x, y\}$.

Then d is called a rectangular metric on X and (X, d) is called a rectangular metric space.

R. George et al.([11]) defined rectangular b-metric space as follows:

Definition 1.8. *([11]) Let X be a non-empty set and mapping $d : X \times X \to [0, \infty)$ satisfies:*

(i) $d(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$,
(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,
(iii) *there exist a real number* $s \geq 1$ *such that* $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ *for all* $x, y \in X$ *and all distinct points* $u, v \in X \setminus \{x, y\}$.

Then d is called a rectangular b-metric on X and (X, d) is called a rectangular b-metric space with coefficient s.

Example 1.9. *([6]) Let $A = \{0, 2\}$,$B = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $X = A \cup B$ define $d : X \times X \to [0, \infty)$ by*

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \text{ and } \{x, y\} \subset A \text{ or } \{x, y\} \subset B, \\ y^2, & \text{if } x \in A \text{ and } y \in B, \\ x^2, & \text{if } x \in B \text{ and } y \in A, \end{cases}$$

then (X, d) is rectangular b-metric space with coefficient $k = 3$.

P.G.Golhare and C.T.Aage[13] have defined dislocated quasi rectangular b-metric space. It is also called as dq-rectangular b-metric space as follows

Definition 1.10. *[13] Let X be a non-empty set and mapping $d : X \times X \to [0, \infty)$ satisfies:*

(i) $d(x, y) = 0 = d(y, x)$ *then* $x = y$ *for all* $x, y \in X$,
(ii) *there exist a real number* $k \geq 1$ *such that* $d(x, y) \leq k[d(x, u) + d(u, v) + d(v, y)]$ *for all* $x, y \in X$ *and all points* $u, v \in X \setminus \{x, y\}$.

Then d is called a dislocated quasi or dq-rectangular b-metric on X and (X, d) is called a dislocated quasi or dq-rectangular b-metric space with coefficient k.

Example 1.11. *Let $X = \mathbb{N}$, define $d : X \times X \to [0, \infty)$ by*

$$d(x, y) = \begin{cases} 4\alpha, & \text{if } x = 1, y = 2, \\ 3\alpha, & \text{if } x = 2, y = 1, \\ \frac{\alpha}{2}, & \text{otherwise} \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a dislocated quasi rectangular b-metric space with coefficient $k = 3 > 1$. Note that for any $x \in \mathbb{N}$, $d(x, x) = \frac{\alpha}{2} \neq 0$. Therefore (X, d) is not a rectangular b-metric space. Also $d(1, 2) = 4\alpha \neq 3\alpha = d(2, 1)$.

Definition 1.12. *[13] An open ball $B_r(x)$ of radius r about x in dislocated quasi rectangular b-metric space (X, d) is*

$$\Big\{ y \in X : \max\{|d(x, y) - d(x, x)|, |d(y, x) - d(x, x)|\} < r \Big\}.$$

Definition 1.13. *A subset G of a dislocated quasi rectangular b-metric space (X, d) is said to be open if for every $x \in G$ there exists $r > 0$ such that $B_r(x) \subset G$.*

Definition 1.14. *A subset F of a dislocated quasi rectangular b-metric space (X, d) is said to be closed if its complement $X \setminus F$ is open.*

Definition 1.15. *A sequence $\{x_n\}$ in a dislocated quasi rectangular b-metric space (X, d) is said to be convergent to $x \in X$ if and only if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = d(x, x)$. In this case, we say that x is limit of sequence $\{x_n\}$.*

Proposition 1.16. *Every subsequence of a convergent sequence in dislocated quasi rectangular b-metric space also converges to the same limit.*

Definition 1.17. *Let F be a subset of dislocated quasi rectangular b-metric space (X, d). A point $x \in X$ is said to be limit point of F if and only if for every $\epsilon > 0$ there exists an open ball $B_\epsilon(x)$ such that $B_\epsilon(x) \cap F \neq \emptyset$.*

Proposition 1.18. *A subset F of a dislocated quasi rectangular b-metric space (X, d) is closed if and only if F contains all of its limit points.*

Proposition 1.19. *A subset F of a dislocated quasi rectangular b-metric space (X, d) is closed if and only if following statement holds:*
If $\{x_n\}$ is a sequence of points in F converging to some $x \in X$ implies that $x \in F$.

Definition 1.20. *Let (X, d_1) and (Y, d_2) be two dislocated quasi rectangular b-metric spaces. A mapping $T : X \to Y$ is said to be continuous at $u \in X$ if and only if given $\epsilon > 0$ there exists $\delta > 0$ such that $\max\{|d_2(Tx, Tu) - d_1(u, u)|, |d_2(Tu, Tx) - d_1(u, u)|\} < \epsilon$ whenever $\max\{|d_1(x, u) - d_1(u, u)|, |d_1(u, x) - d_1(u, u)|\} < \delta$.*

Definition 1.21. *A sequence $\{x_n\}$ in a dislocated quasi rectangular b-metric space (X, d) is called as Cauchy sequence if and only if $\lim_{n\to\infty} d(x_n, x_{n+i})$ and $\lim_{n\to\infty} d(x_{n+i}, x_n)$ exists and is finite for all $i \in \mathbb{N}$.*

Definition 1.22. *A dislocated quasi rectangular b-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.*

2. Main Results

Definition 2.1. *Let A and B be non-empty subsets of a dislocated quasi rectangular b-metric space (X, d) with coefficient k, then a cyclic mapping $T : A \cup B \to A \cup B$ is called a dq -rectangular b-cyclic-Banach mapping if there exists $\alpha \in [0, 1/k)$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y), \tag{2.1}$$

for all $x \in A, y \in B$.

Our first result is given below.

Theorem 2.2. *Let (X, d) be a complete dislocated quasi rectangular b-metric space with coefficient $k > 1$ and A, B be two non-empty closed subsets of X. If $T : A \cup B \to A \cup B$ is a dq-rectangular b-cyclic-Banach mapping then T has a unique fixed point in $A \cap B$.*

Proof . We choose any arbitrary point $x_0 \in A$. Now we can find $x_1 \in B$ such that $x_1 = Tx_0$. Similarly we can find $x_2 \in A$ such that $x_2 = Tx_1$. Thus we get a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Also note that $x_n \in A$ if n is even and $x_n \in B$ if n is odd. That is $\{x_{2n}\}$ is sequence in A and $\{x_{2n-1}\}$ is sequence in B for $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, $x_{n-1} = x_n$, then x_{n-1} becomes fixed point of T and we have nothing to prove. Therefore, we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. From inequality (2.1), we have

$$d(x_{n-1}, x_n) = d(Tx_{n-2}, Tx_{n-1}) \leq \alpha d(x_{n-2}, x_{n-1}). \tag{2.2}$$

Applying inequality (2.2) repeatedly, we get,

$$d(x_{n-1}, x_n) \leq \alpha d(x_{n-2}, x_{n-1}) \leq \cdots \leq \alpha^{n-1} d(x_0, x_1). \tag{2.3}$$

Similarly,

$$d(x_n, x_{n-1}) \leq \alpha d(x_{n-1,x_{n-2}}) \leq \cdots \leq \alpha^{n-1} d(x_1, x_0). \tag{2.4}$$

We also assume that $x_0 \neq x_n$ for any $2 \leq n \in \mathbb{N}$. If not, then for some $n \geq 2$ in view of (2.3), we have

$$d(x_0, Tx_0) = d(x_n, Tx_n)$$
$$d(x_0, x_1) = d(x_n, x_{n+1}).$$

It implies that

$$d(x_0, x_1) \leq \alpha^n d(x_0, x_1),$$

which is a contradiction unless $d(x_0, x_1) = 0$. Thus $x_0 = x_1$ and x_0 turns out to be a fixed point of T. So, we assume that $x_n \neq x_m$ for all $n \neq m \in \mathbb{N}$. In view of (2.1), for any $n \in \mathbb{N}$, we can write,

$$d(x_{n-1}, x_{n+1}) = d(Tx_{n-2}, Tx_n) \leq \alpha d(x_{n-2}, x_n). \tag{2.5}$$

Applying (2.1) repeatedly, we get,

$$d(x_{n-1}, x_{n+1}) \leq \alpha^{n-1} d(x_0, x_2). \tag{2.6}$$

Similarly,

$$d(x_{n+1}, x_{n-1}) \leq \alpha^{n-1} d(x_2, x_0). \tag{2.7}$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence in X, equivalently, we will show

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 = \lim_{n\to\infty} d(x_{n+m}, x_n),$$

for all $n, m \in \mathbb{N}$.

Case (i): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be even or odd. Using inequalities (2.3), (2.4) and rectangular inequality, we get,

$$\begin{aligned}
d(x_n, x_{n+m}) &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i})] \\
&\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i})] \\
&\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
&+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6}) +] + \cdots \\
&+ k^{i-1}[d(x_{n-4+2i}, x_{n-3+2i}) + d(x_{n-3+2i}, x_{n-2+2i})] + k^{i-1}[d(x_{n-2+2i}, x_{n+2i})] \\
&\leq k[\alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1)] + k^2[\alpha^{n+2} d(x_0, x_1) + \alpha^{n+3} d(x_0, x_1)] \\
&+ k^3[\alpha^{n+4} d(x_0, x_1) + \alpha^{n+5} d(x_0, x_1)] + \cdots + k^{i-1}[\alpha^{n-4+2i} d(x_0, x_1) + \alpha^{n-3+2i} d(x_0, x_1)] \\
&+ k^{i-1}\alpha^{n-2+2i} d(x_0, x_2) \\
&\leq k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots] d(x_0, x_1) + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots] d(x_0, x_1) \\
&+ k^{i-1}\alpha^{n-2+2i} d(x_0, x_2) \\
&\leq \Big[\frac{(1+\alpha)}{1 - k\alpha^2}\Big] k\alpha^n d(x_0, x_1) + k^{i-1}\alpha^{n-2+2i} d(x_0, x_2) \\
&\leq \Big[\frac{(1+\alpha)}{1 - k\alpha^2}\Big] k\alpha^n d(x_0, x_1) + \alpha^{n-2} d(x_0, x_2).
\end{aligned}$$

Letting $n \to \infty$ in last inequality above, we get,

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0,$$

for all even $m \in \mathbb{N}$.

Case (ii): Suppose m is odd i.e. $m = 2i - 1$ for some $i \in \mathbb{N}$ and n may be even or odd. Using inequalities (2.3), (2.4) and rectangular inequality, we get,

$$\begin{aligned}
d(x_n, x_{n+m}) &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i-1})] \\
&\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i-1})] \\
&\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
&+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \cdots \\
&+ k^i[d(x_{n+2i-2}, x_{n+2i-1})] \\
&\leq k[\alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1)] + k^2[\alpha^{n+2} d(x_0, x_1) + \alpha^{n+3} d(x_0, x_1)] \\
&+ k^3[\alpha^{n+4} d(x_0, x_1) + \alpha^{n+5} d(x_0, x_1)] + \cdots + k^i\alpha^{n+2i-2} d(x_0, x_1) \\
&\leq k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots] d(x_0, x_1) + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots] d(x_0, x_1) \\
&\leq \Big[\frac{(1+\alpha)}{1 - k\alpha^2}\Big] k\alpha^n d(x_0, x_1).
\end{aligned}$$

Letting $n \to \infty$ in last inequality above, we see that limit on the right hand side exist and is finite. Therefore, $\lim_{n\to\infty} d(x_n, x_{n+m})$ exists and is finite for all odd $m \in \mathbb{N}$. Thus from the case(i) and case(ii), it follows that $\lim_{n\to\infty} d(x_n, x_{n+m})$ exists and,

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0, \ \forall m \in \mathbb{N}. \tag{2.8}$$

Now, we will prove that $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$, for all $m, n \in \mathbb{N}$ with $m > n$. We consider two cases:

Case (a): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be odd or even. Then

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq k^{i-2} d(x_{n+2i}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq k^{i-2}\alpha^{n+2i-2} d(x_2, x_0) + k^{i-2}[\alpha^{n+2i-3} d(x_1, x_0) + \alpha^{n+2i-4} d(x_1, x_0)] \\
&+ k^{i-3}[\alpha^{n+2i-4} d(x_1, x_0) + \alpha^{n+2i-5} d(x_1, x_0)] + \cdots \\
&+ k[\alpha^{n+1} d(x_1, x_0) + \alpha^n d(x_1, x_0)] \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\} d(x_1, x_0) \\
&\leq (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1} d(x_1, x_0).
\end{aligned}$$

It gives that

$$\lim_{n\to\infty} d(x_{n+m}, x_n) = 0.$$

Case (b): Suppose m is odd i.e. $m = 2i - 1$, for some $i \in \mathbb{N}$ and n may be odd or even. Then

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq k^{i-2} d(x_{n+2i-1}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq k^{i-2}\alpha^{n+2i-2} d(x_1, x_0) + k^{i-2}[\alpha^{n+2i-3} d(x_1, x_0) + \alpha^{n+2i-4} d(x_1, x_0)] \\
&+ k^{i-3}[\alpha^{n+2i-4} d(x_1, x_0) + \alpha^{n+2i-5} d(x_1, x_0)] + \cdots \\
&+ k[\alpha^{n+1} d(x_1, x_0) + \alpha^n d(x_1, x_0)] \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\} d(x_1, x_0) \\
&\leq (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1} d(x_1, x_0).
\end{aligned}$$

It gives that $\lim_{n,m\to\infty} d(x_{n+m}, x_n) = 0$. Thus from case (a) and case (b), it follows that

$$lim_{n\to\infty} d(x_{n+m}, x_n) = 0, \ \forall n, m \in \mathbb{N}. \tag{2.9}$$

Thus from (2.8) and (2.9), we conclude that $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 = \lim_{n\to\infty} d(x_{n+m}, x_n) = 0, \ \forall n, m \in \mathbb{N}$. Hence $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete dislocated quasi rectangular b-metric space, there exists some $u \in X$ such that $x_n \to u$. In view of proposition 1.16, subsequences $\{x_{2n}\}$ in A and $\{x_{2n-1}\}$ in B also converge to $u \in X$. As A and B are closed subsets of X, $u \in A \cap B$. We will show that u is fixed point of T. For any given $n \in \mathbb{N}$, we can write,

$$\begin{aligned} d(u, Tu) &\leq k[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] \\ &= k[d(u, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tu)] \\ &\leq k[d(u, x_n) + d(x_n, x_{n+1}) + \alpha d(x_n, u)]. \end{aligned}$$

Letting $n \to \infty$, using fact that $x_n \to u$ and (2.6), we get $d(u, Tu) = 0$. Also,

$$\begin{aligned} d(Tu, u) &\leq k[d(Tu, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, u)] \\ &= k[d(Tu, Tx_n) + d(x_{n+1}, x_n) + d(x_n, u)] \\ &\leq k[\alpha d(u, x_n) + d(x_{n+1}, x_n) + d(x_n, u)]. \end{aligned}$$

Letting $n \to \infty$, using fact that $x_n \to u$ and (2.6), we get $d(Tu, u) = 0$. Thus $d(u, Tu) = 0 = d(Tu, u)$. This gives that $Tu = u$. Hence u is fixed point of T in X.

Now, we prove that u is unique fixed point of T in X. Suppose u' be another fixed point of T in X. In view of (2.1), we have

$$d(u, u') = d(Tu, Tu') \leq \alpha d(u, u') < d(u, u').$$

This a contradiction unless $d(u, u') = 0$.

Similarly,

$$d(u', u) = d(Tu', Tu) \leq \alpha d(u', u) < d(u', u).$$

This a contradiction unless $d(u', u) = 0$. Thus $d(u, u') = 0 = d(u', u)$. Hence $u = u'$. Thus uniqueness of u is established. □

Example 2.3. *Let* $X = A \cup B$ *where* $A = \{0\} \cup \{2n : n \in \mathbb{N}\}, B = \{0\} \cup \{2n-1 : n \in \mathbb{N}\}$ *define* $d : X \times X \to [0, \infty)$ *by*

$$d(x, y) = \begin{cases} 4\alpha, & \text{if } x \in B - \{0\} \text{ and } y \in A - \{0\}, \\ 3\alpha, & \text{if } x \in A - \{0\} \text{ and } y \in B - \{0\}, \\ \frac{5\alpha}{4}, & \text{Otherwise} \end{cases}$$

where $\alpha > 0$ *is a constant. Then* (X, d) *is a complete dislocated quasi rectangular b-metric space with coefficient* $k = \frac{6}{5} > 1$. *If* $T : X \to X$ *is defined as follows:*

$$Tx = \begin{cases} x - 1, & \text{if } x \in A - \{0\}, \\ x + 1, & \text{if } x \in B - \{0\}, \\ 0, & \text{Otherwise} \end{cases}$$

then T *is dq-rectangular b-cyclic-Banach contraction in complete dislocated quasi rectangular b-metric space* (X, d) *and* T *has unique fixed point* $x = 0 \in X$.

Definition 2.4. *Let A and B be non-empty subsets of a dislocated quasi rectangular b-metric space (X,d) with coefficient k, then a cyclic mapping $T : A \cup B \to A \cup B$ is called a dq -rectangular b-cyclic-Kannan mapping if there exists $\gamma \in [0, 1/2k]$ such that*

$$d(Tx,Ty) \leq \gamma[d(x,Tx) + d(y,Ty)], \tag{2.10}$$

for all $x \in A, y \in B$.

Theorem 2.5. *Let (X,d) be a complete dislocated quasi rectangular b-metric space with coefficient $k > 1$ and A, B be two non-empty closed subsets of X. If $T : A \cup B \to A \cup B$ is a dq-rectangular b-cyclic-Kannan mapping then T has a unique fixed point in $A \cap B$.*

Proof . We choose any arbitrary point $x_0 \in A$. Now we can find $x_1 \in B$ such that $x_1 = Tx_0$. Similarly we can find $x_2 \in A$ such that $x_2 = Tx_1$. Thus we get a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Also note that $x_n \in A$ if n is even and $x_n \in B$ if n is odd. That is $\{x_{2n}\}$ is sequence in A and $\{x_{2n-1}\}$ is sequence in B for $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, $x_{n-1} = x_n$, then x_{n-1} becomes fixed point of T and we have nothing to prove. Therefore, we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. From inequality (2.10), we have

$$\begin{aligned} d(x_{n-1},x_n) = d(Tx_{n-2},Tx_{n-1}) &\leq \gamma[d(x_{n-2},Tx_{n-2}) + d(x_{n-1},Tx_{n-1})] \\ &= \gamma[d(x_{n-2},x_{n-1}) + d(x_{n-1},x_n)]. \end{aligned}$$

This gives that

$$d(x_{n-1},x_n) \leq \frac{\gamma}{1-\gamma} d(x_{n-2},x_{n-1}) = \alpha d(x_{n-2},x_{n-1}) \tag{2.11}$$

where $\alpha = \frac{\gamma}{1-\gamma}$.

Applying inequality (2.11) repeatedly we get,

$$d(x_{n-1},x_n) \leq \alpha d(x_{n-2},x_{n-1}) \leq \cdots \leq \alpha^{n-1} d(x_0,x_1). \tag{2.12}$$

We also assume that $x_0 \neq x_n$ for any $2 \leq n \in \mathbb{N}$. If not, then for some $n \geq 2$ in view of (2.12), we have

$$\begin{aligned} d(x_0,Tx_0) &= d(x_n,Tx_n) \\ d(x_0,x_1) &= d(x_n,x_{n+1}). \end{aligned}$$

It implies that

$$d(x_0,x_1) \leq \alpha^n d(x_0,x_1),$$

which is a contradiction unless $d(x_0,x_1) = 0$. Thus $x_0 = x_1$ and x_0 turns out to be a fixed point of T. Hence, we assume that $x_n \neq x_m$ for all $n \neq m \in \mathbb{N}$. In view of (2.10), for any $n \in \mathbb{N}$, we can write

$$\begin{aligned} d(x_{n-1},x_{n+1}) = d(Tx_{n-2},Tx_n) &\leq \gamma[d(x_{n-2},Tx_{n-2}) + d(x_n,Tx_n)] \\ &= \gamma[d(x_{n-2},x_{n-1}) + d(x_n,x_{n+1})] \\ &\leq \gamma[\alpha^{n-2} d(x_0,x_1) + \alpha^n d(x_0,x_1)] \\ &= \gamma\alpha^{n-2}[1+\alpha^2] d(x_0,x_1) \\ &= \beta\alpha^{n-2} d(x_0,x_1), \end{aligned}$$

where $\beta = \gamma[1+\alpha^2]$. Thus
$$d(x_{n-1}, x_{n+1}) \leq \beta\alpha^{n-2} d(x_0, x_1). \tag{2.13}$$
In order to show $\{x_n\}$ is a Cauchy sequence in X, it is sufficient to show that
$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 = \lim_{n\to\infty} d(x_{n+m}, x_n),$$
for all $n, m \in \mathbb{N}$.

Case (i): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be even or odd. Then using inequalities (2.12), (2.13) and rectangular inequality, we get
$$\begin{aligned}
d(x_n, x_{n+2i}) &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i})]\\
&\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i})]\\
&\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})]\\
&+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6}) +] + \cdots\\
&+ k^{i-1}[d(x_{n+2i-4}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-2})] + k^{i-1}[d(x_{n+2i-2}, x_{n+2i})]\\
&\leq k[\alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1)] + k^2[\alpha^{n+2} d(x_0, x_1) + \alpha^{n+3} d(x_0, x_1)]\\
&+ k^3[\alpha^{n+4} d(x_0, x_1) + \alpha^{n+5} d(x_0, x_1)] + \cdots + k^{i-1}[\alpha^{n+2i-4} d(x_0, x_1) + \alpha^{n+2i-3} d(x_0, x_1)]\\
&+ k^{i-1}\alpha^{n+2i-2}\beta d(x_0, x_2)\\
&\leq k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1) + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1)\\
&+ k^{i-1}\alpha^{n-3+2i} d(x_0, x_2)\\
&\leq \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big] k\alpha^{n-1} d(x_0, x_1) + k^{i-1}\alpha^{n-3+2i}\beta d(x_0, x_2)\\
&\leq \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big] k\alpha^{n-1} d(x_0, x_1) + \alpha^{n-3}\beta d(x_0, x_2).
\end{aligned}$$
Letting $n \to \infty$ in last inequality above, we get
$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0,$$
for all even $m \in \mathbb{N}$.

Case (ii): Suppose m is even i.e. $m = 2i - 1$ for some $i \in \mathbb{N}$ and n may be even or odd. Using inequalities (2.12), (2.13) and rectangular inequality, we get
$$\begin{aligned}
d(x_n, x_{n+2i-1}) &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i-1})]\\
&\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i-1})]\\
&\leq k[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})]\\
&+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \cdots\\
&+ k^i[d(x_{n+2i}, x_{n+2i-1})]\\
&\leq k[\alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1)] + k^2[\alpha^{n+2} d(x_0, x_1) + \alpha^{n+3} d(x_0, x_1)]\\
&+ k^3[\alpha^{n+4} d(x_0, x_1) + \alpha^{n+5} d(x_0, x_1)] + \cdots + k^i\alpha^{n+2i} d(x_0, x_1)\\
&\leq k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1) + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots]d(x_0, x_1)\\
&\leq \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big] k\alpha^{n-1} d(x_0, x_1).
\end{aligned}$$

Letting $n \to \infty$ in last inequality above, we get

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0,$$

for all odd $m \in \mathbb{N}$. Thus from case (i) and case (ii), it follows that for all $m, n \in \mathbb{N}$.

$$\lim_{n,m\to\infty} d(x_n, x_{n+m}) = 0. \tag{2.14}$$

Now, we prove that $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$ for all $m, n \in \mathbb{N}$. We consider two cases.

Case (a): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be odd or even. Then

$$\begin{aligned}
d(x_{n+m}, x_n) &\le k^{i-2} d(x_{n+2i}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\le k^{i-2}\alpha^{n+2i-2} d(x_2, x_0) + k^{i-2}[\alpha^{n+2i-3} d(x_1, x_0) + \alpha^{n+2i-4} d(x_1, x_0)] \\
&+ k^{i-3}[\alpha^{n+2i-4} d(x_1, x_0) + \alpha^{n+2i-5} d(x_1, x_0)] + \cdots \\
&+ k[\alpha^{n+1} d(x_1, x_0) + \alpha^n d(x_1, x_0)] \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\} d(x_1, x_0) \\
&\le (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_2, x_0) + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1} d(x_1, x_0).
\end{aligned}$$

It gives that

$$\lim_{n\to\infty} d(x_{n+m}, x_n) = 0.$$

Case (b): Suppose m is odd i.e. $m = 2i-1$ for some $i \in \mathbb{N}$ and n may be odd or even.

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq k^{i-2} d(x_{n+2i-1}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq k^{i-2}\alpha^{n+2i-2} d(x_1, x_0) + k^{i-2}[\alpha^{n+2i-3} d(x_1, x_0) + \alpha^{n+2i-4} d(x_1, x_0)] \\
&+ k^{i-3}[\alpha^{n+2i-4} d(x_1, x_0) + \alpha^{n+2i-5} d(x_1, x_0)] + \cdots \\
&+ k[\alpha^{n+1} d(x_1, x_0) + \alpha^n d(x_1, x_0)] \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\} d(x_1, x_0) \\
&\leq (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\} d(x_1, x_0) \\
&= (k\alpha)^{2i-2}\alpha^n d(x_1, x_0) + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1} d(x_1, x_0).
\end{aligned}$$

It gives that

$$\lim_{n\to\infty} d(x_{n+m}, x_n) = 0.$$

Thus

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 = \lim_{n\to\infty} d(x_{n+m}, x_n)$$

for all $n, m \in \mathbb{N}$. Hence $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete dislocated quasi rectangular b-metric space, there exists some $u \in X$ such that $x_n \to u$. In view of proposition 1.16, subsequences $\{x_{2n}\}$ in A and $\{x_{2n-1}\}$ in B also converge to $u \in X$. As A and B are closed subsets of X, $u \in A \cap B$. We claim that u is fixed point of T. For any given $n \in \mathbb{N}$, we can write

$$\begin{aligned}
d(u, Tu) &\leq k[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] \\
&= k[d(u, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tu)] \\
&\leq k\Big\{d(u, x_n) + d(x_n, x_{n+1}) + \gamma[d(x_n, Tx_n) + d(u, Tu)]\Big\} \\
&= k\Big\{d(u, x_n) + d(x_n, x_{n+1}) + \gamma[d(x_n, x_{n+1}) + d(u, Tu)]\Big\},
\end{aligned}$$

which gives that,

$$d(u, Tu) \leq \frac{1}{1-\gamma}\Big\{d(u, x_n) + d(x_n, x_{n+1}) + \gamma d(x_n, x_{n+1})\Big\}. \tag{2.15}$$

Letting $n \to \infty$, the sequence $x_n \to u$, we get $d(u, Tu) = 0$. Also,

$$\begin{aligned}
d(Tu, u) &\leq k[d(Tu, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, u)] \\
&= k[d(Tu, Tx_n) + d(x_{n+1}, x_n) + d(x_n, u)] \\
&\leq k\Big\{\gamma[d(u, Tu) + d(x_n, Tx_n)] + d(x_{n+1}, x_n) + d(x_n, u)\Big\} \\
&= k\Big\{\gamma[d(u, Tu) + d(x_n, x_{n+1})] + d(x_{n+1}, x_n) + d(x_n, u)\Big\}.
\end{aligned}$$

Letting $n \to \infty$, the sequence $x_n \to u$, we get $d(Tu, u) = 0$. Thus $d(u, Tu) = 0 = d(Tu, u)$. It gives that $Tu = u$. Hence u is fixed point of T in X.

Note that

$$d(u, u) = d(Tu, Tu) \leq \gamma[d(u, Tu) + d(u, Tu)] = 2\gamma d(u, u) < d(u, u) \tag{2.16}$$

which is a contradiction unless $d(u, u) = 0$. Thus if u is fixed point of T, then we have $d(v, v) = 0$. Suppose u' be another fixed point of T in X. In view of (2.10), we have

$$\begin{aligned} d(u, u') = d(Tu, Tu') &\leq \gamma[d(u, Tu) + d(u', Tu')] \\ &= \gamma[d(u, u) + d(u', u')] = 0. \end{aligned}$$

Also,

$$\begin{aligned} d(u', u) = d(Tu', Tu) &\leq \gamma[d(u', Tu') + d(u, Tu)] \\ &= \gamma[d(u', u') + d(u, u)] = 0. \end{aligned}$$

Thus $d(u, u') = d(u', u) = 0$ i.e. $u = u'$. u is a unique fixed point of T. □

Example 2.6. *Let $A = \{0, 2, 1/2\}$, $B = \{\frac{1}{n} : n \in \mathbb{N}, n \geq 2\}$ and $X = A \cup B$ define $d : X \times X \to [0, \infty)$ by*

$$d(x, y) = \begin{cases} \frac{1}{2}, & \text{if } x, y \in A - \{\frac{1}{2}\}, \\ \frac{1}{3}, & \text{if } x, y \in B - \{\frac{1}{2}\}, \\ \frac{1}{4}, & \text{if } x \in A - \{\frac{1}{2}\} \text{ and } y \in B - \{\frac{1}{2}\}, \\ \frac{3}{4}, & \text{if } x \in B - \{\frac{1}{2}\} \text{ and } y \in A - \{\frac{1}{2}\}, \\ 0, & \text{Otherwise} \end{cases}$$

then (X, d) is complete dislocated quasi rectangular b-metric space with coefficient $k = \frac{3}{2} > 1$. If $T : X \to X$ is defined as follows:

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x \in A, \\ 0, & \text{if } x \in B - \{\frac{1}{2}\}, \end{cases}$$

then T is dq-rectangular b-cyclic-Kannan contraction in complete dislocated quasi rectangular b-metric space (X, d) and T has unique fixed point $x = \frac{1}{2} \in X$.

We define,
$\Phi = \{\phi : [0, \infty) \to [0, \infty) | \phi$ is continuous, non-decreasing and $\phi(\alpha) = 0$ if and only if $\alpha = 0\}$. Now, we define quasi-like weak cyclic $\phi-$contraction in dislocated quasi rectangular b-metric space $(A \cup B = X, d)$ with coefficient k as follows:

Definition 2.7. *A mapping $T : A \cup B \to A \cup B$ said to be a quasi-like weak cyclic $\phi-$contraction if,*

$$d(Tx, Ty) \leq \alpha Q(x, y) - \phi(Q(x, y)) \tag{2.17}$$

for all $x, y \in X$, where $Q(x, y) = \max\{d(x, y), d(Tx, x), d(y, Ty)\}$, $0 \leq \alpha < \frac{1}{k}$ and $\phi \in \Phi$.

Theorem 2.8. *Let (X, d) be a complete dislocated quasi rectangular b-metric space with coefficient $k \geq 1$ such that $X = A \cup B$ where A and B are closed subsets of X. Let $T : X \to X$ be a quasi-like weak cyclic $\phi-$contraction. Then T has unique fixed point in X.*

Proof . We consider any arbitrary point $x_0 \in X$. Now define sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. For an obvious reason we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. In the light of inequality (2.17), we see that,

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Tx_1) \\ &\leq \alpha Q(x_0, x_1) - \phi(Q(x_0, x_1)) \\ &\leq \alpha Q(x_0, x_1) \\ &\leq \alpha \max\{d(x_0, x_1), d(x_1, x_0)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_2, x_1) &= d(Tx_1, Tx_0) \\ &\leq \alpha Q(x_1, x_0) - \phi(Q(x_1, x_0) \\ &\leq \alpha Q(x_1, x_0) \\ &\leq \alpha \max\{d(x_1, x_0), d(x_0, x_1)\}. \end{aligned}$$

Let $\eta = \max\{d(x_1, x_0), d(x_0, x_1)\}$. Then

$$d(x_1, x_2) \leq \alpha\eta \tag{2.18}$$

and

$$d(x_2, x_1) \leq \alpha\eta. \tag{2.19}$$

Now consider,

$$\begin{aligned} d(x_2, x_3) &= d(Tx_1, Tx_2) \\ &\leq \alpha Q(x_1, x_2) - \phi(Q(x_1, x_2)) \\ &\leq \alpha Q(x_1, x_2) \\ &\leq \alpha \max\{d(x_1, x_2), d(x_2, x_1)\} \\ &\leq \alpha^2\eta. \end{aligned} \tag{2.20}$$

Similarly,

$$\begin{aligned} d(x_3, x_2) &= d(Tx_2, Tx_1) \\ &\leq \alpha Q(x_2, x_1) - \phi(Q(x_2, x_1)) \\ &\leq \alpha Q(x_2, x_1) \\ &\leq \alpha \max\{d(x_2, x_1), d(x_1, x_2)\} \\ &\leq \alpha^2\eta. \end{aligned} \tag{2.21}$$

Applying above inequalities (2.20) and (2.21),repeatedly, we get,

$$d(x_n, x_{n+1}) \leq \alpha^n\eta \tag{2.22}$$

and

$$d(x_{n+1}, x_n) \leq \alpha^n\eta. \tag{2.23}$$

If for some n such that $2 \leq n \in \mathbb{N},\ x_0 = x_n$ then in view of (2.22), we have

$$\begin{aligned} d(x_0, Tx_0) &= d(x_n, Tx_n) \\ d(x_0, x_1) &= d(x_n, x_{n+1}) \\ d(x_0, x_1) &\leq \alpha^n\eta. \end{aligned}$$

If $\eta = d(x_0, x_1)$, then we get $d(x_0, x_1) \leq \alpha^n d(x_0, x_1)$, which is a contradiction unless $d(x_0, x_1) = 0$. And hence $d(x_1, x_0) = 0$. This yields that $x_0 = x_1$. And thus x_0 turns out to be a fixed point of T.Therefore we assume that $x_0 \neq x_n$ for any $2 \leq n \in \mathbb{N}$.
Similarly,we assume that $x_n \neq x_m$, for all $n \neq m \in \mathbb{N}$. Since, if not

$$\begin{aligned} d(Tx_0, x_0) &= d(Tx_n, x_n) \\ d(x_1, x_0) &= d(x_{n+1}, x_n) \\ d(x_1, x_0) &\leq \alpha^n \eta. \end{aligned}$$

If $\eta = d(x_1, x_0)$, then we get $d(x_1, x_0) \leq \alpha^n d(x_1, x_0)$, which is a contradiction unless $d(x_1, x_0) = 0$. And hence $d(x_0, x_1) = 0$. This yields that $x_0 = x_1$ and thus x_0 turns out to be a fixed point of T.

Let $\beta = \max\{d(x_2, x_0), d(x_0, x_2), \eta\}$. We claim that $d(x_n, x_{n+2}) \leq \alpha^n \beta$ and $d(x_{n+2}, x_n) \leq \alpha^n \beta$, for all $n \in \mathbb{N}$. We first prove $d(x_n, x_{n+2}) \leq \alpha^n \beta$. We proceed by induction. For $n = 1$,

$$\begin{aligned} d(x_1, x_3) &= d(Tx_0, Tx_2) \\ &\leq \alpha Q(x_0, x_2) - \phi((x_0, x_2)) \\ &\leq \alpha(x_0, x_2) \\ &\leq \alpha \max\{d(x_0, x_2), \eta, \alpha^2 \eta\} \\ &\leq \alpha \max\{d(x_0, x_2), \eta\} \\ &= \alpha\beta. \end{aligned}$$

Assume that $d(x_{n-1}, x_{n+1}) \leq \alpha^{n-1}\beta$. Now consider

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \alpha Q(x_{n-1}, x_{n+1}) - \phi(Q(x_{n-1}, x_{n+1})) \\ &\leq \alpha Q(x_{n-1}, x_{n+1}) \\ &\leq \alpha \max\{\alpha^{n-1}\beta, \alpha^{n-1}\eta, \alpha^{n+1}\eta\} \\ &\leq \alpha\alpha^{n-1}\beta \\ &= \alpha^n \beta. \end{aligned}$$

Thus for all $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+2}) \leq \alpha^n \beta. \tag{2.24}$$

Now, we prove that $d(x_{n+2}, x_n) \leq \alpha^n \beta$. Again we proceed by induction. For $n = 1$,

$$\begin{aligned} d(x_3, x_1) &= d(Tx_2, Tx_0) \\ &\leq \alpha Q(x_2, x_0) - \phi(Q(x_2, x_0)) \\ &\leq \alpha Q(x_2, x_0) \\ &\leq \alpha \max\{d(x_2, x_0), \alpha^2 \eta, \eta\} \\ &\leq \alpha \max\{d(x_2, x_0), \eta\} \\ &= \alpha\beta. \end{aligned}$$

Assume that $d(x_{n+1}, x_{n-1}) \leq \alpha^{n-1}\beta$. Now, we consider

$$\begin{aligned} d(x_{n+2}, x_n) &= d(Tx_{n+1}, Tx_{n-1}) \\ &\leq \alpha Q(x_{n+1}, x_{n-1}) - \phi(Q(x_{n+1}, x_{n-1})) \\ &\leq \alpha Q(x_{n+1}, x_{n-1}) \\ &\leq \alpha \max\{\alpha^{n-1}\beta, \alpha^{n+1}\eta, \alpha^{n-1}\eta\} \\ &\leq \alpha\alpha^{n-1}\beta \\ &= \alpha^n\beta. \end{aligned}$$

Thus, for all $n \in \mathbb{N}$, we have

$$d(x_{n+2}, x_n) \leq \alpha^n\beta. \tag{2.25}$$

Now, we will prove, $\{x_n\}$ is a Cauchy sequence in X, we prove that $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 = \lim_{n\to\infty} d(x_{n+m}, x_n)$, for all $n, m \in \mathbb{N}$. At first, we prove that $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$. For this purpose, consider the following cases:

Case (i): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be even or odd. Using inequalities (2.22), (2.24) and rectangular inequality, we get

$$\begin{aligned} d(x_n, x_{n+2i}) &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i})] \\ &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i})] \\ &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\ &+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6}) +] + \cdots \\ &+ k^{i-1}[d(x_{n+2i-4}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-2})] + k^{i-1}[d(x_{n+2i-2}, x_{n+2i})] \\ &\leq k[\alpha^n\eta + \alpha^{n+1}\eta] + k^2[\alpha^{n+2}\eta + \alpha^{n+3}\eta] \\ &+ k^3[\alpha^{n+4}\eta + \alpha^{n+5}\eta] + \cdots + k^{i-1}[\alpha^{n+2i-4}\eta + \alpha^{n+2i-3}\eta] \\ &+ k^{i-1}\alpha^{n+2i-2}\beta \\ &\leq k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots]\eta + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots]\eta \\ &+ k^{i-1}\alpha^{n-3+2i}\beta \\ &\leq \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big]k\alpha^{n-1}\eta + k^{i-1}\alpha^{n-3+2i}\beta \\ &\leq \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big]k\alpha^{n-1}\eta + \alpha^{n-3}\beta. \end{aligned}$$

Letting $n \to \infty$ in last inequality above, we get $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$, for all even $m \in \mathbb{N}$.

Case (ii): m is odd i.e. $m = 2i - 1$ for some $i \in \mathbb{N}$ and n may be even or odd. Using inequality

(2.22) and rectangular inequality, we get

$$\begin{aligned}
d(x_n, x_{n+2i-1}) &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2i-1})] \\
&\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2i-1})] \\
&\leq k[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + k^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
&+ k^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \cdots \\
&+ k^i[d(x_{n+2i}, x_{n+2i-1})] \\
&\leq k[\alpha^n\eta + \alpha^{n+1}\eta] + k^2[\alpha^{n+2}\eta + \alpha^{n+3}\eta] \\
&+ k^3[\alpha^{n+4}\eta + \alpha^{n+5}\eta] + \cdots + k^i\alpha^{n+2i}\eta \\
&\leq k\alpha^n[1 + k\alpha^2 + k^2\alpha^4 + \cdots]\eta + k\alpha^{n+1}[1 + k\alpha^2 + k^2\alpha^4 + \cdots]\eta \\
&\leq \Big[\frac{(1+\alpha)}{1-k\alpha^2}\Big]k\alpha^{n-1}\eta.
\end{aligned}$$

Taking limit as $n \to \infty$ in last inequality above, we get $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$, for all odd $m \in \mathbb{N}$. Thus from case (i) and case (ii), it follows that, for all $m, n \in \mathbb{N}$,

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0. \tag{2.26}$$

we prove that $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$ for all $m, n \in \mathbb{N}$. So, we consider two cases:

Case (a): Suppose m is even i.e. $m = 2i$ for some $i \in \mathbb{N}$ and n may be odd or even. Using inequalities (2.23), (2.25) and rectangular inequality, we get

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq k^{i-2}d(x_{n+2i}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq k^{i-2}\alpha^{n+2i-2}\beta + k^{i-2}[\alpha^{n+2i-3}\eta + \alpha^{n+2i-4}\eta] \\
&+ k^{i-3}[\alpha^{n+2i-4}\eta + \alpha^{n+2i-5}\eta] + \cdots \\
&+ k[\alpha^{n+1}\eta + \alpha^n\eta] \\
&= (k\alpha)^{2i-2}\alpha^n\beta + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\}\eta \\
&\leq (k\alpha)^{2i-2}\alpha^n\beta + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\}\eta \\
&= (k\alpha)^{2i-2}\alpha^n\beta + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\}\eta \\
&= (k\alpha)^{2i-2}\alpha^n\beta + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\}\eta \\
&= (k\alpha)^{2i-2}\alpha^n\beta + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1}\eta.
\end{aligned}$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$.

Case (b): m is odd i.e. $m = 2i - 1$ for some $i \in \mathbb{N}$ and n may be odd or even. Using inequality

(2.23) and rectangular inequality, we get

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq k^{i-2} d(x_{n+2i-1}, x_{n+2i-2}) + k^{i-2}[d(x_{n+2i-2}, x_{n+2i-3}) + d(x_{n+2i-3}, x_{n+2i-4})] \\
&+ k^{i-3}[d(x_{n+2i-4}, x_{n+2i-5}) + d(x_{n+2i-5}, x_{n+2i-6})] + \cdots + k[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq k^{i-2}\alpha^{n+2i-2}\eta + k^{i-2}[\alpha^{n+2i-3}\eta + \alpha^{n+2i-4}\eta] \\
&+ k^{i-3}[\alpha^{n+2i-4}\eta + \alpha^{n+2i-5}\eta] + \cdots \\
&+ k[\alpha^{n+1}\eta + \alpha^n\eta] \\
&= (k\alpha)^{2i-2}\alpha^n\eta + \Big\{(k\alpha)^{i-2}\alpha^{n+i-1} + (k\alpha)^{i-2}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-2} + (k\alpha)^{i-3}\alpha^{n+i-3} \\
&+ \cdots + (k\alpha)\alpha^n + (k\alpha)\alpha^{n-1}\Big\}\eta \\
&\leq (k\alpha)^{2i-2}\alpha^n\eta + \Big\{\alpha^{n+i-1} + \alpha^{n+i-2} + \alpha^{n+i-2} + \alpha^{n+i-3} + \cdots \\
&+ \alpha^n + \alpha^{n-1}\Big\}\eta \\
&= (k\alpha)^{2i-2}\alpha^n\eta + \Big\{\alpha^n[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1] + \alpha^{n-1}[\alpha^{i-1} + \alpha^{i-2} + \cdots + 1]\Big\}\eta \\
&= (k\alpha)^{2i-2}\alpha^n\eta + \Big\{\frac{\alpha^n}{1-\alpha} + \frac{\alpha^{n-1}}{1-\alpha}\Big\}\eta \\
&= (k\alpha)^{2i-2}\alpha^n\eta + \Big\{\frac{1+\alpha}{1-\alpha}\Big\}\alpha^{n-1}\eta.
\end{aligned}$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d(x_{n+m}, x_n) = 0$. Thus, from case(a) and case(b), it follows that, for all $m, n \in \mathbb{N}$,

$$\lim_{n\to\infty} d(x_{n+m}, x_n) = 0. \tag{2.27}$$

It shows that $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete dislocated quasi rectangular b-metric space, there exists some $u \in X$ such that $x_n \to u$. In view of proposition 1.16, subsequences $\{x_{2n}\}$ in A and $\{x_{2n-1}\}$ in B also converge to $u \in X$. As A and B are closed subsets of X, $u \in A \cap B$.Since (X, d) is a complete dislocated quasi rectangular b-metric space, there exists some $u \in X$ such that $x_n \to u$. Now, we show that u is fixed point of T. For any given $n \in \mathbb{N}$, we can write

$$\begin{aligned}
d(u, Tu) &\leq k[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] \\
&= k[d(u, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tu)] \\
&\leq k\Big\{d(u, x_n) + d(x_n, x_{n+1}) + \alpha Q(x_n, u) - \phi(Q(x_n, u))\Big\}.
\end{aligned}$$

Letting $n \to \infty$, using fact that $x_n \to u$, and inequalities (2.22), (2.23), we get,

$$d(u, Tu) \leq k\alpha d(u, Tu),$$

which is a contradiction unless $d(u, Tu) = 0$. Also,

$$\begin{aligned}
d(Tu, u) &\leq k[d(Tu, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, u)] \\
&= k[d(Tu, Tx_n) + d(x_{n+1}, x_n) + d(x_n, u)] \\
&\leq k\Big\{\alpha Q(u, x_n) - \phi(Q(u, x_n)) + d(x_{n+1}, x_n) + d(x_n, u)\Big\}.
\end{aligned}$$

Letting $n \to \infty$, utilizing fact that $x_n \to u$, and inequalities (2.22), (2.23), we get,

$$d(Tu, u) \leq k\alpha d(Tu, u).$$

This is a contradiction unless $d(Tu, u) = 0$. Hence, we get $d(u, Tu) = 0 = d(Tu, u)$. It yields that $Tu = u$. That is, u is fixed point of T in X.

Note that,

$$\begin{aligned} d(u, u) = d(Tu, Tu) &\leq \alpha Q(u, u) - \phi(Q(u, u)) \\ &\leq \alpha Q(u, u) \\ &\leq \alpha d(u, u) < d(u, u), \end{aligned}$$

which is a contradiction unless $d(u, u) = 0$. Thus,in general, if v is fixed point of T then, $d(v, v) = 0$. Now, we prove that u is unique fixed point of T in X. Suppose, u' is another fixed point of T in X such that $d(u, u') \neq 0 \neq d(u', u)$. Now, in view of (2.17), we have

$$\begin{aligned} d(u, u') = d(Tu, Tu') &\leq \alpha Q(u, u') - \phi(Q(u, u')) \\ &\leq \alpha Q(u, u') \\ &\leq \alpha d(u, u'). \end{aligned}$$

This yields a contradiction, $\alpha \geq 1$, unless $d(u, u') = 0$. Also,

$$\begin{aligned} d(u', u) = d(Tu', Tu) &\leq \alpha Q(u', u) - \phi(Q(u', u)) \\ &\leq \alpha Q(u', u) \\ &\leq \alpha d(u', u) < d(u', u). \end{aligned}$$

This too gives same contradiction as in previous case unless $d(u', u) = 0$. Hence we must have, $d(u, u') = 0 = d(u', u)$ i.e. $u = u'$. So u is a unique fixed point of T in X. □

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