Variation of the first eigenvalue of \((p, q)\)-Laplacian along the Ricci-harmonic flow

Shahroud Azami

Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran.

(Communicated by Hamid Khodaei)

Abstract

In this paper, we study monotonicity for the first eigenvalue of a class of \((p, q)\)-Laplacian. We find the first variation formula for the first eigenvalue of \((p, q)\)-Laplacian on a closed Riemannian manifold evolving by the Ricci-harmonic flow and construct various monotonic quantities by imposing some conditions on initial manifold.

Keywords: Laplace, Ricci flow, Harmonic map

2010 MSC: Primary 58C40; Secondary 53C43.

1. Introduction

The study on eigenvalue problem has received remarkable attention. Recently, many mathematicians considered the eigenvalue problem of geometric operators under various geometric flows, because it is a very powerful tool for the understanding Riemannian manifold. The fundamental study of this works began when Perelman [10] showed that the functional

\[ F = \int_M (R + |\nabla f|^2) e^{-f} \, d\mu \]

is nondecreasing along the Ricci flow coupled to a backward heat-type equation, where \(R\) is the scalar curvature with respect to the metric \(g(t)\) and \(d\mu\) denotes the volume form of the metric \(g(t)\). The nondecreasing of the functional \(F\) implies that the first eigenvalue of the geometric operator \(-4\Delta + R\) is nondecreasing under the Ricci flow. Then, Li [7] and Zeng et al [12] extended the

*Corresponding author

Email address: azami@sci.ikiu.ac.ir, shahrood78@yahoo.com (Shahroud Azami)

Received: July 2019    Accepted: January 2020
geometric operator $-4\Delta + R$ to the operator $-\Delta + cR$ and studied the monotonicity of eigenvalues of the operator $-\Delta + cR$ along Ricci flow and the Ricci-Bourguignon flow, respectively.

Also, in [1, 11, 13] has been investigated the evolution for the first eigenvalue of $p$-Laplacian along the Ricci-harmonic flow, Ricci flow and $m$th mean curvature flow, respectively. A generalization of $p$-Laplacian is a class of $(p, q)$-Laplacian which has applications in physics and related sciences such as non-Newtonian fluids, pseudoplastics [4, 5] that we introduce it in later section.

On the other hand, geometric flows for instance, Ricci-harmonic flow have been a topic of active research interest in mathematics and physics. A geometric flow is an evolution of a geometric structure. Let $M$ be a closed $m$-dimensional Riemannian manifold with a Riemannian metric $g_0$. Hamilton for the first time in 1982 introduced the Ricci flow as follows

$$\frac{\partial g(t)}{\partial t} = -2\text{Ric}(g(t)), \quad g(0) = g_0,$$

where $\text{Ric}$ is the Ricci tensor of $g(t)$. The Ricci flow has been proved to be a very useful tool to improve metrics in Riemannian geometry, when $M$ is compact. Now, let $(M^m, g)$ and $(N^n, \gamma)$ be closed Riemannian manifolds. By Nash’s embedding theorem, assume that $N$ is isometrically embedded into Euclidean space $e_N : (N^n, \gamma) \hookrightarrow \mathbb{R}^d$ for a sufficiently large $d$. We identify map $\phi : M \rightarrow N$ with $e_N \circ \phi : M \rightarrow \mathbb{R}^d$. Müller [9] considered a generalization of Ricci flow as

$$\begin{cases}
\frac{\partial g(t)}{\partial t} = -2\text{Ric}(g(t)) + 2\eta \nabla \phi \otimes \nabla \phi, \quad g(0) = g_0, \\
\frac{\partial \phi}{\partial t} = \tau_g \phi, \quad \phi(0) = \phi_0,
\end{cases}
$$

(1.1)

where $\eta$ is a positive coupling constant, $\phi(t)$ is a family of smooth maps from $M$ to some closed target manifold $N$ and $\tau_g \phi$ is the intrinsic Laplacian of $\phi$ which denotes the tension field of $\phi$ with respect to the evolving metric $g(t)$. This evolution equation system called Ricci flow coupled with harmonic map flow or $(RH)_\eta$ flow for short. Müller in [9] shown that system (1.1) has unique solution with initial data $(g(0), \phi(0)) = (g_0, \phi_0)$. Also, the normalized $(RH)_\eta$ flow defined as

$$\begin{cases}
\frac{\partial g(t)}{\partial t} = -2\text{Ric}(g(t)) + 2\eta \nabla \phi \otimes \nabla \phi + \frac{2}{m} rg(t), \quad g(0) = g_0, \\
\frac{\partial \phi}{\partial t} = \tau_g \phi, \quad \phi(0) = \phi_0,
\end{cases}
$$

(1.2)

where $r = \frac{\int_M (R - \eta |\nabla \phi|^2) d\mu}{\int_M \eta d\mu}$ is the average of $R - \eta |\nabla \phi|^2$. Under this normalized flow, the volume of the solution metrics remains constant in time.

2. Preliminaries

2.1. Eigenvalues of $p$-Laplacian

Let $(M, g)$ be a closed Riemannian manifold and $f : M \rightarrow \mathbb{R}$ be a smooth function on $M$ or $f \in W^{1, p}(M)$. The Laplace-Beltrami operator acting on a smooth function $f$ on $M$ is the divergence of gradient of $f$, written as

$$\Delta f = \text{div}(\text{grad } f) = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} \partial_j f),$$

where $\partial_i f = \frac{\partial f}{\partial x^i}$. The $p$-Laplacian of $f$ for $1 < p < \infty$ is defined as

$$\begin{align*}
\Delta_p f &= \text{div}(|\nabla f|^{p-2} \nabla f) \\
&= |\nabla f|^{p-2} \Delta f + (p - 2)|\nabla f|^{p-4}(\text{Hess } f)(\nabla f, \nabla f),
\end{align*}
$$

(2.1)
where \((Hessf)(X,Y) = \nabla(\nabla f)(X,Y) = X.(Y.f) - (\nabla_X Y).f, \ X, Y \in \mathcal{X}(M)\) and in local coordinate, we get
\[
(Hessf)(\partial_i, \partial_j) = \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f.
\]
Notice that when \(p = 2, p\)-Laplacian is the Laplace-Beltrami operator. Let \((M^n, g)\) be a closed Riemannian manifold. In this paper, we consider the nonlinear system introduced in [6], that is
\[
\begin{cases}
\Delta_p u = -\lambda |u|^{\alpha} |v|^\beta v \text{ in } M \\
\Delta_q v = -\lambda |u|^{\alpha} |v|^\beta u \text{ in } M \\
(u, v) \in W^{1,p}(M) \times W^{1,q}(M)
\end{cases}
\tag{2.2}
\]
where \(p > 1, q > 1\) and \(\alpha, \beta\) are real numbers satisfying
\[
\alpha > 0, \beta > 0, \frac{\alpha + 1}{p} + \frac{\beta + 1}{q} = 1.
\tag{2.3}
\]
In (2.2), we say that \(\lambda\) is an eigenvalue whenever for some \(u \in W_0^{1,p}(M)\) and \(v \in W_0^{1,q}(M)\),
\[
\int_M |\nabla u|^{p-2} < \nabla u, \nabla \phi > d\mu = \lambda \int_M |u|^{\alpha} |v|^\beta v \phi d\mu, \tag{2.4}
\]
\[
\int_M |\nabla v|^{q-2} < \nabla v, \nabla \psi > d\mu = \lambda \int_M |u|^{\alpha} |v|^\beta u \psi d\mu, \tag{2.5}
\]
where \(\phi \in W^{1,p}(M), \psi \in W^{1,q}(M)\) and \(W_0^{1,p}(M)\) is the closure of \(C_0^\infty(M)\) in Sobolev space \(W^{1,p}(M)\). The pair \((u, v)\) is called an eigenfunction corresponding to eigenvalue \(\lambda\). A first positive eigenvalue of (2.2) obtained as
\[
in\{A(u, v) : (u, v) \in W_0^{1,p}(M) \times W_0^{1,q}(M), \ B(u, v) = 1\}
\]
where
\[
A(u, v) = \frac{\alpha + 1}{p} \int_M |\nabla u|^p d\mu + \frac{\beta + 1}{q} \int_M |\nabla v|^q d\mu,
\]
\[
B(u, v) = \int_M |u|^{\alpha} |v|^\beta u v d\mu.
\]
Let \((M^m, g(t), \phi(t))\) be a solution of the \((RH)\eta\) flow (1.1) on the smooth manifold \((M^m, g_0, \phi_0)\) in the interval \([0, T)\) then
\[
\lambda(t) = \frac{\alpha + 1}{p} \int_M |\nabla u|^p d\mu_t + \frac{\beta + 1}{q} \int_M |\nabla v|^q d\mu_t, \tag{2.6}
\]
defines the evolution of an eigenvalue of (2.2), under the variation of \((g(t), \phi(t))\) where the eigenfunction associated to \(\lambda(t)\) is normalized that is \(B(u, v) = 1\). Motivated by the above works, in this paper we will study the first eigenvalue of a class of \((p, q)\)-Laplacian (2.2) whose metric satisfies the \((RH)\eta\) flow. Throughout of paper we write \(\frac{\partial}{\partial t} = \partial_t u = u', S = Ric_g - \eta \nabla \phi \otimes \nabla \phi, S_{ij} = Ric_{ij} - \eta \nabla_i \phi \nabla_j \phi\) and \(S = R - \eta |\nabla \phi|^2\).
3. Variation of $\lambda(t)$

In this section, we will give some useful evolution formulas for $\lambda(t)$ under the Ricci-harmonic flow. Now, we give a useful statement about the variation of the first eigenvalue of (2.2) under the $(RH)_\eta$ flow.

**Lemma 3.1.** If $g_1$ and $g_2$ are two metrics on Riemannian manifold $M^m$ which satisfy $(1 + \epsilon)^{-1}g_1 < g_2 < (1 + \epsilon)g_1$ then for any $p \geq q > 1$, we have

$$\lambda(g_2) - \lambda(g_1) \leq \left(1 + \epsilon\right)^{\frac{p+m}{2}} - \left(1 + \epsilon\right)^{-\frac{m}{2}} \lambda(g_1)$$

in particular, $\lambda(t)$ is a continues function respect to $t$-variable.

**Proof.** By direct computation we complete the proof of lemma. In local coordinate we have $d\mu = \sqrt{\det g} \, dx^1 \wedge \ldots \wedge dx^m$, therefore

$$(1 + \epsilon)^{-\frac{m}{2}} d\mu_{g_1} < d\mu_{g_2} < (1 + \epsilon)^{\frac{m}{2}} d\mu_{g_1}.$$ 

Let

$$G(g, u, v) = \frac{\alpha + 1}{p} \int_M |\nabla u|^p_g d\mu_g + \frac{\beta + 1}{q} \int_M |\nabla v|^q_g d\mu_g.$$ 

(3.1)

then

$$\int_M |u|^\alpha |v|^\beta uv d\mu_{g_2} G(g_2, u, v) - \int_M |u|^\alpha |v|^\beta uv d\mu_{g_2} G(g_1, u, v)$$

$$= \frac{\alpha + 1}{p} \left( \int_M |u|^\alpha |v|^\beta uv d\mu_{g_1} \left( \int_M |\nabla u|^p_{g_2} d\mu_{g_2} - \int_M |\nabla u|^p_{g_1} d\mu_{g_1} \right) \right)$$

$$+ \frac{\alpha + 1}{p} \left( \int_M |u|^\alpha |v|^\beta uv d\mu_{g_2} \left( \int_M |\nabla u|^p_{g_2} d\mu_{g_2} - \int_M |\nabla u|^p_{g_1} d\mu_{g_1} \right) \right)$$

$$+ \frac{\beta + 1}{q} \left( \int_M |u|^\alpha |v|^\beta uv d\mu_{g_1} \left( \int_M |\nabla v|^q_{g_2} d\mu_{g_2} - \int_M |\nabla v|^q_{g_1} d\mu_{g_1} \right) \right)$$

$$+ \frac{\beta + 1}{q} \left( \int_M |u|^\alpha |v|^\beta uv d\mu_{g_1} \left( \int_M |\nabla v|^q_{g_2} d\mu_{g_2} - \int_M |\nabla v|^q_{g_1} d\mu_{g_1} \right) \right)$$

$$\leq \left( \frac{\alpha + 1}{p} \left( (1 + \epsilon)^{\frac{p+m}{2}} - (1 + \epsilon)^{-\frac{m}{2}} \right) \int_M |u|^\alpha |v|^\beta uv d\mu_{g_1} \int_M |\nabla u|^p_{g_1} d\mu_{g_1} \right)$$

$$+ \left( \frac{\beta + 1}{q} \left( (1 + \epsilon)^{\frac{p+m}{2}} - (1 + \epsilon)^{-\frac{m}{2}} \right) \int_M |u|^\alpha |v|^\beta uv d\mu_{g_1} \int_M |\nabla v|^q_{g_1} d\mu_{g_1} \right)$$

$$\leq \left( (1 + \epsilon)^{\frac{p+m}{2}} - (1 + \epsilon)^{-\frac{m}{2}} \right) \lambda(g_1)$$

Since the eigenfunction corresponding to $\lambda(t)$ are normalized, thus we get

$$\lambda(g_2) - \lambda(g_1) \leq \left( (1 + \epsilon)^{\frac{p+m}{2}} - (1 + \epsilon)^{-\frac{m}{2}} \right) \lambda(g_1)$$

this completes the proof of Lemma. □
Proposition 3.2. Let \((g(t), \phi(t)), \ t \in [0, T)\), be a solution of the \((RH)_\eta\) flow on a closed manifold \(M^m\) and let \(\lambda(t)\) be the first eigenvalue of the \((p,q)\)-Laplacian along this flow. Then for any \(t_0, t_1 \in [0, T)\) and \(t_1 > t_0\), we have

\[
\lambda(t_1) \geq \lambda(t_0) + \int_{t_0}^{t_1} \mathcal{G}(g(\tau), u(\tau), v(\tau))d\tau
\]

where

\[
\mathcal{G}(g(t), u(t), v(t)) = (\alpha + 1) \int_M (\mathcal{S}(\nabla u, \nabla u) + \langle \nabla u', \nabla u \rangle |\nabla u|^{-2}d\mu
\]

\[
+ (\beta + 1) \int_M (\mathcal{S}(\nabla v, \nabla v) + \langle \nabla v', \nabla v \rangle |\nabla v|^{p-2}d\mu
\]

\[
- \frac{\alpha + 1}{p} \int_M |\nabla u|^pSd\mu - \frac{\beta + 1}{q} \int_M |\nabla v|^qSd\mu.
\]

Proof. Assume that

\[
G(g(t), u(t), v(t)) = \frac{\alpha + 1}{p} \int_M |\nabla u(t)|^p_{g(t)}d\mu_{g(t)} + \frac{\beta + 1}{q} \int_M |\nabla v(t)|^q_{g(t)}d\mu_{g(t)},
\]

at time \(t_1\) we first let \((u_1, v_1) = (u(t_1), v(t_1))\) be the eigenfunction for the eigenvalue \(\lambda(t_1)\) of \((p,q)\)-Laplacian. We consider the following smooth functions

\[
h(t) = u_1 \left[ \det [g_{ij}(t_1)] \right]^{\frac{1}{p-1+\beta}} \left[ \det [g_{ij}(t)] \right]^{\frac{1}{q-1+\alpha}}, \quad l(t) = v_1 \left[ \det [g_{ij}(t_1)] \right]^{\frac{1}{q-1+\alpha}} \left[ \det [g_{ij}(t)] \right]^{\frac{1}{p-1+\beta}},
\]

along the \((RH)_\eta\) flow. We define

\[
u(t) = \frac{h(t)}{\left( \int_M |h(t)|^\alpha |l(t)|^{\beta}h(t)l(t)d\mu \right)^{\frac{1}{\beta}}}, \quad u(t) = \frac{l(t)}{\left( \int_M |h(t)|^\alpha |l(t)|^{\beta}h(t)l(t)d\mu \right)^{\frac{1}{\beta}}}
\]

which \(u(t), v(t)\) are smooth functions under the \((RH)_\eta\) flow, satisfy

\[
\int_M |\nabla|^\alpha |v|^\beta uv d\mu = 1,
\]

and at time \(t_1\), \((u(t_1), v(t_1))\) are the eigenfunctions for \(\lambda(t_1)\) of \((p,q)\)-Laplacian at time \(t_1\) i.e. \(\lambda(t_1) = G(g(t_1), u(t_1), v(t_1))\). If \(f\) is a smooth function respect to time \(t\) then along the \((RH)_\eta\) flow we have

\[
\frac{d}{dt} |\nabla f|^p = \frac{p}{2} \left[ \partial_k g^{ij} \nabla_i f \nabla_j f + 2 g^{ij} \nabla_i f' \nabla_j f \right] |\nabla f|^{p-2}
\]

by (1.1) we have \(\partial_k g^{ij} = 2 g^{ik} g^{jl} S_{kl}\), therefore

\[
\frac{d}{dt} |\nabla f|^p = p |\nabla f|^{p-2} \left( \mathcal{S}(\nabla f, \nabla f) + \langle \nabla f', \nabla f \rangle \right),
\]

and

\[
\partial_t d\mu = \frac{1}{2} tr_g (\partial_t g) d\mu = -S d\mu.
\]
Since $u(t)$ and $v(t)$ are smooth functions, hence $G(g(t), u(t), v(t))$ is a smooth function with respect to $t$. If we set
\[
\mathcal{G}(g(t), u(t), v(t)) := \frac{d}{dt} G(g(t), u(t), v(t)),
\]
then
\[
\mathcal{G}(g(t), u(t), v(t)) = (\alpha + 1) \int_M (S(\nabla u, \nabla v) + \nabla u', \nabla u >) |\nabla u|^p d\mu
\]
\[
+ (\beta + 1) \int_M (S(\nabla v, \nabla v') + \nabla v', \nabla v >) |\nabla v|^q d\mu
\]
\[
- \frac{\alpha + 1}{p} \int_M |\nabla u|^p Sd\mu - \frac{\beta + 1}{q} \int_M |\nabla v|^q Sd\mu.
\]
Taking integration on the both sides of (3.5) between $t_0$ and $t_1$, we conclude that
\[
G(g(t_1), u(t_1), v(t_1)) - G(g(t_0), u(t_0), v(t_0)) = \int_{t_0}^{t_1} \mathcal{G}(g(\tau), u(\tau), v(\tau)) d\tau
\]
where $t_0 \in [0, T)$ and $t_1 > t_0$. Noticing $G(g(t_0), u(t_0), v(t_0)) \geq \lambda(t_0)$ and plugin $\lambda(t_1) = G(g(t_1), u(t_1), v(t_1))$ in (3.7), yields (3.2) and $\mathcal{G}(g(t), u(t), v(t))$ satisfies in (3.3). \(\square\)

**Theorem 3.3.** Let $(M^m, g(t), \phi(t))$ be a solution of the $(RH)_n$ flow on the smooth closed manifold $(M^m, g_0, \phi_0)$ and $\lambda(t)$ denotes the evolution of the first eigenvalue under the $(RH)_n$ flow. Suppose that $k = \min\{p, q\}$ and
\[
S - \frac{1}{k} Sg \geq 0 \text{ in } M^m \times [0, T).
\]
If $S_{\min}(0) \geq 0$, then $\lambda(t)$ is nondecreasing and differentiable almost everywhere along the $(RH)_n$ flow on $[0, T)$.

**Proof.** Let for any $t_1 \in [0, T)$, $u(t_1), v(t_1)$ be the eigenfunctions for $\lambda(t_1)$ of $(p, q)$-Laplacian. Then
\[
\int_M |u(t_1)|^a |v(t_1)|^b u(t_1)v(t_1) d\mu = 1 \text{ and }
\]
\[
\mathcal{G}(g(t_1), u(t_1), v(t_1)) = (\alpha + 1) \int_M (S(\nabla u, \nabla v) + \nabla u', \nabla u >) |\nabla u|^p d\mu
\]
\[
+ (\beta + 1) \int_M (S(\nabla v, \nabla v') + \nabla v', \nabla v >) |\nabla v|^q d\mu
\]
\[
- \frac{\alpha + 1}{p} \int_M |\nabla u|^p Sd\mu - \frac{\beta + 1}{q} \int_M |\nabla v|^q Sd\mu.
\]
Now, by the time derivative of the condition
\[
\int_M |u|^a |v|^b uvd\mu = 1
\]
we can get
\[
(\alpha + 1) \int_M |u|^a |v|^b u'vd\mu + (\beta + 1) \int_M |u|^a |v|^b uv'd\mu = \int_M S |u|^a |v|^b uvd\mu.
\]
On the other hand, (2.4) and (2.5) imply that
\[ \int_M <\nabla u', \nabla u > |\nabla u|^{p-2} d\mu = \lambda(t_1) \int_M |u|^\alpha|v|^\beta u'v d\mu, \] (3.11)
\[ \int_M <\nabla v', \nabla v > |\nabla v|^{q-2} d\mu = \lambda(t_1) \int_M |u|^\alpha|v|^\beta uv' d\mu. \] (3.12)
Therefore from (3.10), (3.11) and (3.12) we have
\[ (\alpha + 1) \int_M <\nabla u', \nabla u > |\nabla u|^{p-2} d\mu + (\beta + 1) \int_M <\nabla v', \nabla v > |\nabla v|^{q-2} d\mu = \lambda(t_1) \int_M S|u|^\alpha|v|^\beta uv d\mu, \] (3.13)
and the replacing (3.13) in (3.9), results that
\[ \mathcal{G}(g(t_1), u(t_1), v(t_1)) = \lambda(t_1) \int_M S|u|^\alpha|v|^\beta uv d\mu + (\alpha + 1) \int_M S(\nabla u, \nabla u)|\nabla u|^{p-2} d\mu - \frac{\alpha + 1}{p} \int_M |\nabla u|^p S d\mu + (\beta + 1) \int_M S(\nabla v, \nabla v)|\nabla v|^{q-2} d\mu - \frac{\beta + 1}{q} \int_M |\nabla v|^q S d\mu. \] (3.14)
From (3.14) and (3.8) we have
\[ \mathcal{G}(g(t_1), u(t_1), v(t_1)) \geq \lambda(t_1) \int_M S|u|^\alpha|v|^\beta uv d\mu + (\alpha + 1)(\frac{1}{k} - \frac{1}{p}) \int_M |\nabla u|^p S d\mu + (\beta + 1)(\frac{1}{k} - \frac{1}{q}) \int_M |\nabla v|^q S d\mu. \] (3.15)
Since
\[ \frac{\partial}{\partial t} S = \Delta S + 2|S_{ij}|^2 + 2\eta|\tau_\eta \phi|^2 \]
and \(|S_{ij}|^2 \geq \frac{1}{m}S^2\), it follows that
\[ \frac{\partial}{\partial t} S \geq \Delta S + \frac{2}{m}S^2. \] (3.16)
The solution to
\[ \frac{d}{dt} y(t) = \frac{2}{m}y^2(t), \quad y(t) = S_{\min}(0), \]
is
\[ y(t) = \frac{S_{\min}(0)}{1 - \frac{2}{m}S_{\min}(0)t}, \quad t \in [0, T'], \] (3.17)
where \(T' = \min\{T, \frac{m}{2S_{\min}(0)}\}\). Using maximum principle to (3.16), we get \(S \geq y(t)\) along the \((RH)_q\) flow. If \(S_{\min}(0) \geq 0\) then the nonnegativity of \(S\) preserved along the \((RH)_q\) flow. Therefore (3.15) becomes \(\mathcal{G}(g(t_1), u(t_1), v(t_1)) \geq 0\). Thus we get \(\mathcal{G}(g(t), u(t), v(t)) > 0\) in any small enough neighborhood of \(t_1\). Hence \(\int_{t_0}^{t_1} \mathcal{G}(g(\tau), u(\tau), v(\tau)) d\tau > 0\) for any \(t_0 < t_1\) sufficiently close to \(t_1\). Since \(t_1 \in [0, T)\) is arbitrary, the Proposition \(3.2\) completes the proof of the first part of theorem. For the differentiability for \(\lambda(t)\), since \(\lambda(t)\) is increasing and continues on the interval \([0, T)\), the classical Lebesgue’s theorem (see [8]), \(\lambda(t)\) is differentiable almost everywhere on \([0, T)\). □ Motivated by the works of X.-D. Cao [2, 3] and J. Y. Wu [11], similar to proof of Proposition \(3.2\) we first introduce a
new smooth eigenvalue function along the \((RH)_\eta\) flow and then we give evolution formula for it. Let \(M\) be an \(m\)-dimensional closed Riemannian manifold and \(g(t)\) be a smooth solution of the \((RH)_\eta\) flow. Suppose that
\[
\lambda(u, v, t) := \frac{\alpha + 1}{p} \int_M |\nabla u|^p d\mu + \frac{\beta + 1}{q} \int_M |\nabla v|^q d\mu
\]
where \(u, v\) are smooth functions and satisfy
\[
\int_M |u|^\alpha |v|^\beta uv d\mu = 1, \quad \int_M |u|^\alpha |v|^\beta v d\mu = 0, \quad \int_M |u|^\alpha |v|^\beta u d\mu = 0.
\]
The function \(\lambda(u, v, t)\) is a smooth eigenvalue function respect to \(t\)-variable. If \((u, v)\) are the corresponding eigenfunctions of the first eigenvalue \(\lambda(t_1)\) then \(\lambda(u, v, t_1) = \lambda(t_1)\). As proof of Proposition 3.2 and Theorem 3.3 we have the following propositions.

**Proposition 3.4.** Let \((M^m, g(t), \phi(t))\) be a solution of the \((RH)_\eta\) flow on the smooth closed manifold \((M^m, g_0, \phi_0)\). If \(\lambda(t)\) denotes the evolution of the first eigenvalue under the \((RH)_\eta\) flow, then
\[
\frac{d\lambda}{dt}(u, v, t)|_{t=t_1} = \lambda(t_1) \int_M S|u|^{\alpha}|v|^{\beta} uv d\mu + (\alpha + 1) \int_M S(\nabla u, \nabla u)|\nabla u|^{p-2} d\mu
\]
\[- \frac{\alpha + 1}{p} \int_M |\nabla u|^p S d\mu + (\beta + 1) \int_M S(\nabla v, \nabla v)|\nabla v|^{q-2} d\mu\]
\[- \frac{\beta + 1}{q} \int_M |\nabla v|^q S d\mu,
\]
where \((u, v)\) is the associated normalized evolving eigenfunction.

Now, we give a variation of \(\lambda(t)\) under the normalized \((RH)_\eta\) flow which is similar to the previous Proposition.

**Proposition 3.5.** Let \((M^m, g(t), \phi(t))\) be a solution of the normalized \((RH)_\eta\) flow on the smooth closed manifold \((M^m, g_0, \phi_0)\). If \(\lambda(t)\) denotes the evolution of the first eigenvalue under the normalized \((RH)_\eta\) flow, then
\[
\frac{d\lambda}{dt}(u, v, t)|_{t=t_1} = \lambda(t_1) \int_M S|u|^{\alpha}|v|^{\beta} uv d\mu + (\alpha + 1) \int_M S(\nabla u, \nabla u)|\nabla u|^{p-2} d\mu
\]
\[+ (\beta + 1) \int_M S(\nabla v, \nabla v)|\nabla v|^{q-2} d\mu - \frac{\beta + 1}{q} \int_M |\nabla v|^q S d\mu\]
\[- \frac{\alpha + 1}{p} \int_M |\nabla u|^p S d\mu - \frac{\alpha + 1}{m} r(t_1) \int_M |\nabla u|^\alpha d\mu
\]
\[- \frac{\beta + 1}{m} r(t_1) \int_M |\nabla v|^\beta d\mu,
\]
where \((u, v)\) is the associated normalized evolving eigenfunction.

**Proof.** In the normalized case, derivative of the integrability condition \(\int_M |u|^{\alpha}|v|^{\beta} uv d\mu = 1\) respect to \(t\), results that
\[
(\alpha + 1) \int_M |u|^{\alpha}|v|^{\beta} u' v d\mu + (\beta + 1) \int_M |u|^{\alpha}|v|^{\beta} u v' d\mu = -r(t_1) + \int_M S|u|^{\alpha}|v|^{\beta} u v d\mu.
\]

}\]
On the other hand

\[
\frac{d}{dt}(d\mu) = \frac{1}{2} tr_g \left( \frac{\partial g}{\partial t} \right) d\mu = \frac{1}{2} tr_g \left( \frac{2}{m} r g - 2S \right) d\mu = (r - S) d\mu, \tag{3.21}
\]

hence we can then write

\[
\frac{d\lambda}{dt}(u, v, t)|_{t=t_1} = \frac{\alpha + 1}{p} \left( \frac{p}{2} \left\{ -\frac{2}{m} r |\nabla u|^2 + 2S(\nabla u, \nabla u) + 2 < \nabla u', \nabla u > \right\} |\nabla u|^{p-2} d\mu \right) + \frac{\beta + 1}{q} \left( \frac{q}{2} \left\{ -\frac{2}{m} r |\nabla v|^2 + 2S(\nabla v, \nabla v) + 2 < \nabla v', \nabla v > \right\} |\nabla v|^{q-2} d\mu \right) + \frac{\alpha + 1}{p} \int_M |\nabla u|^p (r - S) d\mu + \frac{\beta + 1}{q} \int_M |\nabla v|^q (r - S) d\mu,
\]

but

\[
(\alpha + 1) \int_M < \nabla u', \nabla u > |\nabla u|^{p-2} d\mu + (\beta + 1) \int_M < \nabla v', \nabla v > |\nabla v|^{q-2} d\mu = -\lambda(t_1) r(t_1) + \lambda(t_1) \int_M S |u|^{\alpha} |v|^\beta uv d\mu. \tag{3.22}
\]

Therefore the proposition is obtained by replacing (3.22) in previous relation. □

**Theorem 3.6.** Let \((M^m, g(t), \phi(t))\) be a solution of the \((RH)_{\eta}\) flow on the smooth closed manifold \((M^m, g_0, \phi_0)\) and \(\lambda(t)\) denotes the evolution of the first eigenvalue under the \((RH)_{\eta}\) flow. If \(k = \min\{p, q\},\)

\[
S - \frac{S}{k} g > 0 \text{ in } M^m \times [0, T) \tag{3.23}
\]

and \(S_{\min}(0) > 0\), then the quantity \(\lambda(t)(1 - \frac{2}{m} S_{\min}(0)t)^\frac{m}{2}\) is nondecreasing along the \((RH)_{\eta}\) flow on \([0, T')\), where \(T' := \min\{\frac{m}{2 S_{\min}(0)}, T\}\).

**Proof.** According to (3.18) and (3.23) we have

\[
\frac{d\lambda}{dt}(u, v, t)|_{t=t_1} > \lambda(t_1) \int_M S |u|^{\alpha} |v|^\beta uv d\mu + (\alpha + 1)(\frac{1}{k} - \frac{1}{p}) \int_M |\nabla u|^p S d\mu + (\beta + 1)(\frac{1}{k} - \frac{1}{q}) \int_M |\nabla v|^q S d\mu. \tag{3.24}
\]

If \(S_{\min}(0) > 0\), then (3.17) results that the positive of \(S\) remains under the \((RH)_{\eta}\) flow, therefore

\[
\frac{d\lambda}{dt}(u, v, t)|_{t=t_1} \geq \lambda(t_1) \frac{S_{\min}(0)}{1 - \frac{2}{m} S_{\min}(0)t_1}. \tag{3.25}
\]

Then in any small enough neighborhood of \(t_1\) as \(I\), we get

\[
\frac{d\lambda}{dt}(u, v, t) \geq \lambda(u, v, t) \frac{S_{\min}(0)}{1 - \frac{2}{m} S_{\min}(0)t}. \tag{3.26}
\]

Integrating the last inequality with respect to \(t\) on \([t_0, t_1] \subset I\), we have

\[
\ln \frac{\lambda(u(t_1), v(t_1), t_1)}{\lambda(u(t_0), v(t_0), t_0)} \geq \ln \left( \frac{1 - \frac{2}{m} S_{\min}(0)t_1}{1 - \frac{2}{m} S_{\min}(0)t_0} \right)^{-\frac{m}{2}}. \tag{3.27}
\]
Since \( \lambda(u(t_1), v(t_1), t_1) = \lambda(t_1) \) and \( \lambda(u(t_0), v(t_0), t_0) \geq \lambda(t_0) \) we conclude that

\[
\ln \frac{\lambda(t_1)}{\lambda(t_0)} \geq \ln \left( \frac{1 - \frac{2}{m} S_{\min}(0) t_1}{1 - \frac{2}{m} S_{\min}(0) t_0} \right)^{-\frac{m}{2}},
\]

that is the quantity \( \lambda(t)(1 - \frac{2}{m} S_{\min}(0) t)^{\frac{m}{2}} \) is nondecreasing in any sufficiently small neighborhood of \( t_1 \). Since \( t_1 \) is arbitrary, hence \( \lambda(t)(1 - \frac{2}{m} S_{\min}(0) t)^{\frac{m}{2}} \) is nondecreasing along the \((RH)_{\eta}\) flow on \([0, T')\).

Now, if in the \((RH)_{\eta}\) flow, we suppose that \( \eta = 0 \), then the \((RH)_{\eta}\) flow reduce to the Ricci flow and we have the following corollary

**Corollary 3.7.** Let \( g(t), \quad t \in [0, T) \) be a solution of the Ricci flow on a closed Riemannian manifold \( M \) and \( \lambda(t) \) denotes the first eigenvalue of the \((p, q)\)-Laplacian \([2.3]\). Suppose that \( k = \min\{p, q\} \) and \( \text{Ric} - \frac{R}{k} g \geq 0 \) along the Ricci flow.

1. If \( R_{\min}(0) \geq 0 \), then \( \lambda(t) \) is nondecreasing along the Ricci flow for any \( t \in [0, T) \).
2. If \( R_{\min}(0) > 0 \), then the quantity \( (1 - R_{\min}(0) t) \lambda(t) \) is nondecreasing along the Ricci flow for any \( t \in [0, T') \) where \( T' = \min\{T, \frac{1}{R_{\min}(0)}\} \).

In dimension two we have

**Proposition 3.8.** Let \((g(t), \phi(t))\), \( t \in [0, T) \) be a solution of the \((RH)_{\eta}\) flow on a closed Riemannian surface \( M \) and \( \lambda(t) \) denotes the first eigenvalue of the \((p, q)\)-Laplacian \([2.2]\).

1. Suppose that \( \text{Ric} \geq \epsilon \nabla \phi \otimes \nabla \phi \) where \( \epsilon \geq 2\eta \frac{k-1}{k} \) and \( 2 \leq k = \min\{p, q\} \).
2. \( R_{\min}(0) \geq 0 \), then \( \lambda(t) \) is nondecreasing along the \((RH)_{\eta}\) flow for any \( t \in [0, T) \).
3. \( R_{\min}(0) > 0 \), then the quantity \( (1 - R_{\min}(0) t) \lambda(t) \) is nondecreasing along the \((RH)_{\eta}\) flow on \([0, T') \) where \( T' = \min\{T, \frac{1}{R_{\min}(0)}\} \).

**Proof.** In the case of surface, we have \( R_{ij} = \frac{R}{2} g_{ij} \). Then

\[
T_{ij} := S_{ij} - \frac{S}{k} g_{ij} = \frac{R}{2} g_{ij} - \eta \nabla_i \phi \nabla_j \phi - \frac{1}{k} (R - \eta |\nabla \phi|^2) g_{ij} = \frac{1}{2} R g_{ij} - \eta \nabla_i \phi \nabla_j \phi + \frac{\alpha}{k} |\nabla \phi|^2 g_{ij}.
\]

For any vector \( V = (V^i) \) we get

\[
T_{ij} V^i V^j = \left( \frac{1}{2} - \frac{1}{k} \right) R |V|^2 - \eta (\nabla_i \phi \nabla_j \phi)^2 + \frac{\eta}{k} |\nabla \phi|^2 |V|^2 \\
\geq \left( \frac{1}{2} - \frac{1}{k} \right) R |V|^2 + \eta \left( \frac{1}{k} - 1 \right) |\nabla \phi|^2 |V|^2.
\]

If \( \text{Ric} \geq \epsilon \nabla \phi \otimes \nabla \phi \) where \( \epsilon \geq 2\eta \frac{k-1}{k^2} \) then \( R \geq \epsilon |\nabla \phi|^2 \) and

\[
T_{ij} V^i V^j \geq \left[ \left( \frac{1}{2} - \frac{1}{k} \right) \epsilon + \eta \left( \frac{1}{k} - 1 \right) \right] |\nabla \phi|^2 |V|^2 \geq 0.
\]
The first eigenvalue of \((p, q)\)-Laplacian 12 (2021) No. 2, 193-204

203

For second case, we have

\[
T_{ij}V^iV^j = R_{ij}V_iV_j - \eta \nabla_i V^i \nabla_j V^j - \frac{R}{k} |V|^2 + \frac{\eta}{k} |\nabla \phi|^2 |V|^2 \geq R_{ij}V_iV_j - \frac{\eta}{k} |\nabla \phi|^2 |V|^2 - \frac{R}{k} |V|^2 + \frac{\eta}{k} |\nabla \phi|^2 |V|^2 = 0.
\]

Hence the corresponding results follows by Theorems 3.3 and 3.6 \(\Box\) When we restrict the \((RH)_\eta\) flow to the Ricci flow, we obtain

**Corollary 3.9.** Let \(g(t), \ t \in [0, T)\) be a solution of the Ricci flow on a closed Riemannian surface \(M\) and \(\lambda(t)\) denotes the first eigenvalue of the \((p, q)\)-Laplacian \([2,3]\).

1. If \(R_{\text{min}}(0) \geq 0\), then \(\lambda(t)\) is nondecreasing along the Ricci flow for any \(t \in [0, T)\).
2. If \(R_{\text{min}}(0) > 0\), then the quantity \((1 - R_{\text{min}}(0)t)\lambda(t)\) is nondecreasing along the Ricci flow for any \(t \in [0, T')\) where \(T' = \min\{T, \frac{1}{R_{\text{min}}(0)}\}\).

**Example 3.10.** Let \((M^n, g_0)\) be an Einstein manifold i.e. there exists a constant such that Ric\((g_0) = ag_0\). Assume that \((N, \gamma) = (M, g_0)\), then \(\phi_0\) is the identity map. With the assumption \(g(t) = c(t)g_0, \ c(0) = 1\) and the fact that \(\phi(t) = \phi(0)\) is harmonic map for all \(g(t)\), the \((RH)_\eta\) flow reduces to

\[
\frac{\partial c(t)}{\partial t} = -2a + 2\eta, \ c(0) = 1,
\]

then the solution of the initial value problem is given by

\[
c(t) = (-2a + 2\eta)t + 1.
\]

Therefore the solution of the \((RH)_\eta\) flow remains Einstein and we have

\[
S = \text{Ric}_{g(t)} - \eta \nabla \phi \otimes \nabla \phi = (a - \eta)g_0 = \frac{a - \eta}{-2(a - \eta)t + 1} g(t),
\]

\[
S = R - \eta |\nabla \phi|^2 = \frac{am}{-2(a - \eta)t + 1} - \eta \frac{m}{-2(a - \eta)t + 1} = \frac{(a - \eta)m}{-2(a - \eta)t + 1}.
\]

Using equation (3.18), we have

\[
\frac{d\lambda}{dt}(u, v, t)|_{t=t_1} = \frac{a - \eta}{-2(a - \eta)t + 1} \left( (\alpha + 1) \int_M |\nabla u|^p d\mu + (\beta + 1) \int_M |\nabla v|^q d\mu \right).
\]

Now if assume that \(p \leq q\) then for \(\eta < a\) and \(t_1 \in [0, T'')\) where \(T'' = \min\{\frac{1}{2(a - \eta)}, T\}\), we have

\[
\frac{d\lambda}{dt}(u, v, t)|_{t=t_1} \geq \frac{a - \eta}{-2(a - \eta)t_1 + 1} \lambda(t_1).
\]

This results that in any sufficiently small neighborhood of \(t_1\) as \(I_1\), we get

\[
\frac{d\lambda}{dt}(u, v, t) \geq \frac{a - \eta}{-2(a - \eta)t + 1} \lambda(u, v, t).
\]

Integrating the last inequality with respect to \(t\) on \([t_0, t_1] \subset I_1\) we have

\[
\ln \frac{\lambda(u(t_1), v(t_1), t_1)}{\lambda(u(t_0), v(t_0), t_0)} \geq \ln \left( \frac{-2(a - \eta)t_1 + 1}{-2(a - \eta)t_0 + 1} \right)^{\frac{q}{2}},
\]

but \(t_1 \in [0, T'')\) is arbitrary, \(\lambda(u(t_1), v(t_1), t_1) = \lambda(t_1)\) and \(\lambda(u(t_0), v(t_0), t_0) \geq \lambda(t_0)\), then \(\lambda(t)(-2(a - \eta)t + 1)^{\frac{q}{2}}\) is nondecreasing along the \((RH)_\eta\) flow on \([0, T'')\).
References


