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Variation of the first eigenvalue of (p,q)-Laplacian along the Ricci-harmonic flow

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Abstract

In this paper, we study monotonicity for the first eigenvalue of a class of (p,q)-Laplacian. We find the first variation formula for the first eigenvalue of (p,q)-Laplacian on a closed Riemannian manifold evolving by the Ricci-harmonic flow and construct various monotonic quantities by imposing some conditions on initial manifold.

Keywords: Laplace, Ricci flow, Harmonic map
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1. Introduction

The study on eigenvalue problem has received remarkable attention. Recently, many mathematicians considered the eigenvalue problem of geometric operators under various geometric flows, because it is a very powerful tool for the understanding Riemannain manifold. The fundamental study of this works began when Perelman [10] showed that the functional

$$F = \int_M (R + |\nabla f|^2) e^{-f}\, d\mu$$

is nondecreasing along the Ricci flow coupled to a backward heat-type equation, where R is the scalar curvature with respect to the metric $g(t)$ and $d\mu$ denotes the volume form of the metric $g(t)$. The nondecreasing of the functional F implies that the first eigenvalue of the geometric operator $-4\Delta + R$ is nondecreasing under the Ricci flow. Then, Li [7] and Zeng et al [12] extended the

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geometric operator $-4\Delta + R$ to the operator $-\Delta + cR$ and studied the monotonicity of eigenvalues of the operator $-\Delta + cR$ along Ricci flow and the Ricci-Bourguignon flow, respectively.

Also, in [1, 11, 13] has been investigated the evolution for the first eigenvalue of p-Laplacian along the Ricci-harmonic flow, Ricci flow and mth mean curvature flow, respectively. A generalization of p-Laplacian is a class of (p, q)-Laplacian which has applications in physics and related sciences such as non-Newtonnian fluids, pseudoplastics [4, 5] that we introduce it in later section.

On the other hand, geometric flows for instance, Ricci-harmonic flow have been a topic of active research interest in mathematics and physics. A geometric flow is an evolution of a geometric structure. Let M be a closed m-dimensional Riemannian manifold with a Riemannian metric g_0. Hamilton for the first time in 1982 introduced the Ricci flow as follows

$$\frac{\partial g(t)}{\partial t} = -2Ric(g(t)), \qquad g(0) = g_0,$$

where Ric is the Ricci tensor of $g(t)$. The Ricci flow has been proved to be a very useful tool to improve metrics in Riemannian geometry, when M is compact. Now, let (M^m, g) and (N^n, γ) be closed Riemannain manifolds. By Nash's embedding theorem, assume that N is isometrically embedded into Euclidean space $e_N : (N^n, \gamma) \hookrightarrow \mathbb{R}^d$ for a sufficiently large d. We identify map $\phi : M \to N$ with $e_N \circ \phi : M \to \mathbb{R}^d$. Müller [9] considered a generalization of Ricci flow as

$$\begin{cases} \frac{\partial g(t)}{\partial t} = -2Ric(g(t)) + 2\eta \nabla\phi \otimes \nabla\phi, & g(0) = g_0, \\ \frac{\partial \phi}{\partial t} = \tau_g \phi & \phi(0) = \phi_0, \end{cases} \tag{1.1}$$

where η is a positive coupling constant, $\phi(t)$ is a family of smooth maps from M to some closed target manifold N and $\tau_g\phi$ is the intrinsic Laplacian of ϕ which denotes the tension field of ϕ with respect to the evolving metric $g(t)$. This evolution equation system called Ricci flow coupled with harmonic map flow or $(RH)_\eta$ flow for short. Müller in [9] shown that system (1.1) has unique solution with initial data $(g(0), \phi(0)) = (g_0, \phi_0)$. Also, the normalized $(RH)_\eta$ flow defined as

$$\begin{cases} \frac{\partial g(t)}{\partial t} = -2Ric(g(t)) + 2\eta \nabla\phi \otimes \nabla\phi + \frac{2}{m} r g(t), & g(0) = g_0, \\ \frac{\partial \phi}{\partial t} = \tau_g \phi & \phi(0) = \phi_0, \end{cases} \tag{1.2}$$

where $r = \frac{\int_M (R - \eta|\nabla\phi|^2) d\mu}{\int_M d\mu}$ is the average of $R - \eta|\nabla\phi|^2$. Under this normalized flow, the volume of the solution metrics remains constant in time.

2. Preliminaries

2.1. Eigenvalues of p-Laplacian

Let (M, g) be a closed Riemannian manifold and $f : M \longrightarrow \mathbb{R}$ be a smooth function on M or $f \in W^{1,p}(M)$. The Laplace-Beltrami operator acting on a smooth function f on M is the divergence of gradient of f, written as

$$\Delta f = div(grad\, f) = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g}\, \partial_j f),$$

where $\partial_i f = \frac{\partial f}{\partial x^i}$. The p-Laplacian of f for $1 < p < \infty$ is defined as

$$\begin{aligned} \triangle_p f &= div(|\nabla f|^{p-2} \nabla f) \\ &= |\nabla f|^{p-2} \Delta f + (p-2)|\nabla f|^{p-4} (Hess f)(\nabla f, \nabla f), \end{aligned} \tag{2.1}$$

where

$$(Hessf)(X,Y) = \nabla(\nabla f)(X,Y) = X.(Y.f) - (\nabla_X Y).f, \quad X,Y \in \mathcal{X}(M)$$

and in local coordinate, we get

$$(Hessf)(\partial_i,\partial_j) = \partial_i\partial_j f - \Gamma^k_{ij}\partial_k f.$$

Notice that when $p = 2$, p-Laplacian is the Laplace-Beltrami operator. Let (M^n, g) be a closed Riemannian manifold. In this paper, we consider the nonlinear system introduced in [6], that is

$$\begin{cases} \Delta_p u = -\lambda |u|^\alpha |v|^\beta v \text{ in } M \\ \Delta_q v = -\lambda |u|^\alpha |v|^\beta u \text{ in } M \\ (u,v) \in W^{1,p}(M) \times W^{1,q}(M) \end{cases} \tag{2.2}$$

where $p > 1,\ q > 1$ and α, β are real numbers satisfying

$$\alpha > 0,\ \beta > 0, \quad \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1. \tag{2.3}$$

In (2.2), we say that λ is an eigenvalue whenever for some $u \in W_0^{1,p}(M)$ and $v \in W_0^{1,q}(M)$,

$$\int_M |\nabla u|^{p-2} < \nabla u, \nabla\phi > d\mu \ = \ \lambda \int_M |u|^\alpha |v|^\beta v\phi d\mu, \tag{2.4}$$

$$\int_M |\nabla v|^{q-2} < \nabla v, \nabla\psi > d\mu \ = \ \lambda \int_M |u|^\alpha |v|^\beta u\psi d\mu, \tag{2.5}$$

where $\phi \in W^{1,p}(M)$, $\psi \in W^{1,q}(M)$ and $W_0^{1,p}(M)$ is the closure of $C_0^\infty(M)$ in Sobolev space $W^{1,p}(M)$. The pair (u,v) is called a eigenfunction corresponding to eigenvalue λ. A first positive eigenvalue of (2.2) obtained as

$$inf\{A(u,v) : (u,v) \in W_0^{1,p}(M) \times W_0^{1,q}(M),\ B(u,v) = 1\}$$

where

$$A(u,v) \ = \ \frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu + \frac{\beta+1}{q}\int_M |\nabla v|^q d\mu,$$

$$B(u,v) \ = \ \int_M |u|^\alpha |v|^\beta uv d\mu.$$

Let $(M^m, g(t), \phi(t))$ be a solution of the $(RH)_\eta$ flow (1.1) on the smooth manifold (M^m, g_0, ϕ_0) in the interval $[0,T)$ then

$$\lambda(t) = \frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu_t + \frac{\beta+1}{q}\int_M |\nabla v|^q d\mu_t, \tag{2.6}$$

defines the evolution of an eigenvalue of (2.2), under the variation of $(g(t), \phi(t))$ where the eigenfunction associated to $\lambda(t)$ is normalized that is $B(u,v) = 1$. Motivated by the above works, in this paper we will study the first eigenvalue of a class of (p,q)-Laplacian (2.2) whose metric satisfies the $(RH)_\eta$ flow. Throughout of paper we write $\frac{\partial u}{\partial t} = \partial_t u = u'$, $\mathcal{S} = Ric_g - \eta\nabla\phi \otimes \nabla\phi$, $\mathcal{S}_{ij} = Ric_{ij} - \eta\nabla_i\phi\nabla_j\phi$ and $S = R - \eta|\nabla\phi|^2$.

3. Variation of $\lambda(t)$

In this section, we will give some useful evolution formulas for $\lambda(t)$ under the Ricci-harmonic flow. Now, we give a useful statement about the variation of the first eigenvalue of (2.2) under the $(RH)_\eta$ flow.

Lemma 3.1. *If g_1 and g_2 are two metrics on Riemannian manifold M^m which satisfy $(1+\epsilon)^{-1}g_1 < g_2 < (1+\epsilon)g_1$ then for any $p \geq q > 1$, we have*

$$\lambda(g_2) - \lambda(g_1) \leq \left((1+\epsilon)^{\frac{p+m}{2}} - (1+\epsilon)^{-\frac{m}{2}}\right)\lambda(g_1)$$

in particular, $\lambda(t)$ is a continues function respect to t-variable.

Proof . By direct computation we complete the proof of lemma. In local coordinate we have $d\mu = \sqrt{\det g}\, dx^1 \wedge ... \wedge dx^m$, therefore

$$(1+\epsilon)^{-\frac{m}{2}} d\mu_{g_1} < d\mu_{g_2} < (1+\epsilon)^{\frac{m}{2}} d\mu_{g_1}.$$

Let

$$G(g,u,v) = \frac{\alpha+1}{p}\int_M |\nabla u|_g^p d\mu_g + \frac{\beta+1}{q}\int_M |\nabla v|_g^q d\mu_g, \tag{3.1}$$

then

$$\begin{aligned}
& \int_M |u|^\alpha |v|^\beta uv d\mu_{g_1} G(g_2,u,v) - \int_M |u|^\alpha |v|^\beta uv d\mu_{g_2} G(g_1,u,v) \\
= & \frac{\alpha+1}{p}\int_M |u|^\alpha |v|^\beta uv d\mu_{g_1}\left(\int_M |\nabla u|_{g_2}^p d\mu_{g_2} - \int_M |\nabla u|_{g_1}^p d\mu_{g_1}\right) \\
& + \frac{\alpha+1}{p}\left(\int_M |u|^\alpha |v|^\beta uv d\mu_{g_1} - \int_M |u|^\alpha |v|^\beta uv d\mu_{g_2}\right)\int_M |\nabla u|_{g_1}^p d\mu_{g_1} \\
& + \frac{\beta+1}{q}\int_M |u|^\alpha |v|^\beta uv d\mu_{g_1}\left(\int_M |\nabla v|_{g_2}^q d\mu_{g_2} - \int_M |\nabla v|_{g_1}^q d\mu_{g_1}\right) \\
& + \frac{\beta+1}{q}\left(\int_M |u|^\alpha |v|^\beta uv d\mu_{g_1} - \int_M |u|^\alpha |v|^\beta uv d\mu_{g_2}\right)\int_M |\nabla v|_{g_1}^q d\mu_{g_1} \\
\leq & \frac{\alpha+1}{p}\left((1+\epsilon)^{\frac{p+m}{2}} - (1+\epsilon)^{-\frac{m}{2}}\right)\int_M |u|^\alpha |v|^\beta uv d\mu_{g_1}\int_M |\nabla u|_{g_1}^p d\mu_{g_1} \\
& + \frac{\beta+1}{q}\left((1+\epsilon)^{\frac{q+m}{2}} - (1+\epsilon)^{-\frac{m}{2}}\right)\int_M |u|^\alpha |v|^\beta uv d\mu_{g_1}\int_M |\nabla v|_{g_1}^q d\mu_{g_1} \\
\leq & \left((1+\epsilon)^{\frac{p+m}{2}} - (1+\epsilon)^{-\frac{m}{2}}\right) G(g_1,u,v)\int_M |u|^\alpha |v|^\beta uv d\mu_{g_1}.
\end{aligned}$$

Since the eigenfunction corresponding to $\lambda(t)$ are normalized, thus we get

$$\lambda(g_2) - \lambda(g_1) \leq \left((1+\epsilon)^{\frac{p+m}{2}} - (1+\epsilon)^{-\frac{m}{2}}\right)\lambda(g_1)$$

this completes the proof of Lemma. □

Proposition 3.2. *Let $(g(t),\phi(t))$, $t\in[0,T)$, be a solution of the $(RH)_\eta$ flow on a closed manifold M^m and let $\lambda(t)$ be the first eigenvalue of the (p,q)-Laplacian along this flow. Then for any t_0, $t_1\in[0,T)$ and $t_1>t_0$, we have*

$$\lambda(t_1)\geq\lambda(t_0)+\int_{t_0}^{t_1}\mathcal{G}(g(\tau),u(\tau),v(\tau))d\tau \tag{3.2}$$

where

$$\begin{aligned}\mathcal{G}(g(t),u(t),v(t)) &= (\alpha+1)\int_M\left(\mathcal{S}(\nabla u,\nabla u)+<\nabla u',\nabla u>\right)|\nabla u|^{p-2}d\mu\\ &\quad+(\beta+1)\int_M\left(\mathcal{S}(\nabla v,\nabla v)+<\nabla v',\nabla v>\right)|\nabla v|^{q-2}d\mu\\ &\quad-\frac{\alpha+1}{p}\int_M|\nabla u|^pSd\mu-\frac{\beta+1}{q}\int_M|\nabla v|^qSd\mu.\end{aligned} \tag{3.3}$$

Proof . Assume that

$$G(g(t),u(t),v(t))=\frac{\alpha+1}{p}\int_M|\nabla u(t)|^p_{g(t)}d\mu_{g(t)}+\frac{\beta+1}{q}\int_M|\nabla v(t)|^q_{g(t)}d\mu_{g(t)},$$

at time t_1 we first let $(u_1,v_1)=(u(t_1),v(t_1))$ be the eigenfunction for the eigenvalue $\lambda(t_1)$ of (p,q)-Laplacian. We consider the following smooth functions

$$h(t)=u_1\left[\frac{\det[g_{ij}(t_1)]}{\det[g_{ij}(t)]}\right]^{\frac{1}{2(\alpha+\beta+1)}},\qquad l(t)=v_1\left[\frac{\det[g_{ij}(t_1)]}{\det[g_{ij}(t)]}\right]^{\frac{1}{2(\alpha+\beta+1)}},$$

along the $(RH)_\eta$ flow. We define

$$u(t)=\frac{h(t)}{\left(\int_M|h(t)|^\alpha|l(t)|^\beta h(t)l(t)d\mu\right)^{\frac{1}{p}}},\qquad u(t)=\frac{l(t)}{\left(\int_M|h(t)|^\alpha|l(t)|^\beta h(t)l(t)d\mu\right)^{\frac{1}{q}}}$$

which $u(t)$, $v(t)$ are smooth functions under the $(RH)_\eta$ flow, satisfy

$$\int_M|u|^\alpha|v|^\beta uvd\mu=1,$$

and at time t_1, $(u(t_1),v(t_1))$ are the eigenfunctions for $\lambda(t_1)$ of (p,q)-Laplacian at time t_1 i.e. $\lambda(t_1)=G(g(t_1),u(t_1),v(t_1))$. If f is a smooth function respect to time t then along the $(RH)_\eta$ flow we have

$$\frac{d}{dt}\left(|\nabla f|^p\right)=\frac{p}{2}\left[\partial_tg^{ij}\nabla_if\nabla_jf+2g^{ij}\nabla_if'\nabla_jf\right]|\nabla f|^{p-2}$$

by (1.1) we have $\partial_tg^{ij}=2g^{ik}g^{jl}\mathcal{S}_{kl}$, therefore

$$\frac{d}{dt}\left(|\nabla f|^p\right)=p|\nabla f|^{p-2}\Big(\mathcal{S}(\nabla f,\nabla f)+<\nabla f',\nabla f>\Big), \tag{3.4}$$

and

$$\partial_td\mu=\frac{1}{2}tr_g(\partial_tg)d\mu=-Sd\mu.$$

Since $u(t)$ and $v(t)$ are smooth functions, hence $G(g(t),u(t),v(t))$ is a smooth function with respect to t. If we set

$$\mathcal{G}(g(t),u(t),v(t)) := \frac{d}{dt}G(g(t),u(t),v(t)), \tag{3.5}$$

then

$$\begin{aligned}\mathcal{G}(g(t),u(t),v(t)) &= (\alpha+1)\int_M \left(\mathcal{S}(\nabla u,\nabla u)+<\nabla u',\nabla u>\right)|\nabla u|^{p-2}d\mu \\ &\quad +(\beta+1)\int_M \left(\mathcal{S}(\nabla v,\nabla v)+<\nabla v',\nabla v>\right)|\nabla v|^{q-2}d\mu \\ &\quad -\frac{\alpha+1}{p}\int_M |\nabla u|^p S d\mu - \frac{\beta+1}{q}\int_M |\nabla v|^q S d\mu.\end{aligned} \tag{3.6}$$

Taking integration on the both sides of (3.5) between t_0 and t_1, we conclude that

$$G(g(t_1),u(t_1),v(t_1)) - G(g(t_0),u(t_0),v(t_0)) = \int_{t_0}^{t_1}\mathcal{G}(g(\tau),u(\tau),v(\tau))d\tau \tag{3.7}$$

where $t_0 \in [0,T)$ and $t_1 > t_0$. Noticing $G(g(t_0),u(t_0),v(t_0)) \geq \lambda(t_0)$ and plugin $\lambda(t_1) = G(g(t_1),u(t_1),v(t_1))$ in (3.7), yields (3.2) and $\mathcal{G}(g(t),u(t),v(t))$ satisfies in (3.3). $\square$

Theorem 3.3. *Let $(M^m, g(t), \phi(t))$ be a solution of the $(RH)_\eta$ flow on the smooth closed manifold (M^m, g_0, ϕ_0) and $\lambda(t)$ denotes the evolution of the first eigenvalue under the $(RH)_\eta$ flow. Suppose that $k = \min\{p,q\}$ and*

$$\mathcal{S} - \frac{1}{k}Sg \geq 0 \text{ in } M^m \times [0,T). \tag{3.8}$$

If $S_{\min}(0) \geq 0$, then $\lambda(t)$ is nondecreasing and differentiable almost everywhere along the $(RH)_\eta$ flow on $[0,T)$.

Proof . Let for any $t_1 \in [0,T)$, $u(t_1)$, $v(t_1)$ be the eigenfunctions for $\lambda(t_1)$ of (p,q)-Laplacian. Then $\int_M |u(t_1)|^\alpha |v(t_1)|^\beta u(t_1)v(t_1)d\mu_{g(t_1)} = 1$ and

$$\begin{aligned}\mathcal{G}(g(t_1),u(t_1),v(t_1)) &= (\alpha+1)\int_M \left(\mathcal{S}(\nabla u,\nabla u)+<\nabla u',\nabla u>\right)|\nabla u|^{p-2}d\mu \\ &\quad +(\beta+1)\int_M \left(\mathcal{S}(\nabla v,\nabla v)+<\nabla v',\nabla v>\right)|\nabla v|^{q-2}d\mu \\ &\quad -\frac{\alpha+1}{p}\int_M |\nabla u|^p S d\mu - \frac{\beta+1}{q}\int_M |\nabla v|^q S d\mu.\end{aligned} \tag{3.9}$$

Now, by the time derivative of the condition

$$\int_M |u|^\alpha |v|^\beta uv d\mu = 1$$

we can get

$$(\alpha+1)\int_M |u|^\alpha |v|^\beta u'v d\mu + (\beta+1)\int_M |u|^\alpha |v|^\beta uv' d\mu = \int_M S|u|^\alpha |v|^\beta uv d\mu. \tag{3.10}$$

On the other hand, (2.4) and (2.5) imply that

$$\int_M <\nabla u', \nabla u> |\nabla u|^{p-2} d\mu = \lambda(t_1) \int_M |u|^\alpha |v|^\beta u' v d\mu, \tag{3.11}$$

$$\int_M <\nabla v', \nabla v> |\nabla v|^{q-2} d\mu = \lambda(t_1) \int_M |u|^\alpha |v|^\beta u v' d\mu. \tag{3.12}$$

Therefore from (3.10), (3.11) and (3.12) we have

$$\begin{aligned}(\alpha+1)\int_M <\nabla u', \nabla u> |\nabla u|^{p-2} d\mu + (\beta+1)\int_M <\nabla v', \nabla v> |\nabla v|^{q-2} d\mu \\ = \lambda(t_1)\int_M S|u|^\alpha |v|^\beta uv d\mu,\end{aligned} \tag{3.13}$$

and the replacing (3.13) in (3.9), results that

$$\begin{aligned}\mathcal{G}(g(t_1), u(t_1), v(t_1)) \ = \ & \lambda(t_1)\int_M S|u|^\alpha |v|^\beta uv d\mu + (\alpha+1)\int_M \mathcal{S}(\nabla u, \nabla u)|\nabla u|^{p-2} d\mu \\ & -\frac{\alpha+1}{p}\int_M |\nabla u|^p S d\mu + (\beta+1)\int_M \mathcal{S}(\nabla v, \nabla v)|\nabla v|^{q-2} d\mu \\ & -\frac{\beta+1}{q}\int_M |\nabla v|^q S d\mu.\end{aligned} \tag{3.14}$$

From (3.14) and (3.8) we have

$$\begin{aligned}\mathcal{G}(g(t_1), u(t_1), v(t_1)) \ \geq \ & \lambda(t_1)\int_M S|u|^\alpha |v|^\beta uv d\mu + (\alpha+1)(\frac{1}{k}-\frac{1}{p})\int_M |\nabla u|^p S d\mu \\ & +(\beta+1)(\frac{1}{k}-\frac{1}{q})\int_M |\nabla v|^q S d\mu.\end{aligned} \tag{3.15}$$

Since

$$\frac{\partial}{\partial t}S = \Delta S + 2|\mathcal{S}_{ij}|^2 + 2\eta|\tau_g \phi|^2$$

and $|\mathcal{S}_{ij}|^2 \geq \frac{1}{m}S^2$, it follows that

$$\frac{\partial}{\partial t}S \geq \Delta S + \frac{2}{m}S^2. \tag{3.16}$$

The solution to

$$\frac{d}{dt}y(t) = \frac{2}{m}y^2(t), \qquad y(t) = S_{\min}(0),$$

is

$$y(t) = \frac{S_{\min}(0)}{1-\frac{2}{m}S_{\min}(0)t}, \quad t \in [0, T'), \tag{3.17}$$

where $T' = \min\{T, \frac{m}{2S_{\min}(0)}\}$. Using maximum principle to (3.16), we get $S \geq y(t)$ along the $(RH)_\eta$ flow. If $S_{\min}(0) \geq 0$ then the nonnegativity of S preserved along the $(RH)_\eta$ flow. Therefore (3.15) becomes $\mathcal{G}(g(t_1), u(t_1), v(t_1)) \geq 0$. Thus we get $\mathcal{G}(g(t), u(t), v(t)) > 0$ in any small enough neighborhood of t_1. Hence $\int_{t_0}^{t_1} \mathcal{G}(g(\tau), u(\tau), v(\tau))d\tau > 0$ for any $t_0 < t_1$ sufficiently close to t_1. Since $t_1 \in [0, T)$ is arbitrary the Proposition 3.2 completes the proof of the first part of theorem. For the differentiability for $\lambda(t)$, since $\lambda(t)$ is increasing and continues on the interval $[0, T)$, the classical Lebesgue's theorem (see [8]), $\lambda(t)$ is differentiable almost everywhere on $[0, T)$. □ Motivated by the works of X.-D. Cao [2, 3] and J. Y. Wu [11], similar to proof of Proposition 3.2 we first introduce a new smooth eigenvalue function along the $(RH)_\eta$ flow and then we give evolution formula for it. Let M be an m-dimensional closed Riemannian manifold and $g(t)$ be a smooth solution of the $(RH)_\eta$ flow. Suppose that

$$\lambda(u,v,t) := \frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu + \frac{\beta+1}{q}\int_M |\nabla v|^q d\mu$$

where u, v are smooth functions and satisfy

$$\int_M |u|^\alpha |v|^\beta uv d\mu = 1, \quad \int_M |u|^\alpha |v|^\beta v d\mu = 0, \quad \int_M |u|^\alpha |v|^\beta u d\mu = 0.$$

The function $\lambda(u,v,t)$ is a smooth eigenvalue function respect to t-variable. If (u,v) are the corresponding eigenfunctions of the first eigenvalue $\lambda(t_1)$ then $\lambda(u,v,t_1) = \lambda(t_1)$. As proof of Proposition 3.2 and Theorem 3.3 we have the following propositions.

Proposition 3.4. *Let $(M^m, g(t), \phi(t))$ be a solution of the $(RH)_\eta$ flow on the smooth closed manifold (M^m, g_0, ϕ_0). If $\lambda(t)$ denotes the evolution of the first eigenvalue under the $(RH)_\eta$ flow, then*

$$\begin{aligned} \frac{d\lambda}{dt}(u,v,t)|_{t=t_1} &= \lambda(t_1)\int_M S|u|^\alpha |v|^\beta uv d\mu + (\alpha+1)\int_M \mathcal{S}(\nabla u, \nabla u)|\nabla u|^{p-2} d\mu \\ &\quad -\frac{\alpha+1}{p}\int_M |\nabla u|^p S d\mu + (\beta+1)\int_M \mathcal{S}(\nabla v, \nabla v)|\nabla v|^{q-2} d\mu \\ &\quad -\frac{\beta+1}{q}\int_M |\nabla v|^q S d\mu, \end{aligned} \tag{3.18}$$

where (u,v) is the associated normalized evolving eigenfunction.

Now, we give a variation of $\lambda(t)$ under the normalized $(RH)_\eta$ flow which is similar to the previous Proposition.

Proposition 3.5. *Let $(M^m, g(t), \phi(t))$ be a solution of the normalized $(RH)_\eta$ flow on the smooth closed manifold (M^m, g_0, ϕ_0). If $\lambda(t)$ denotes the evolution of the first eigenvalue under the normalized $(RH)_\eta$ flow, then*

$$\begin{aligned} \frac{d\lambda}{dt}(u,v,t)|_{t=t_1} &= \lambda(t_1)\int_M S|u|^\alpha |v|^\beta uv d\mu + (\alpha+1)\int_M \mathcal{S}(\nabla u, \nabla u)|\nabla u|^{p-2} d\mu \\ &\quad +(\beta+1)\int_M \mathcal{S}(\nabla v, \nabla v)|\nabla v|^{q-2} d\mu - \frac{\beta+1}{q}\int_M |\nabla v|^q S d\mu \\ &\quad -\frac{\alpha+1}{p}\int_M |\nabla u|^p S d\mu - \frac{\alpha+1}{m} r(t_1)\int_M |\nabla u|^p d\mu \\ &\quad -\frac{\beta+1}{m} r(t_1)\int_M |\nabla v|^q d\mu, \end{aligned} \tag{3.19}$$

where (u,v) is the associated normalized evolving eigenfunction.

Proof . In the normalized case, derivative of the integrability condition $\int_M |u|^\alpha |v|^\beta uv d\mu = 1$ respect to t, results that

$$(\alpha+1)\int_M |u|^\alpha |v|^\beta u' v d\mu + (\beta+1)\int_M |u|^\alpha |v|^\beta u v' d\mu = -r(t_1) + \int_M S|u|^\alpha |v|^\beta uv d\mu. \tag{3.20}$$

On the other hand

$$\frac{d}{dt}(d\mu_t) = \frac{1}{2}tr_g(\frac{\partial g}{\partial t})d\mu = \frac{1}{2}tr_g(\frac{2}{m}rg - 2\mathcal{S})d\mu = (r - S)d\mu, \tag{3.21}$$

hence we can then write

$$\begin{aligned}\frac{d\lambda}{dt}(u,v,t)|_{t=t_1} &= \frac{\alpha+1}{p}\left(\frac{p}{2}\int_M\left\{-\frac{2}{m}r|\nabla u|^2 + 2\mathcal{S}(\nabla u,\nabla u) + 2 < \nabla u', \nabla u >\right\}|\nabla u|^{p-2}d\mu\right)\\ &+\frac{\beta+1}{q}\left(\frac{q}{2}\int_M\left\{-\frac{2}{m}r|\nabla v|^2 + 2\mathcal{S}(\nabla v,\nabla v) + 2 < \nabla v', \nabla v >\right\}|\nabla v|^{q-2}d\mu\right)\\ &+\frac{\alpha+1}{p}\int_M|\nabla u|^p(r-S)d\mu + \frac{\beta+1}{q}\int_M|\nabla v|^q(r-S)d\mu,\end{aligned}$$

but

$$\begin{aligned}(\alpha+1)\int_M < \nabla u', \nabla u > |\nabla u|^{p-2}d\mu \;&+\; (\beta+1)\int_M < \nabla v', \nabla v > |\nabla v|^{q-2}d\mu\\ &= -\lambda(t_1)r(t_1) + \lambda(t_1)\int_M S|u|^\alpha|v|^\beta uv d\mu.\end{aligned} \tag{3.22}$$

Therefore the proposition is obtained by replacing (3.22) in previous relation. □

Theorem 3.6. *Let $(M^m, g(t), \phi(t))$ be a solution of the $(RH)_\eta$ flow on the smooth closed manifold (M^m, g_0, ϕ_0) and $\lambda(t)$ denotes the evolution of the first eigenvalue under the $(RH)_\eta$ flow. If $k = \min\{p, q\}$,*

$$\mathcal{S} - \frac{S}{k}g > 0 \text{ in } M^m \times [0, T) \tag{3.23}$$

and $S_{\min}(0) > 0$, then the quantity $\lambda(t)(1 - \frac{2}{m}S_{\min}(0)t)^{\frac{m}{2}}$ is nondecreasing along the $(RH)_\eta$ flow on $[0, T')$, where $T' := \min\{\frac{m}{2S_{\min}(0)}, T\}$.

Proof . According to (3.18) and (3.23) we have

$$\begin{aligned}\frac{d\lambda}{dt}(u,v,t)|_{t=t_1} &> \lambda(t_1)\int_M S|u|^\alpha|v|^\beta uv d\mu + (\alpha+1)(\frac{1}{k} - \frac{1}{p})\int_M|\nabla u|^p S d\mu\\ &+(\beta+1)(\frac{1}{k} - \frac{1}{q})\int_M|\nabla v|^q S d\mu.\end{aligned} \tag{3.24}$$

If $S_{\min}(0) > 0$, then (3.17) results that the positive of S remains under the $(RH)_\eta$ flow, therefore

$$\frac{d\lambda}{dt}(u,v,t)|_{t=t_1} \geq \lambda(t_1)\frac{S_{\min}(0)}{1 - \frac{2}{m}S_{\min}(0)t_1}. \tag{3.25}$$

Then in any small enough neighborhood of t_1 as I, we get

$$\frac{d\lambda}{dt}(u,v,t) \geq \lambda(u,v,t)\frac{S_{\min}(0)}{1 - \frac{2}{m}S_{\min}(0)t}. \tag{3.26}$$

Integrating the last inequality with respect to t on $[t_0, t_1] \subset I$, we have

$$\ln\frac{\lambda(u(t_1), v(t_1), t_1)}{\lambda(u(t_0), v(t_0), t_0)} \geq \ln\left(\frac{1 - \frac{2}{m}S_{\min}(0)t_1}{1 - \frac{2}{m}S_{\min}(0)t_0}\right)^{-\frac{m}{2}}. \tag{3.27}$$

Since $\lambda(u(t_1), v(t_1), t_1) = \lambda(t_1)$ and $\lambda(u(t_0), v(t_0), t_0) \geq \lambda(t_0)$ we conclude that

$$\ln \frac{\lambda(t_1)}{\lambda(t_0)} \geq \ln \left(\frac{1 - \frac{2}{m} S_{\min}(0) t_1}{1 - \frac{2}{m} S_{\min}(0) t_0} \right)^{-\frac{m}{2}}, \tag{3.28}$$

that is the quantity $\lambda(t)(1 - \frac{2}{m} S_{\min}(0)t)^{\frac{m}{2}}$ is nondecreasing in any sufficiently small neighborhood of t_1. Since t_1 is arbitrary, hence $\lambda(t)(1 - \frac{2}{m} S_{\min}(0)t)^{\frac{m}{2}}$ is nondecreasing along the $(RH)_\eta$ flow on $[0, T')$. □ Now, if in the $(RH)_\eta$ flow, we suppose that $\eta = 0$, then the $(RH)_\eta$ flow reduce to the Ricci flow and we have the following corollary

Corollary 3.7. *Let $g(t)$, $t \in [0, T)$ be a solution of the Ricci flow on a closed Riemannain manifold M and $\lambda(t)$ denotes the first eigenvalue of the (p, q)-Laplacian (2.2). Suppose that $k = \min\{p, q\}$ and $Ric - \frac{R}{k} g \geq 0$ along the Ricci flow.*
(1) If $R_{\min}(0) \geq 0$, then $\lambda(t)$ is nondecreasing along the Ricci flow for any $t \in [0, T)$.
(2) If $R_{\min}(0) > 0$, then the quantity $(1 - R_{\min}(0)t)\lambda(t)$ is nondecreasing along the Ricci flow for any $t \in [0, T')$ where $T' = \min\{T, \frac{1}{R_{\min}(0)}\}$.

In dimension two we have

Proposition 3.8. *Let $(g(t), \phi(t))$, $t \in [0, T)$ be a solution of the $(RH)_\eta$ flow on a closed Riemannian surface M and $\lambda(t)$ denotes the first eigenvalue of the (p, q)-Laplacian (2.2).*
(1) Suppose that $Ric \geq \epsilon \nabla\phi \otimes \nabla\phi$ where $\epsilon \geq 2\eta \frac{k-1}{k-2}$ and $2 \leq k = \min\{p, q\}$.
$(1-1)$ If $S_{\min}(0) \geq 0$, then $\lambda(t)$ is nondecreasing along the $(RH)_\eta$ for any $t \in [0, T)$.
$(1-2)$ If $S_{\min}(0) > 0$, then the quantity $(1 - S_{\min}(0)t)\lambda(t)$ is nondecreasing along the $(RH)_\eta$ flow on $[0, T')$ where $T' = \min\{T, \frac{1}{S_{\min}(0)}\}$.
(2) Suppose that $k = \min\{p, q\}$ and $|\nabla\phi|^2 \geq k \nabla\phi \otimes \nabla\phi$.
$(2-1)$ If $S_{\min}(0) \geq 0$, then $\lambda(t)$ is nondecreasing along the $(RH)_\eta$ for any $t \in [0, T)$.
$(2-2)$ If $S_{\min}(0) > 0$, then the quantity $(1 - S_{\min}(0)t)\lambda(t)$ is nondecreasing along the $(RH)_\eta$ flow on $[0, T')$ where $T' = \min\{T, \frac{1}{S_{\min}(0)}\}$.

Proof . In the case of surface, we have $R_{ij} = \frac{R}{2} g_{ij}$. Then

$$\begin{aligned} T_{ij} := \mathcal{S}_{ij} - \frac{S}{k} g_{ij} &= \frac{R}{2} g_{ij} - \eta \nabla_i \phi \nabla_j \phi - \frac{1}{k}(R - \eta |\nabla\phi|^2) g_{ij} \\ &= (\frac{1}{2} - \frac{1}{k}) R g_{ij} - \eta \nabla_i \phi \nabla_j \phi + \frac{\alpha}{k} |\nabla\phi|^2 g_{ij}. \end{aligned}$$

For any vector $V = (V^i)$ we get

$$\begin{aligned} T_{ij} V^i V^j &= (\frac{1}{2} - \frac{1}{k}) R |V|^2 - \eta (\nabla_i \phi V^i)^2 + \frac{\eta}{k} |\nabla\phi|^2 |V|^2 \\ &\geq (\frac{1}{2} - \frac{1}{k}) R |V|^2 + \eta (\frac{1}{k} - 1) |\nabla\phi|^2 |V|^2. \end{aligned}$$

If $Ric \geq \epsilon \nabla\phi \otimes \nabla\phi$ where $\epsilon \geq 2\eta \frac{k-1}{k-2}$ then $R \geq \epsilon |\nabla\phi|^2$ and

$$T_{ij} V^i V^j \geq \left[(\frac{1}{2} - \frac{1}{k}) \epsilon + \eta (\frac{1}{k} - 1) \right] |\nabla\phi|^2 |V|^2 \geq 0.$$

For second case, we have

$$\begin{aligned} T_{ij}V^iV^j &= R_{ij}V_iV^j - \eta\nabla_iV^i\nabla_jV^j - \frac{R}{k}|V|^2 + \frac{\eta}{k}|\nabla\phi|^2|V|^2 \\ &\geq R_{ij}V^iV^j - \frac{\eta}{k}|\nabla\phi|^2|V|^2 - \frac{R}{k}|V|^2 + \frac{\eta}{k}|\nabla\phi|^2|V|^2 = 0. \end{aligned}$$

Hence the corresponding results follows by Theorems 3.3 and 3.6. □ When we restrict the $(RH)_\eta$ flow to the Ricci flow, we obtain

Corollary 3.9. *Let $g(t)$, $t \in [0, T)$ be a solution of the Ricci flow on a closed Riemannain surface M and $\lambda(t)$ denotes the first eigenvalue of the (p, q)-Laplacian (2.2).*
(1) If $R_{\min}(0) \geq 0$, then $\lambda(t)$ is nondecreasing along the Ricci flow for any $t \in [0, T)$.
(2) If $R_{\min}(0) > 0$, then the quantity $(1 - R_{\min}(0)t)\lambda(t)$ is nondecreasing along the Ricci flow for any $t \in [0, T')$ where $T' = \min\{T, \frac{1}{R_{\min}(0)}\}$.

Example 3.10. *Let (M^m, g_0) be an Einstein manifold i.e. there exists a constant such that $Ric(g_0) = ag_0$. Assume that $(N, \gamma) = (M, g_0)$, then ϕ_0 is the identity map. With the assumption $g(t) = c(t)g_0$, $c(0) = 1$ and the fact that $\phi(t) = \phi(0)$ is harmonic map for all $g(t)$, the $(RH)_\eta$ flow reduces to*

$$\frac{\partial c(t)}{\partial t} = -2a + 2\eta, \quad c(0) = 1,$$

then the solution of the initial value problem is given by

$$c(t) = (-2a + 2\eta)t + 1.$$

Therefore the solution of the $(RH)_\eta$ flow remains Einstein and we have

$$\begin{aligned} \mathcal{S} &= Ric_{g(t)} - \eta\nabla\phi \otimes \nabla\phi = (a - \eta)g_0 = \frac{a - \eta}{-2(a - \eta)t + 1}g(t), \\ S &= R - \eta|\nabla\phi|^2 = \frac{am}{-2(a - \eta)t + 1} - \eta\frac{m}{-2(a - \eta)t + 1} = \frac{(a - \eta)m}{-2(a - \eta)t + 1}. \end{aligned}$$

Using equation (3.18), we have

$$\frac{d\lambda}{dt}(u, v, t)|_{t=t_1} = \frac{a - \eta}{-2(a - \eta)t + 1}\left((\alpha + 1)\int_M |\nabla u|^p d\mu + (\beta + 1)\int_M |\nabla v|^q d\mu\right).$$

Now if assume that $p \leq q$ then for $\eta < a$ and $t_1 \in [0, T'')$ where $T'' = \min\{\frac{1}{2(a-\eta)}, T\}$, we have

$$\frac{d\lambda}{dt}(u, v, t)|_{t=t_1} \geq \frac{a - \eta}{-2(a - \eta)t_1 + 1}\lambda(t_1).$$

This results that in any sufficiently small neighborhood of t_1 as I_1, we get

$$\frac{d\lambda}{dt}(u, v, t) \geq \frac{a - \eta}{-2(a - \eta)t + 1}\lambda(u, v, t).$$

Integrating the last inequality with respect to t on $[t_0, t_1] \subset I_1$ we have

$$\ln\frac{\lambda(u(t_1), v(t_1), t_1)}{\lambda(u(t_0), v(t_0), t_0)} \geq \ln\left(\frac{-2(a - \eta)t_1 + 1}{-2(a - \eta)t_0 + 1}\right)^{-\frac{p}{2}},$$

but $t_1 \in [0, T'')$ is arbitrary, $\lambda(u(t_1), v(t_1), t_1) = \lambda(t_1)$ and $\lambda(u(t_0), v(t_0), t_0) \geq \lambda(t_0)$, then $\lambda(t)(-2(a-\eta)t + 1)^{\frac{p}{2}}$ is nondecreasing along the $(RH)_\eta$ flow on $[0, T'')$.

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