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Hyers-Ulam types stability of nonlinear summation equations with delay

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Abstract

In this manuscript, we present existence and uniqueness theorem for the solution of nonlinear summation equation with delay. Furthermore, we present Hyers–Ulam stability, generalized Hyers–Ulam stability, Hyers–Ulam–Rassias stability and generalized Hyers–Ulam–Rassias stability of mentioned equation by utilizing discrete Grönwall Lemma. We finalized our manuscript through examples to help our primary outcomes.

Keywords: Hyers–Ulam stability, nonlinear summation equations, summation inequality of Grönwall type.
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1. Introduction

Stability of functional equations was analyzed by Ulam [36] in 1940 and raised some questions. The answer to these question was first given by Hyers in [19]. Then the well known result was established named as Hyers–Ulam stability ($\mathcal{HUS}$) and then generalized it by Aoki [5] and Rassias [31]. The concept of this stability for differential equation was extended by Obloza [28, 29]. The various methods used for the analysis of $\mathcal{HUS}$ of differential and difference operators can be seen in [7, 17, 18, 20, 23, 37, 38, 41].

Time–delay usually occurs in practical systems which causes instability, bad performance and oscillation [11, 33]. Therefore, from the last few decades the problem of time–delay and stability analysis have been studied, very well in [12, 13, 16], as these have many applications in theoretical as well as in practical systems. But still there are some limitation, which are to be over come. In

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practical systems, we examine the solution of some challenging equations like differential, integral and integro differential equations, which can be manipulated by using Volterra equations e.g. in [26, 34, 40]. Furthermore, direct Lyapunov methods help in studying the theory of stability, regarding Volterra summation equations via boundedness [8, 10, 14]. In [4] Agarval *et al.* studied the solutions (existence and approximation) for Lyapunov summation equations. The stability criteria for Volterra summation equations with degenerate Kernels is studied in [9]. However, Kolmanowski and Myshkis in [21] gave brief details about the stability problems and circumstances in respect of characteristic equations of some Volterra summation equations. On the other hand weighted norms are exploited for the occurrence of unique solutions of Volterra equation in [22, 24]. Also in [27], the weighted norms are utilized for the asymptotic equivalence of Volterra summation equations. Baker *et al.* in [6] discussed convolution and non-convolution types periodic solutions of linear and nonlinear summation equations. Oscillation and asymptotic behavior of the solution of summation equation plays an essential character in the qualitative theory of dynamic equation, see [2, 3]. In [15, 25, 26, 34, 35], there is a brief discussion on qualitative analysis and the properties of Volterra summation equations.

Recently, Zada *et al.* studied the $\mathcal{HUS}$ of linear summation equation [39]. To the best of our knowledge, in the literature so for there is no paper loyal to the study of $\mathcal{HUS}$ of non-linear summation equation. Our interest in this paper is to prove the $\mathcal{HUS}$, $\mathcal{GHUS}$, $\mathcal{HURS}$ and $\mathcal{GHURS}$ of the non–linear summation equations with delay

$$y_n = \xi_n + \lambda \sum_{a=0}^{n} \mathcal{K}(n, a, y_a, y_{h(a)}), \quad \forall \;\; n \in \mathcal{Z}^+ \tag{1.1}$$

where $h : \mathcal{Z}^+ \to \mathcal{Z}^+$ such that $h(n) \leq n$ on $\mathcal{Z}^+$, $\xi_n \in \mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$, $\mathcal{K} \in [{\mathcal{Z}^+}^2 \times \Phi^2, \Phi]$ and $y \in \mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$ is called a solution of equation (1.1). For validity, we checked our main results through the following non–linear summation equation having delay operator

$$y_n = \xi_n + \sum_{a=0}^{n} (n-a)^p \Big(y_a + y_{h(a)} + f_a \Big), \quad \forall \;\; n \in \mathcal{Z}^+ \tag{1.2}$$

where $\xi, f \in \mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$ and p is any arbitrary fixed integer.

Following is the organization of this manuscript. Section 2 contains basic definitions, auxiliary lemma and theorems regarding of the problem (1.1). Four types $\mathcal{HU}$ stabilities are presented in Section 3 and in section 4, we present examples for section 3.

2. Preliminaries

Let Φ be a Banach space, $\mathcal{Z}^+$ is the set of all positive integers and $\mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$ denote the space of all bounded linear operators with norm $\|\cdot\|_\infty$, defined by

$$\|f\|_\infty = \sup_{a \in \mathcal{Z}^+} \|f_a\|, \qquad f \in \mathcal{B}\big(\mathcal{Z}^+, \Phi\big). \tag{2.1}$$

Here, we give the definitions of $\mathcal{HUS}$, $\mathcal{GHUS}$, $\mathcal{HURS}$ and $\mathcal{GHURS}$ of the non–linear summation equation (1.1). For this, we focus on the following inequalities:

$$\left\| x_n - \xi_n - \lambda \sum_{a=0}^{n} \mathcal{K}(n, a, x(a), x(h(a))) \right\| \leq \epsilon, \tag{2.2}$$

$$\left\| x_n - \xi_n - \lambda \sum_{a=0}^{n} \mathcal{K}(n, a, x(a), x(h(a))) \right\| \leq \varphi_n \tag{2.3}$$

and

$$\left\| x_n - \xi_n - \lambda \sum_{a=0}^{n} \mathcal{K}(n, a, x(a), x(h(a))) \right\| \leq \epsilon \varphi_n, \tag{2.4}$$

where $\epsilon > 0$ and φ be a non–decreasing function.

Definition 2.1. *The summation equation* (1.1) *is said to be $\mathcal{HU}$ stable on $\mathcal{Z}^+$ if there exists a real number $\mathcal{N}_{\mathcal{K}}$ such that, for every $\epsilon > 0$ and for every sequence $x_n \in \mathcal{B}(\mathcal{Z}^+, \Phi)$ of* (2.2)*, there exists a solution $u_n \in \mathcal{B}(\mathcal{Z}^+, \Phi)$ of* (1.1) *with*

$$\|x_n - u_n\| \leq \mathcal{N}_{\mathcal{K}} \epsilon,$$

for all $n \in \mathcal{Z}^+$.

Definition 2.2. *The summation equation* (1.1) *is said to be $\mathcal{GHU}$ stable on $\mathcal{Z}^+$ if there is a non–decreasing function $\Upsilon_{\mathcal{K}}$ with $\Upsilon_{\mathcal{K}}(0) = 0$ such that, for every $\epsilon > 0$ and for every solution $x_n \in \mathcal{B}(\mathcal{Z}^+, \Phi)$ of* (2.2)*, there exists a solution $u_n \in \mathcal{B}(\mathcal{Z}^+, \Phi)$ of* (1.1) *with*

$$\|x_n - u_n\| \leq \Upsilon_{\mathcal{K}}(\epsilon),$$

for all $n \in \mathcal{Z}^+$.

Definition 2.3. *The summation equation* (1.1) *is said to be $\mathcal{HUR}$ stable on $\mathcal{Z}^+$ if there exists $\mathcal{N}_{\mathcal{K},\varphi} > 0$ such that, for every $\epsilon > 0$ and for every solution $x_n \in \mathcal{B}(\mathcal{Z}^+, \Phi)$ of* (2.4)*, there exists a solution $u_n \in \mathcal{B}(\mathcal{Z}^+, \Phi)$ of* (1.1) *with*

$$\|x_n - u_n\| \leq \mathcal{N}_{\mathcal{K},\varphi} \epsilon \varphi_n,$$

for all $n \in \mathcal{Z}^+$.

Definition 2.4. *The summation equation* (1.1) *is said to be $\mathcal{GHUR}$ stable on $\mathcal{Z}^+$ if there exists $\mathcal{N}_{\mathcal{K},\varphi} > 0$ such that, for every solution $x_n \in \mathcal{B}(\mathcal{Z}^+, \Phi)$ of* (2.3)*, there exists a solution $u_n \in \mathcal{B}(\mathcal{Z}^+, \Phi)$ of* (1.1) *with*

$$\|x_n - u_n\| \leq \mathcal{N}_{\mathcal{K},\varphi} \varphi_n,$$

for all $n \in \mathcal{Z}^+$.

Remark 2.5. *It is clear that: Definition 2.2$\Rightarrow$ Definition 2.1, Definition 2.3$\Rightarrow$ Definition 2.4 and for $\varphi = 1$ Definition 2.3$\Rightarrow$ Definition 2.1.*

Now, we present discrete Grönwall lemma (see [1]).

Lemma 2.6. *Let for $n \in \mathcal{Z}^+$ the inequality*

$$y_n \leq \xi_n + \sum_{a=0}^{n} \mathcal{H}(n, a) y_a,$$

holds, where $\xi \in \mathcal{B}(\mathcal{Z}^+, \Phi)$ is a nondecreasing sequence, $y \in \mathcal{B}(\mathcal{Z}^+, \Phi)$ and $\mathcal{H}(n, a)$ is a convergent sequence for $n \geq a \geq 0$. Then, for $n \geq 0$ the following inequality is hold:

$$y_n \leq \xi_n + \sum_{a=0}^{n} \mathcal{H}(n, a) \xi_a \prod_{r=a+1}^{n} (\mathcal{H}(n, r) + 1).$$

Lemma 2.7. *[32] Let $(\Phi, d, \leq)$ be an ordered metric space and $\Delta : \Phi \to \Phi$ be an increasing Picard operator with fixed point $y^* \in \Phi$. Then for any $y \in \Phi$, $y \leq \Delta(y)$ implies $y \leq y^*$ and $y \geq \Delta(y)$ implies $y \geq y^*$.*

Lemma 2.8. *[30] Let $T : \Phi \to \Phi$ is an operator on a metric space Φ such that for some $n \in \mathcal{Z}^+$ the operator T^n is strictly contractive on Φ, then T and T^n have same unique fixed point in Φ.*

3. Results

Our first result has given in this section.

Theorem 3.1. *Assume that $\xi \in \mathcal{B}(\mathcal{Z}^+, \Phi)$ and $\mathcal{K} \in {\mathcal{Z}^+}^2 \times \Phi^2 \to \Phi$ are convergent sequences with $\mathcal{K}$ satisfying the Lipschitz condition*

$$\Big\|\mathcal{K}(n, a, x_1, x_2) - \mathcal{K}(n, a, y_1, y_2)\Big\| \leq \mathcal{J}_{\mathcal{K}} \sum_{j=1}^{2} \|x_j - y_j\|,$$

for all $x_1, x_2, y_1, y_2 \in \Phi$ and $n \geq a$, $n, a \in \mathcal{Z}^+$, then
(C_1) The equation (1.1) has unique solution in $\mathcal{B}(\mathcal{Z}^+, \Phi)$.
(C_2) The equation (1.1) has $\mathcal{GHURS}$ on $\mathcal{Z}^+$.

Proof .(C_1) For $n \in \mathcal{Z}^+$, we defined an operator $\Delta : \mathcal{B}(\mathcal{Z}^+, \Phi) \to \mathcal{B}(\mathcal{Z}^+, \Phi)$ by

$$(\Delta y)(n) = \xi_n + \lambda \sum_{a=0}^{n} \mathcal{K}(n, a, y(a), y(h(a))). \tag{3.1}$$

Clearly, Δ is well-defined. We claim that

$$\sup_{n \in \mathcal{Z}^+} \Big\|(\Delta^j u)(n) - (\Delta^j x)(n)\Big\| \leq 2^j |\lambda|^j \mathcal{J}_{\mathcal{K}}^j (n+1)^j \|u - x\|_\infty, \tag{3.2}$$

for all $u, x \in \mathcal{B}(\mathcal{Z}^+, \Phi)$, $n \in \mathcal{Z}^+$ and positive integers j. To prove (3.2), we use mathematical induction:
For $j = 1$, consider

$$\begin{aligned}
\sup_{n \in \mathcal{Z}^+} \Big\|(\Delta u)(n) - (\Delta x)(n)\Big\| &= \sup_{n \in \mathcal{Z}^+} \Big\|\lambda \sum_{a=0}^{n} \mathcal{K}(n, a, u(a), u(h(a))) - \lambda \sum_{a=0}^{n} \mathcal{K}(n, a, x(a), x(h(a)))\Big\| \\
&\leq \sup_{n \in \mathcal{Z}^+} |\lambda| \sum_{a=0}^{n} \Big\|\mathcal{K}(n, a, u(a), u(h(a))) - \mathcal{K}(n, a, x(a), x(h(a)))\Big\| \\
&\leq \sup_{n \in \mathcal{Z}^+} |\lambda| \sum_{a=0}^{n} \mathcal{J}_{\mathcal{K}} \Big\{\Big\|u(a) - x(a)\Big\| + \Big\|u(h(a)) - x(h(a))\Big\|\Big\} \\
&\leq |\lambda| \mathcal{J}_{\mathcal{K}} \sum_{a=0}^{n} \Big\{ \sup_{a \in \mathcal{Z}^+} \Big\|u(a) - x(a)\Big\| + \sup_{a \in \mathcal{Z}^+} \Big\|u(h(a)) - x(h(a))\Big\|\Big\} \\
&\leq 2|\lambda| \mathcal{J}_{\mathcal{K}} (n+1) \|u - x\|_\infty.
\end{aligned}$$

Which agrees with (3.2) for $j = 1$. Now assume that (3.2) holds for $j = s$, i.e.

$$\sup_{n \in \mathcal{Z}^+} \Big\|(\Delta^s u)(n) - (\Delta^s x)(n)\Big\| \leq 2^s |\lambda|^s \mathcal{J}_{\mathcal{K}}^s (n+1)^s \|u - x\|_\infty, \tag{3.3}$$

is true for all $n \in \mathcal{Z}^+$. For $j = s+1$, using (3.3), we proceed as follows:

$$\begin{aligned}
\sup_{n\in\mathcal{Z}^+}\Big\|(\Delta^{s+1}u)(n)-(\Delta^{s+1}x)(n)\Big\| &= \sup_{n\in\mathcal{Z}^+}\Big\|\Delta(\Delta^s u)(n)-\Delta(\Delta^s x)(n)\Big\| \\
&= \sup_{n\in\mathcal{Z}^+}\Big\|\lambda\sum_{a=0}^{n}\mathcal{K}\big(n,a,(\Delta^s u)(a),(\Delta^s u)(h(a))\big) \\
&\quad -\lambda\sum_{a=0}^{n}\mathcal{K}\big(n,a,(\Delta^s x)(a),(\Delta^s x)(h(a))\big)\Big\| \\
&\le \sup_{n\in\mathcal{Z}^+}|\lambda|\sum_{a=0}^{n}\Big\|\mathcal{K}\big(n,a,(\Delta^s u)(a),(\Delta^s u)(h(a))\big) \\
&\quad -\mathcal{K}\big(n,a,(\Delta^s x)(a),(\Delta^s x)(h(a))\big)\Big\| \\
&\le \sup_{n\in\mathcal{Z}^+}|\lambda|\sum_{a=0}^{n}\mathcal{J}_{\mathcal{K}}\Big\{\big\|(\Delta^s u)(a)-(\Delta^s x)(a)\big\| \\
&\quad +\big\|(\Delta^s u)(h(a))-(\Delta^s x)(h(a))\big\|\Big\} \\
&\le 2|\lambda|\mathcal{J}_{\mathcal{K}}\sum_{a=0}^{n}\sup_{a\in\mathcal{Z}^+}\big\|(\Delta^s u)(a)-(\Delta^s x)(a)\big\| \\
&\le 2|\lambda|\mathcal{J}_{\mathcal{K}}2^s|\lambda|^s\mathcal{J}_{\mathcal{K}}^s(n+1)^s\|u-x\|_\infty\sum_{a=0}^{n} \\
&\le 2^{s+1}|\lambda|^{s+1}\mathcal{J}_{\mathcal{K}}^{s+1}(n+1)^{s+1}\|u-x\|_\infty.
\end{aligned}$$

Thus (3.3) holds for all integers j. Finally, we write

$$\sup_{n\in\mathcal{Z}^+}\Big\|(\Delta^j u)(n)-(\Delta^j x)(n)\Big\| \le \gamma_j\|u-x\|_\infty,$$

where

$$\gamma_j = 2^j|\lambda|^j\mathcal{J}_{\mathcal{K}}^j(n+1)^j.$$

For any fixed λ, $\mathcal{J}_{\mathcal{K}}$ and sufficiently large j we have $\gamma_j < 1$. Hence the corresponding operator Δ^j is strictly contractive on $\mathcal{B}\big(\mathcal{Z}^+,\Phi\big)$. Using Lemma 2.8, we conclude that Δ has a unique fixed point in $\mathcal{B}\big(\mathcal{Z}^+,\Phi\big)$. Following from (3.1), therefore the unique fixed point is in fact the unique solution of (1.1).

(C_2) Let $u \in \mathcal{B}\big(\mathcal{Z}^+,\Phi\big)$ be such that

$$\Big\|u_n-\xi_n-\lambda\sum_{a=0}^{n}\mathcal{K}(n,a,u(a),u(h(a)))\Big\| \le \varphi_n,\ \ n\in\mathcal{Z}^+,$$

where $\varphi : \mathcal{Z}^+ \to \mathcal{R}^+$ is non-decreasing sequence. We consider the absolute difference

$$\begin{aligned}\|u_n - y_n\| \leq& \bigg\| u_n - \xi_n - \lambda \sum_{a=0}^{n} \mathcal{K}(n, a, u(a), u(h(a))) \bigg\| \\ &+ \bigg\| \lambda \sum_{a=0}^{n} \mathcal{K}(n, a, u(a), u(h(a))) - \lambda \sum_{a=0}^{n} \mathcal{K}(n, a, y(a), y(h(a))) \bigg\| \\ \leq& \varphi_n + |\lambda| \mathcal{J}_{\mathcal{K}} \left[\sum_{a=0}^{n} \|u(a) - y(a)\| + \sum_{a=0}^{n} \|u(h(a)) - y(h(a))\| \right].\end{aligned}$$

For last inequality, we define an operator $\mathtt{T} : \mathcal{B}\big(\mathcal{Z}^+, \Phi\big) \to \mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$ by

$$(\mathtt{T}x)(n) = \varphi_n + |\lambda| \mathcal{J}_{\mathcal{K}} \left[\sum_{a=0}^{n} x(a) + \sum_{a=0}^{n} x(h(a)) \right], \quad n \in \mathcal{Z}^+.$$

We show that $\mathtt{T}$ is an increasing Picard operator. To do this, similarly as above we assert that

$$\sup_{n \in \mathcal{Z}^+} \bigg\| (\mathtt{T}^j u)(n) - (\mathtt{T}^j y)(n) \bigg\| \leq 2^j |\lambda|^j \mathcal{J}_{\mathcal{K}}^j (n+1)^j \|u - x\|_\infty,$$

for all $n \in \mathcal{Z}^+$ and positive integer j. Finally, we write

$$\sup_{n \in \mathcal{Z}^+} \bigg\| (\mathtt{T}^j u)(n) - (\mathtt{T}^j y)(n) \bigg\| \leq \gamma_j \|u - x\|_\infty,$$

where

$$\gamma_j = 2^j |\lambda|^j \mathcal{J}_{\mathcal{K}}^j (n+1)^j.$$

For any fixed λ and $\mathcal{J}_{\mathcal{K}}$, we can find a positive integers j, sufficiently large, such that $\gamma_j < 1$. Hence the corresponding operator $\mathtt{T}^j$ is strictly contractive on $\mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$. Using Lemma 2.8, we conclude that Δ has a unique fixed point u^* in $\mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$ i.e. $\mathtt{T}$ is a Picard operator on $\mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$. Thus,

$$u^*(n) = \varphi_n + |\lambda| \mathcal{J}_{\mathcal{K}} \left[\sum_{a=0}^{n} u^*(a) + \sum_{a=0}^{n} u^*(h(a)) \right], \ n \in \mathcal{Z}^+.$$

We see that $\Delta u^*(n) \geq 0$ for all $n \in \mathcal{Z}^+$, so u^* is increasing and hence $u^*(n) \geq u^*(h(n))$ for all $n \in \mathcal{Z}^+$. This leads us to

$$u^*(n) \leq \varphi_n + 2|\lambda| \mathcal{J}_{\mathcal{K}} \sum_{a=0}^{n} u^*(a), \ n \in \mathcal{Z}^+.$$

Using Grönwall Lemma 2.6, we get

$$\|u_n - y_n\| \leq \varphi_n + 2|\lambda| \mathcal{J}_{\mathcal{K}} \sum_{a=0}^{n} \varphi_a (2|\lambda| \mathcal{J}_{\mathcal{K}} + 1)^a. \tag{3.4}$$

In particular, if $x = \|u_n - y_n\|$, then $x_n \leq (\mathtt{T}x)(n)$ i.e. $x_n \leq u^*(n)$.

Let $\mathcal{N}_{\mathcal{K},\varphi} \varphi_n = 1 + 2|\lambda| \mathcal{J}_{\mathcal{K}} \sum\limits_{a=0}^{n} \frac{\varphi_a}{\varphi_n} (2|\lambda| \mathcal{J}_{\mathcal{K}} + 1)$, then (3.4) is written as:

$$\|u_n - y_n\| \leq \mathcal{N}_{\mathcal{K},\varphi} \varphi_n, \quad \text{for all} \quad n \in \mathcal{Z}^+.$$

□

In the following theorems, we state about the $\mathcal{HUS}$, $\mathcal{GHUS}$ and $\mathcal{HURS}$ of (1.1) on $\mathcal{Z}^+$. The proofs can be established by performing the same steps as that of above theorem.

Theorem 3.2. *Assume that $\xi \in \mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$ and $\mathcal{K} \in {\mathcal{Z}^+}^2 \times \Phi^2 \to \Phi$ are convergent sequences with $\mathcal{K}$ satisfying the Lipschitz condition*

$$\Big\|\mathcal{K}(n, a, x_1, x_2) - \mathcal{K}(n, a, y_1, y_2)\Big\| \leq \mathcal{J}_{\mathcal{K}} \sum_{j=1}^{2} \|x_j - y_j\|,$$

for all $x_1, x_2, y_1, y_2 \in \Phi$ and $n \geq a$, $n, a \in \mathcal{Z}^+$, then
(C_1) The equation (1.1) has unique solution in $\mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$.
(C_2) The equation (1.1) has $\mathcal{HUS}$ on $\mathcal{Z}^+$.

Theorem 3.3. *Assume that $\xi \in \mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$ and $\mathcal{K} \in {\mathcal{Z}^+}^2 \times \Phi^2 \to \Phi$ are convergent sequences with $\mathcal{K}$ satisfying the Lipschitz condition*

$$\Big\|\mathcal{K}(n, a, x_1, x_2) - \mathcal{K}(n, a, y_1, y_2)\Big\| \leq \mathcal{J}_{\mathcal{K}} \sum_{j=1}^{2} \|x_j - y_j\|,$$

for all $x_1, x_2, y_1, y_2 \in \Phi$ and $n \geq a$, $n, a \in \mathcal{Z}^+$, then
(C_1) The equation (1.1) has unique solution in $\mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$.
(C_2) The equation (1.1) has $\mathcal{GHUS}$ on $\mathcal{Z}^+$.

Theorem 3.4. *Assume that $\xi \in \mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$ and $\mathcal{K} \in {\mathcal{Z}^+}^2 \times \Phi^2 \to \Phi$ are convergent sequences with $\mathcal{K}$ satisfying the Lipschitz condition*

$$\Big\|\mathcal{K}(n, a, x_1, x_2) - \mathcal{K}(n, a, y_1, y_2)\Big\| \leq \mathcal{J}_{\mathcal{K}} \sum_{j=1}^{2} \|x_j - y_j\|,$$

for all $x_1, x_2, y_1, y_2 \in \Phi$ and $n \geq a$, $n, a \in \mathcal{Z}^+$, then
(C_1) The equation (1.1) has unique solution in $\mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$.
(C_2) The equation (1.1) has $\mathcal{HURS}$ on $\mathcal{Z}^+$.

4. Example

In this section, we present two examples of non–linear summation equation, having delay.

Example 4.1. *Let $\lambda > 0$, p be any arbitrary fixed constant and h is a delay operator such that $h(n) \leq n$ for all $n \in \mathcal{Z}^+$. Consider the non–linear summation equation having delay*

$$y_n = g_n + \lambda \sum_{a=0}^{n} (n-a)^p (y_a + y_{h(a)} + f_n), \tag{4.1}$$

for all $n \in \mathcal{Z}^+$, where g_n and f_n are convergent sequences.
Let $u_n : \mathcal{Z}^+ \to \Phi$ such that

$$\left\| u_n - g_n - \lambda \sum_{a=0}^{n} (n-a)^p (u_a + u_{h(a)} + f_n) \right\| < \varphi_n, \tag{4.2}$$

where φ be a non-decreasing sequence. Theorem 3.1 *convincing the existence of unique convergent sequence $u_n : \mathcal{Z}^+ \to \Phi$ that solves* (4.1) *and*

$$\begin{aligned} \|y_n - u_n\| &\leq \varphi_n + 2|\lambda| \sum_{a=0}^{n} (n-a)^p \varphi_a \prod_{r=a+1}^{n} \Big(2|\lambda|(n-r)^p + 1 \Big) \\ &\leq \mathcal{N}_{\mathcal{K},\varphi} \varphi_n, \end{aligned}$$

where $\mathcal{N}_{\mathcal{K},\varphi} = 1 + 2|\lambda| \sum\limits_{a=0}^{n} (n-a)^p \frac{\varphi_a}{\varphi_n} \prod\limits_{r=a+1}^{n} \Big(2|\lambda|(n-r)^p + 1 \Big)$. Hence, (4.1) *is $\mathcal{GHUR}$ stable.*

Example 4.2. *Let $\lambda > 0$, p be any arbitrary fixed constant and h is a delay operator such that $h(n) \leq n$ for all $n \in \mathcal{Z}^+$. Consider the non–linear summation equation with delay*

$$y_n = g_n + \lambda \sum_{a=0}^{n} (n-a)^p (y_a + y_{h(a)} + f_n), \tag{4.3}$$

for all $n \in \mathcal{Z}^+$, where g_n and f_n are convergent sequences.
Let $u : \mathcal{Z}^+ \to \Phi$ such that

$$\left\| u_n - g_n - \lambda \sum_{a=0}^{n} (n-a)^p (u_a + u_{h(a)} + f_n) \right\| < \epsilon, \tag{4.4}$$

where $\epsilon > 0$. Theorem 3.3 *ensure the existence of unique convergent sequence $u_n : \mathcal{Z}^+ \to \Phi$ that solves* (4.3) *and*

$$\begin{aligned} \|y_n - u_n\| &\leq \epsilon + 2\epsilon|\lambda| \sum_{a=0}^{n} (n-a)^p \prod_{r=a+1}^{n} \Big(2|\lambda|(n-r)^p + 1 \Big) \\ &\leq \Upsilon_{\mathcal{K}}(\epsilon). \end{aligned}$$

Hence, (4.3) *is $\mathcal{GHU}$ stable.*

Conclusion

In this manuscript, we exhibit the existence and uniqueness theorem for the solutions of a non-linear summation equation with delay on $\mathcal{B}\big(\mathcal{Z}^+, \Phi\big)$. Moreover, with the help of discrete Grönwall Lemma, we investigate four types of stabilities ($\mathcal{HUS}$, $\mathcal{GHUS}$, $\mathcal{HURS}$ and $\mathcal{GHURS}$) of non-linear summation equation with delay. A particular example for nonlinear summation equations are described to uphold our main outcomes.

Conflict of interests

The author says publicly that there is no contending interest concerning the paper.

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