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Convergence theorems of a new multiparametric family of Newton-like method in Banach space

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Abstract

In this work, we have considered a new multi-parametric family of modified Newton-like methods(MNL) of order three to approximate a zero of a nonlinear operator in $\mathbb{B}$-space (Banach space). Here, we studied the semilocal convergence analysis of this family of methods by using a new type of majorant condition. Note that this majorant condition generalizes the earlier majorant conditions used for studying convergence analysis of third order methods. Moreover, by using second-order directional derivative of the majorizing function we obtained an error estimate. We also established relations between our majorant condition and assumption based on Kantorovich, Smale-type and Nesterov-Nemirovskii-type, that will show our result generalize these earlier convergence results.

Keywords: Multi-parametric family of modified Newton-like (MNL) methods, Majorant conditions, Majorizing sequence, Majorizing function Nesterov-Nemirovskii condition, Smale-type assumption, Kantorovich-type assumption.
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1. Introduction

Assume $f : \Omega \subseteq W_1 \longrightarrow W_2$ is a non-linear operator, on Ω which is a non-empty open convex subset in a $\mathbb{B}$-space W_1 to another $\mathbb{B}$-space W_2. Finding solution σ of

$$f(w) = 0 \tag{1.1}$$

is a classical occurring problem that come out in several areas of scientific, engineering and mathematical computing area. Generally, such equations are nonlinear equations, differential equations,

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integral equations, algebraic equations and so on. Conventionally, iterative methods along with semilocal as well as local convergence are used for finding the approximating solution of these types of equations. Semi-local convergence provide information about the initial point, while through local convergence, we find radius of convergence ball and the information about the solution of Eq. (1.1). The most widely used one parametric iterative method is Newton's method [13] defined as $w_{n+1} = w_n - f'(w_n)^{-1} f(w_n)$ with initial approximation w_0, which is quadratic convergent.

Due to less order of convergence, many researchers studied higher order iterative methods, to do the same work. These iterative methods are Chebyshev method, Halley's method, Super-Halley method, Euler-Chebyshev method, Newton-like, Househölder-like method and so on [2, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24], which have third order of convergence or more than two. Many researchers have worked for semilocal and local convergence analysis of these iterative methods for solving the nonlinear operator equation (1.1) by using majorizing sequence, Recurrence relation and so on under the γ-condition, centered Lipschitz condition, Lipschitz condition, Hölder condition, majorant condition and so on[3, 4, 14, 17, 18, 19, 20, 21, 22, 23, 24, 27]. When Eq.(1.1) is a stiff type differential equation, for finding its solution we need to improve the speed of convergence of Newton method. For this, in 1993 Aygrous [1] introduce Newton-type method, which has R-order of convergence at least two

$$w_{n+1} = w_n - \Lambda(\pounds_f(w_n)) \Gamma_n f(w_n), \qquad n \geq 0. \tag{1.2}$$

Here, $f'(w_n)$ is first and $f''(w_n)$ is second Fréchet-derivative of f, $\Gamma_n = f'(w_n)^{-1}$, $\pounds_f(w_n)$ is defined by

$$\pounds_f(w_n) = \Gamma_n f''(w_n) \Gamma_n f(w_n), \qquad w_n \in W_1.$$

and

$$\Lambda : \pounds(W_1, W_1) \longrightarrow \pounds(W_1, W_1),$$

where $\pounds(W_1, W_1)$ is the set of bounded linear operators from W_1 into W_1. When $W_1 = W_2 = \mathbb{R}$, in order to find the function Λ such that the iterative method has R-order of convergence at least three, Gander[6] provided that if $\Lambda(0) = 1$, $\Lambda'(0) = \frac{1}{2}$ and $|\Lambda''(w_1)| < \infty$, the method (1.2) is an iterative method with R-order of convergence at least three. Observe that, the well-known iterative methods with R-order of convergence at least three will fit into the form of algorithm (1.2), when $\Lambda(L_f(w_n))$ is in the following form:

Chebyshev Method [5]:

$$\Lambda(\pounds_f(w_n)) = I + \frac{1}{2}\pounds_f(w_n) :$$

Super-Halley Method[1]:

$$\Lambda(\pounds_f(w_n)) = I + \frac{1}{2}\pounds_f(w_n) + \sum_{m \geq 2} \frac{1}{2}\pounds_f(w_n)^m;$$

Halley Method[8]:

$$\Lambda(\pounds_f(w_n)) = I + \frac{1}{2}\pounds_f(w_n) + \sum_{m \geq 2} \frac{1}{2^m}\pounds_f(w_n)^m.$$

where I is the identity operator. Another famous method for solving Eq. (1.1) is MNL method [9, 10, 11, 15], which is cubically convergent and defined as:

$$w_{n+1} = w_n - \left[I + \frac{1}{2}\pounds_f(w_n)\varLambda_f(w_n)\right]\varGamma_n f(w_n), \quad n \geq 0. \tag{1.3}$$

where

$$\varGamma_n = f'(w_n)^{-1}, \quad \varLambda_f(w_n) = \varLambda(\pounds_f(w_n)) = \left[I + \sum_{m\geq 2} 2S_m \pounds_f(w_n)^{m-1}\right], \qquad S_m \in \mathbb{R}^+, \quad m \geq 2,$$

Here, we suppose that there exist $r > 0$ such that the series $\sum_{m\geq 2} 2S_m t^{m-1}$ is convergent for $|t| < r$. Due to third order of convergence, MNL method (1.3) is far better than Newton's method. In 2005, Hernández and Romero [9] provided the semilocal convergence of this method under Lipschitz condition $\|f''(v) - f''(w)\| \leq K\|v - w\|$, where $v, w \in \varOmega$, $K > 0$, on the second derivative and proved the method is of R-order at least three. The Lipschitz condition is further weakened by ω-condition [10, 11] i.e. $\|f''(v) - f''(w)\| \leq \omega\big(\|v - w\|\big)$, where $v, w \in \varOmega$, and $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ continuous and non-decreasing, $\omega(0) > 0$ to get better results when the operator does not satisfy Lipschitz condition.

Suppose that there exist $\wp > 0$ and $g : (0, \wp) \longrightarrow \mathbb{R}$ be function of class C^2 on $(0, \wp)$ such that $B(w_0, \wp) \subseteq \varOmega$

$$\|\varGamma_0[f''(v) - f''(w)]\| \leq g''(\|v - w\| + \|w - w_0\|) - g''(\|w - w_0\|) \tag{1.4}$$

for $w, v \in B(w_0, \wp)$, $\|v - w\| + \|w - w_0\| < \wp$, and the following assumption hold:

I. $g(0) > 0$, $g''(0) > 0$, $g'(0) = -1$.

II. g'' is strictly increasing and convex in $(0, \wp)$.

III. $g(\alpha) = 0$ for some $\alpha \in (0, \wp)$. Then g has a minimal zero $\bar{\alpha}$ and $g'(\bar{\alpha}) < 0$.

Also, the following initial condition also hold:

$$\|\varGamma_0 f(w_0)\| \leq g(0) \ \ \textit{and} \ \ \|\varGamma_0 f''(w_0)\| \leq g''(0). \tag{1.5}$$

Note that this majorant condition is a generalization of Lipschitz condition, Smale-type condition and Nesterov-Nemirovskii type assumption. The details of these will be discussed in section 3. Using this majorant condition Ling and Xu [16] presented semilocal convergence of Halley's method. On the other hand Argyros and Ren [3] presented local convergence of third order methods such as Halley's method and Chebyshev's method under similar majorant condition on second derivative of f given above. In 2018 [14], we also presented local convergence of Chebyshev's method for solving Eq. (1.1) which is also a third order iterative method by using majorant function and and provided error estimate.

Inspired with the idea of Ling and Xu [16], In this article we present the semilocal convergence of MNL method (1.3) by using the new type of majorant condition. By using majorant function and their condition, we proved that the MNL method (1.3) is cubically convergent. Here, we establish relation between the nonlinear operator and majorant function. Additionally, we obtain a new priori and posteriori error estimate based on twice directional derivative of the majorant function. We also present the convergence result based on Kantorovich, Smale-type and Nesterov-Nemirovskii type assumption which are particular cases of above mentioned majorant condition. Three numerical examples are also presented to show the efficiency of our study.

The organization of our article has been given as follows: In subsection 1.1, we present some auxiliary results and notations. In section 2, we state and prove the main result, establish relation between non-linear operator and majorizing function. In section 3, we provide special cases of our main result. By using Kantorovich-type, Smale-type and Nesterov-Nemirovskii-type assumption, we present semilocal convergence theorems for MNL method (1.3). We also give some examples to show the efficiency of our method in subsection 3.1, 3.2 and 3.3, respectively.

1.1. *Notation and auxiliary results*

Throughout the paper we will assume that W_1 and W_2 be a $\mathbb{B}$-spaces. Let $B(w,\wp) = \{w \in W_1 : \|z-w\| < \wp\}$ be the open ball with $\wp > 0$ and $\bar{B}(w,\wp)$ be its closure. The following auxiliary results are recalled on scalar valued functions which are given in any elementary convex analysis books. These results are important one and will be used in our analysis.

Lemma 1.1. *[16, 12] Let $\wp > 0$. If $p : (0,\wp) \longrightarrow \mathbb{R}$ is continuously differentiable and convex, then*

(i_1) $(1-\phi)p'(\phi\alpha) \leq \dfrac{p(\alpha)-p(\phi\alpha)}{\alpha} \leq (1-\phi)p'(\alpha). \quad \forall\ s \in (0,\wp)$ *and* $0 \leq \phi \leq 1.$

(i_2) $\dfrac{p(u_1)-p(\phi u_1)}{u_1} \leq \dfrac{p(u_2)-p(\phi u_2)}{u_2}. \qquad \forall\ u_1,u_2 \in (0,\wp),\ u_1 < u_2$ *and* $0 \leq \phi \leq 1.$

Lemma 1.2. *([16, 12]) Let $I \subset \mathbb{R}$ be an interval and $p : I \longrightarrow \mathbb{R}$ be convex. Then*

(i_1) For any $u_0 \in int(I)$, there exists (in $\mathbb{R}$)

$$D^{-}p(u_0) := \lim_{z \to u_0^{-}} \frac{p(u_0)-p(u)}{u_0-u} = \sup_{u<u_0} \frac{p(u_0)-p(u)}{u_0-u}. \tag{1.6}$$

(i_2) If $u_1,u_2,u_3 \in I$ and $u_1 \leq u_2 \leq u_3$, then

$$p(u_2)-p(u_1) \leq [p(u_3)-p(u_1)]\frac{u_2-u_1}{u_3-u_1}.$$

2. Main Result

Here, our aim is to state and prove the semilocal convergence analysis of MNL method with the help of majorizing function and their properties. First we prove some basic results about the majorant function and will establish the relation between nonlinear operators and the majorant function. Finally we will show that the MNL method is well defined and converges with Q-cubic. We also shows its uniqueness in suitable region. The statement of the theorem is:

Theorem 2.1. *Let W_1 and W_2 be $\mathbb{B}$-space and $\Omega \subset W_1$, Ω is a non-empty open convex subset in $\mathbb{B}$-space and $f : \Omega \longrightarrow W_2$ be a twice continuously differentiable nonlinear operator. Take $w_0 \in \Omega$, w_0 is a initial point with $\Gamma_0 = f'(w_0)^{-1}$. Suppose that there exist $\wp > 0$ and $g : (0,\wp) \longrightarrow \mathbb{R}$ be function of class C^2 on $(0,\wp)$ such that $B(w_0,\wp) \subseteq \Omega$*

$$\|\Gamma_0[f''(v)-f''(w)]\| \leq g''(\|v-w\|+\|w-w_0\|) - g''(\|w-w_0\|), \tag{2.1}$$

for $w,v \in B(w_0,\wp)$, $\|v-w\|+\|w-w_0\| < \wp$, and the following assumption hold:

I. $g(0) > 0,\ g''(0) > 0,\ g'(0) = -1.$

II. g'' *is strictly increasing and convex in* $(0, \wp)$.

III. $g(\alpha) = 0$ *for some* $\alpha \in (0, \wp)$. *Then* g *has a minimal zero* $\bar{\alpha}$ *and* $g'(\bar{\alpha}) < 0$.

The following initial condition also hold:

$$\|\Gamma_0 f(w_0)\| \leq g(0) \ \ \text{and} \ \ \|\Gamma_0 f''(w_0)\| \leq g''(0). \tag{2.2}$$

Then the sequence $\{w_n\}$ *and* $\{\alpha_n\}$ *obtained by applying MNL method on operator* f *and function* g *respectively, where*

$$\alpha_{n+1} = \alpha_n - \left(1 + \frac{1}{2}\pounds_g(\alpha_n)\Lambda_g(\alpha_n)\right).\frac{g(\alpha_n)}{g'(\alpha_n)} \tag{2.3}$$

with,

$$\Lambda_g(\alpha_n) = 1 + \sum_{m\geq 2} 2S_m \pounds_g(\alpha_n)^{m-1}, \ \ \text{and} \ \ \pounds_g(\alpha_n) = \frac{g(\alpha_n)g''(\alpha_n)}{(g'(\alpha_n))^2} \qquad S_m \in \mathbb{R}^+, \ \ m \geq 2$$

for solving $f(w) = 0$ *and* $g(\alpha) = 0$ *with initial point* w_0 *and* $\alpha_0 = 0$ *are well defined. It is to be mentioned that* $\{\alpha_n\}$ *is strictly increasing, contained in* $(0, \bar{\alpha})$*and converges to* $\bar{\alpha}$. *Also,* $\{w_n\}$ *is contained in* $B(w_0, \bar{\alpha})$ *and converge to a point* $\sigma \in \bar{B}(w_0, \bar{\alpha})$, *which is the solution of Eq.* (1.1).

Remark 2.2. *Under Theorem 2.1, assumption* (I.) – (III.) *on* $g : (0, \wp) \longrightarrow \mathbb{R}$,

1. $g(\alpha) = 0$ *has at most one zero on* $(\bar{\alpha}, \wp)$, *where* $\bar{\alpha}$ *is the minimal zero of* g *in* $[0, \wp)$.

   The condition $g'(\bar{\alpha}) < 0$ *in* (III.) *implies the following properties:*

2. $g(\bar{\alpha}^*) = 0$ *for some* $\bar{\alpha}^* \in (\bar{\alpha}, \wp)$.

3. $g(\alpha) < 0$ *for some* $\alpha \in (\bar{\alpha}, \wp)$.

For proving this theorem we use the following Lemma. In this Lemma we prove some basic properties of the function g.

Lemma 2.3. *Let* $\wp > 0$, *and* $g : (0, \wp) \longrightarrow \mathbb{R}$ *be function of class* C^2 *on* $(0, \wp)$ *which satisfies the condition* (I.) - (III.), *then the following hold:*

(a_1) g' *is strictly convex and strictly increasing in* $(0, \wp)$.

(a_2) g *is strictly convex on* $(0, \wp)$, *for* $\alpha \in (0, \bar{\alpha})$ $g(\alpha) > 0$ *and equation* $g(\alpha) = 0$ *has at most one root in* $(\bar{\alpha}, \wp)$.

(a_3) *for* $\alpha \in (0, \bar{\alpha})$, $g'(\alpha) \in (-1, 0)$.

Proof . From (II.), we have g'' is convex and strictly increasing in $(0,\wp)$ and $g''(0)>0$. Then g' can be concluded as a strictly convex and strictly increasing on $(0,\wp)$, which proves (a_1).
(a_1) implies that g is strictly convex on $(0,\wp)$. By the condition (I.), (a_1) and $g(\bar{\alpha})=0$, we get $g(\alpha)=0$ has at most one root on $(\bar{\alpha},\wp)$. Since $g(\bar{\alpha})=0$ and $g(0)>0$, one has $g(\alpha)>0$ for $\alpha\in(0,\bar{\alpha})$, which proves (a_2).
Since g is strictly convex, from Lemma 1.1, we obtain

$$g'(\alpha)<\frac{g(\bar{\alpha})-g(\alpha)}{\bar{\alpha}-\alpha},\qquad \alpha\in(0,\bar{\alpha}),$$

this implies that $g(\alpha)+g'(\alpha)(\bar{\alpha}-\alpha)<g(\bar{\alpha})$. Here $g(\bar{\alpha})=0$ and $g(\alpha)>0$ in $(0,\bar{\alpha})$ then we get $g'(\alpha)<0$. From (a_1) and (I.), g' is strictly increasing and $g'(\alpha)>1$ for $\alpha\in(0,\bar{\alpha})$, thus $-1<g'(\alpha)<0$ for $\alpha\in(0,\bar{\alpha})$, which proves (a_3).
Hence the Lemma is proved. □
Here, we will consider the majorant function g and prove required results regarding only the sequence $\{\alpha_n\}$. Suppose that g is the majorizing function to f. Then MNL method(1.3) applied to g can be denoted as

$$\alpha_0=0,\qquad \alpha_{n+1}=\Phi_g(\alpha_n),\qquad n=0,1,...,$$

where,

$$\Phi_g(\alpha):=\alpha-\left(1+\frac{1}{2}\pounds_g(\alpha)\Lambda_g(\alpha)\right).\frac{g(\alpha)}{g'(\alpha)},\tag{2.4}$$

$$\Lambda_g(\alpha)=1+\sum_{m\geq 2}2S_m\pounds_g(\alpha)^{m-1},\qquad S_m\in\mathbb{R}^+,\quad m\geq 2$$

and $\pounds_g(\alpha)=\dfrac{g(\alpha)g''(\alpha)}{(g'(\alpha))^2}$. For obtaining the convergence of the majorizing sequence generated by MNL method on the majorizing function, we need some useful Lemma.

Lemma 2.4. *Let $g\colon(0,\wp)\longrightarrow\mathbb{R}$ be a family of $C^2(0,\wp)$ and satisfy conditions* (I.)–(III.). *Then we have $0\leq\pounds_g(\alpha)\leq\frac{1}{2}$ for $\alpha\in[0,\bar{\alpha}]$.*

Proof . Here we define a function

$$\theta(t)=g(\alpha)+g'(\alpha)(t-\alpha)+\frac{1}{2}g''(\alpha)(t-\alpha)^2,\quad t\in[\alpha,\bar{\alpha}].$$

Now,

$$\theta(\alpha)=g(\alpha)+g'(\alpha)(\alpha-\alpha)+\frac{1}{2}g''(\alpha)(\alpha-\alpha)^2=g(\alpha).$$

Then by Lemma 2.3 (a_2), $g(\alpha)>0$ for $\alpha\in(0,\bar{\alpha})$, implies $\theta(\alpha)=g(\alpha)>0$ for $\alpha\in(0,\bar{\alpha})$. Additionally, we have

$$\theta(\bar{\alpha})=g(\alpha)+g'(\alpha)(\bar{\alpha}-\alpha)+\frac{1}{2}g''(\alpha)(\bar{\alpha}-\alpha)^2.\tag{2.5}$$

By Taylor's formula,

$$\begin{aligned} g(\bar{\alpha}) &= g(\alpha) + g'(\alpha)(\bar{\alpha}-\alpha) + \frac{1}{2}g''(\alpha)(\bar{\alpha}-\alpha)^2 \\ &\quad + \int_0^1 (1-\psi)[g''(\alpha+\psi(\bar{\alpha}-\alpha)) - g''(\alpha)](\bar{\alpha}-\alpha)^2 d\psi. \end{aligned} \tag{2.6}$$

As g'' is increasing and $g(\bar{\alpha}) = 0$, then from Eq. (2.5) and (2.6) we get, $\theta(\bar{\alpha}) \leq 0$. Thus, there exist a real root of $\theta(t)$ on $[\alpha, \bar{\alpha}]$. So the discriminant of $\theta(t)$is greater than or equal to zero i.e.

$$g'(\alpha)^2(t-\alpha)^2 - 4 \times \frac{1}{2}g(\alpha)g''(\alpha)(t-\alpha)^2 \geq 0$$

or

$$g'(\alpha)^2 - 2g(\alpha)g''(\alpha) \geq 0$$

which is equivalent to $0 \leq \dfrac{g(\alpha)g''(\alpha)}{(g'(\alpha))^2} \leq \frac{1}{2}$. Hence, $0 \leq \pounds_g(\alpha) \leq \frac{1}{2}$ for $\alpha \in [0, \bar{\alpha}]$. The proof is complete. $\square$

Lemma 2.5. *Let* $g\colon (0, \wp) \longrightarrow \mathbb{R}$ *be a twice continuously differentiable function and satisfy the assumption* (I.)-(III.). *Then,*

(i_1)

$$\forall\alpha \in (0, \bar{\alpha}), \quad g(\alpha) > 0, \quad g'(\alpha) < 0, \quad \alpha < \varPhi_g(\alpha) < \bar{\alpha}. \tag{2.7}$$

Moreover, $g'(\bar{\alpha}) \leq 0$

$$g'(\bar{\alpha}) < 0 \quad \rightleftharpoons \quad \exists\alpha \in (\bar{\alpha}, \wp), \quad \textit{such that} \quad g(\alpha) < 0. \tag{2.8}$$

(i_2)

$$\bar{\alpha} - \varPhi_g \leq -\frac{1}{g'(\bar{\alpha})}\left[\frac{1}{6}D^- g''(\bar{\alpha}) + \frac{1}{8}\left(1 + \sum_{m\geq 2}\left(\frac{1}{2}\right)^{m-2} S_m\right)\frac{g''(\bar{\alpha})}{(\bar{\alpha}-\alpha)}\right](\bar{\alpha}-\alpha)^3, \tag{2.9}$$

$\alpha \in (0, \bar{\alpha}), \quad S_m \in \mathbb{R}^+$ *and* $m \geq 2$.

Proof . For $\alpha \in (0, \bar{\alpha})$, since (from Lemma 2.3) $g(\alpha) > 0$, $-1 < g'(\alpha) < 0$ and (from Lemma 2.6) we have $0 \leq \pounds_g(\alpha) \leq \frac{1}{2}$, thus $\alpha < \varPhi_g(\alpha)$. Furthermore, for any $\alpha \in (0, \bar{\alpha}]$, by the definition of directional derivatives (1.6) and the condition (II.), we get $D^- g''(\alpha) > 0$. Thus we have

$$\begin{aligned} D^-\varPhi_g(\alpha) &= \frac{1}{2}\frac{g(\alpha)g''(\alpha)}{g'(\alpha)^2} - \frac{1}{2}\frac{D^- g''(\alpha)}{g'(\alpha)} + \frac{3}{2}\frac{g(\alpha)^2 g''(\alpha)^2}{g'(\alpha)^4} \\ &\quad + \sum_{m\geq 2} S_m\Bigg(- \frac{(m+1)g(\alpha)g''(\alpha)^{m+1}}{g'(\alpha)^{2m+1}} - \frac{mg(\alpha)^{m+1}g''(\alpha)D^- g''(\alpha)^{m-1}}{g'(\alpha)^{2m+1}} \\ &\quad + \frac{2mg(\alpha)^{m+1}g''(\alpha)^{3m-1}}{g'(\alpha)^{4m}} + \frac{g(\alpha)^{m+1}g''(\alpha)^{m+1}}{g'(\alpha)^{2m+2}}\Bigg) > 0, \end{aligned}$$

$\alpha \in (0, \bar{\alpha}]$, $S_m \in \mathbb{R}^+$ *and* $m \geq 2$. Thus $\Phi_g(\alpha)$ is strictly increasing. This implies that $\Phi_g(\alpha) < \Phi_g(\bar{\alpha}) = \bar{\alpha}$. Hence $\alpha < \Phi_g(\alpha) < \bar{\alpha}$. Thus first part of (i_1) is proved.
For second part of (i_1), if $g'(\bar{\alpha}) < 0$, then it is obvious that there exists $\alpha \in (\bar{\alpha}, \wp)$ such that $g(\alpha) < 0$. Conversely, we know that $g(\bar{\alpha}) = 0$ from (III.). By Lemma 1.2, we have

$$g'(\bar{\alpha}) < \frac{g(\alpha) - g(\bar{\alpha})}{\alpha - \bar{\alpha}} \qquad \bar{\alpha} \in (\bar{\alpha}, \wp)$$

or

$$g(\alpha) > g(\bar{\alpha}) + g'(\bar{\alpha})(\alpha - \bar{\alpha}) \qquad \bar{\alpha} \in (\bar{\alpha}, \wp),$$

which implies $g'(\bar{\alpha}) < 0$. This proves the second part of (i_1). By the definition of Φ_g in (2.4), we obtain

$$\begin{aligned}
\bar{\alpha} - \Phi_g &= (\bar{\alpha} - \alpha) + \left[1 + \frac{1}{2}\pounds_g(\alpha)\left(1 + \sum_{m\geq 2} 2S_m \pounds_g(\alpha)^{m-1}\right)\right] . \frac{g(\alpha)}{g'(\alpha)} \\
&= (\bar{\alpha} - \alpha) + \frac{g(\alpha)}{g'(\alpha)} + \frac{1}{2}\pounds_g(\alpha) . \frac{g(\alpha)}{g'(\alpha)} + \left(\sum_{m\geq 2} S_m \pounds_g(\alpha)^m\right) . \frac{g(\alpha)}{g'(\alpha)} \\
&= \frac{1}{g'(\alpha)}\big[(\bar{\alpha} - \alpha)g'(\alpha) + g(\alpha)\big] + \frac{1}{2}\frac{\pounds_g(\alpha)}{g'(\alpha)} g(\alpha) + \frac{1}{g'(\alpha)}\left(\sum_{m\geq 2} S_m \pounds_g(\alpha)^m\right) g(\alpha) \\
&= -\frac{1}{g'(\alpha)}\int_0^1 [g''(\alpha + \vartheta(\bar{\alpha} - \alpha)) - g''(\alpha)](\bar{\alpha} - \alpha)^2(1 - \vartheta)d\vartheta \\
&\quad - \frac{1}{2}\frac{\pounds_g(\alpha)}{g'(\alpha)}\int_0^1 g''(\alpha + \vartheta(\bar{\alpha} - \alpha))(\bar{\alpha} - \alpha)^2(1 - \vartheta)d\vartheta \\
&\quad - \frac{1}{g'(\alpha)}\left(\sum_{m\geq 2} S_m \pounds_g(\alpha)^m\right)\int_0^1 g''(\alpha + \vartheta(\bar{\alpha} - \alpha))(\bar{\alpha} - \alpha)^2(1 - \vartheta)d\vartheta.
\end{aligned} \tag{2.10}$$

Since g'' is convex and $\alpha < \bar{\alpha}$, then by Lemma 1.2 (i_2) we get,

$$g''(\alpha + \vartheta(\bar{\alpha} - \alpha)) - g''(\alpha) \leq [g''(\bar{\alpha}) - g''(\alpha)]\frac{\vartheta(\bar{\alpha} - \alpha)}{\bar{\alpha} - \alpha}.$$

Thus, by using above inequality in Eq. (2.10),

$$\begin{aligned}
\bar{\alpha} - \Phi &\leq -\frac{1}{g'(\alpha)}\int_0^1 [g''(\bar{\alpha}) - g''(\alpha)](\bar{\alpha} - \alpha)^2\vartheta(1 - \vartheta)d\vartheta \\
&\quad - \frac{1}{2}\frac{\pounds_g(\alpha)}{g'(\alpha)}\int_0^1 g''(\bar{\alpha})(\bar{\alpha} - \alpha)^2(1 - \vartheta)d\vartheta \\
&\quad - \frac{1}{g'(\alpha)}\left(\sum_{m\geq 2} S_m \pounds_g(\alpha)^m\right)\int_0^1 g''(\bar{\alpha})(\bar{\alpha} - \alpha)^2(1 - \vartheta)d\vartheta \\
&\leq -\frac{1}{6}\frac{g''(\bar{\alpha}) - g''(\alpha)}{g'(\alpha)}(\bar{\alpha} - \alpha)^2 - \frac{1}{4}\pounds_g(\alpha)\frac{g''(\bar{\alpha})}{g'(\alpha)}(\bar{\alpha} - \alpha)^2 \\
&\quad - \frac{1}{2}\left(\sum_{m\geq 2} S_m \pounds_g(\alpha)^m\right)\frac{g''(\bar{\alpha})}{g'(\alpha)}(\bar{\alpha} - \alpha)^2.
\end{aligned} \tag{2.11}$$

By condition (I.), (II.) and Lemma 2.3 we have, $g''(0) > 0$, $g'(\alpha) < 0$ and g', g'' are strictly increasing on $(0, \bar{\alpha})$. By Lemma 2.4, $0 \leq \pounds_g(\alpha) \leq \frac{1}{2}$, then above relation in Eq. (2.11) can be reduced to

$$\begin{aligned}
\bar{\alpha} - \varPhi_g \leq & -\frac{1}{6}\frac{g''(\bar{\alpha}) - g''(\alpha)}{g'(\alpha)}(\bar{\alpha} - \alpha)^2 - \frac{1}{8}\frac{g''(\bar{\alpha})}{g'(\alpha)}(\bar{\alpha} - \alpha)^2 \\
& - \frac{1}{2}\left(\sum_{m\geq 2} S_m \left(\frac{1}{2}\right)^m\right)\frac{g''(\bar{\alpha})}{g'(\alpha)}(\bar{\alpha} - \alpha)^2 \\
\leq & -\frac{1}{6}\frac{g''(\bar{\alpha}) - g''(\alpha)}{g'(\alpha)}(\bar{\alpha} - \alpha)^2 \\
& - \frac{1}{8}\left(1 + \sum_{m\geq 2}\left(\frac{1}{2}\right)^{m-2} S_m\right)\frac{g''(\bar{\alpha})}{g'(\alpha)}(\bar{\alpha} - \alpha)^2.
\end{aligned}$$

As g' is increasing, $g'(\bar{\alpha}) < 0$ and $g'(\alpha) < 0$, we have

$$\begin{aligned}
\frac{g''(\bar{\alpha}) - g''(\alpha)}{-g'(\alpha)} \leq \frac{g''(\bar{\alpha}) - g''(\alpha)}{-g'(\bar{\alpha})} &= \frac{1}{-g'(\bar{\alpha})}\frac{g''(\bar{\alpha}) - g''(\alpha)}{\bar{\alpha} - \alpha}(\bar{\alpha} - \alpha) \\
&\leq \frac{D^- g''(\bar{\alpha})}{-g'(\bar{\alpha})}(\bar{\alpha} - \alpha).
\end{aligned}$$

Then,

$$\bar{\alpha} - \varPhi_g \leq -\frac{1}{g'(\bar{\alpha})}\left[\frac{1}{6}D^- g''(\bar{\alpha}) + \frac{1}{8}\left(1 + \sum_{m\geq 2}\left(\frac{1}{2}\right)^{m-2} S_m\right)\frac{g''(\bar{\alpha})}{(\bar{\alpha} - \alpha)}\right](\bar{\alpha} - \alpha)^3,$$

and hence (i_2) is proved. This complete the proof. □

Now, in the following two lemmas we will establish the relation between the majorant function g and the nonlinear operator f.

Lemma 2.6. *Suppose $\|w - w_0\| \leq \alpha < \bar{\alpha}$. If $g\colon (0, \bar{\alpha}) \longrightarrow \mathbb{R}$ be a family of $C^2(0, \bar{\alpha})$ and majorizing function of f at w_0. Then $\varGamma = f'(w)^{-1} \in \pounds(W_2, W_1)$, and*

(i_1)
$$\|\varGamma f'(w_0)\| \leq -g'(\|w - w_0\|)^{-1} \leq -g'(\alpha)^{-1}. \tag{2.12}$$

(i_2)
$$\|\varGamma_0 f''(w)\| \leq g''(\|w - w_0\|) \leq g''(\alpha). \tag{2.13}$$

Proof . Let $w \in \bar{B}(w_0, \alpha)$ and $0 \leq \alpha \leq \bar{\alpha}$. By Taylor series we have

$$\begin{aligned}
f'(w) = f'(w_0) + &\int_0^1 [f''(w^\vartheta) - f''(w_0)](w - w_0)d\vartheta \\
&+ f''(w_0)(w - w_0)
\end{aligned}$$

where $w^\vartheta = w_0 + \vartheta(w - w_0)$ or

$$\begin{aligned}
\varGamma_0[f'(w) - f'(w_0)] = &\int_0^1 \varGamma_0[f''(w^\vartheta) - f''(w_0)](w - w_0)d\vartheta \\
&+ \varGamma_0 f''(w_0)(w - w_0)
\end{aligned}$$

where, $\Gamma_0 = f'(w_0)^{-1}$. Use majorant condition (I.), Eq. (2.1), initial condition given in Eq. (2.2) and Lemma 2.3 (a_3) in above inequality, to get

$$\begin{aligned}\|\Gamma_0\big[f'(w)-f'(w_0)\big]\| &\leq \int_0^1 [g''(\vartheta\|w-w_0\|)-g''(0)]\|w-w_0\|d\vartheta \\ &\quad + g''(0)\|w-w_0\| \\ &\leq g'(\|w-w_0\|)-g'(0) \leq g'(\alpha)-g'(0) < 1.\end{aligned}$$

This implies that,

$$\|\Gamma_0 f'(w) - I\| < 1.$$

Then $\mathbb{B}$-space Lemma on invertible operator [13, 25] imply that $\Gamma \in \pounds(W_2, W_1)$ and

$$\begin{aligned}\|\Gamma f'(w_0)\| &\leq \frac{1}{1-(g'(\|w-w_0\|)-g'(0))} \\ &\leq -\frac{1}{g'(\|w-w_0\|)} \leq -\frac{1}{g'(\alpha)}.\end{aligned}$$

which shows that (i_1) is proved. Now we use Eq. (2.1), initial condition Eq. (2.2) and by condition (II.), g'' is strictly increasing, then we obtain

$$\begin{aligned}\|\Gamma_0 f''(w)\| &\leq \|\Gamma_0[f''(w)-f''(w_0)]\| + \|\Gamma_0 f''(w_0)\| \\ &\leq g''(\|w-w_0\|)-g''(0)+g''(0) \\ &= g''(\|w-w_0\|) \leq g''(\alpha).\end{aligned}$$

which proves (i_2). $\square$

Lemma 2.7. *Assume that $g : (0, \bar{\alpha}) \longrightarrow \mathbb{R}$ be a family of $C^2(0, \bar{\alpha})$ and let the sequence $\{w_n\}$ and $\{\alpha_n\}$ be generated by MNL method given by (1.3) and (2.3) respectively. If g is the majorizing function to f at w_0, then, for all $n \geq 1$, we have*

- *(a_1) Γ_n exists and $\|\Gamma_n f'(w_0)\| \leq -g'(\|w_n - w_0\|)^{-1} \leq -g'(\alpha_n)^{-1}$.*
- *(a_2) $\|\Gamma_0 f''(w_n)\| \leq g''(\alpha_n)$.*
- *(a_3) $\|\Gamma_0 f(w_n)\| \leq g(\alpha_n)\left(\dfrac{\|w_n - w_{n-1}\|}{\alpha_n - \alpha_{n-1}}\right)^3$.*
- *(a_4) $\|w_{n+1} - w_n\| \leq (\alpha_{n+1} - \alpha_n)\left(\dfrac{\|w_n - w_n\|}{\alpha_n - \alpha_{n-1}}\right)^3$.*

Proof . We prove this Lemma by The Principle of Mathematical Induction.
By the hypothesis, it is obvious that (a_1)–(a_4) are true for i=1. Now, assume that (a_1)–(a_4) hold for some $i \in \mathbb{N}$.
For $n = i + 1$, (a_1) and (a_2) hold by Lemma 2.6. For (a_3), notice that we have the relation [7]

$$\begin{aligned}f(w_{i+1}) &= \frac{1}{2} f''(w_i)\pounds_f(w_i)(w_{i+1}-w_i)^2 \\ &\quad + \int_0^1 [f''(w^\vartheta) - f''(w_i)](w_{i+1}-w_i)^2(1-\vartheta)d\vartheta\end{aligned}$$

where, $w_i^{\vartheta} = w_i + \vartheta(w_{i+1} - w_i)$. Then, we get

$$\begin{aligned}\|\Gamma_0 f(w_{i+1})\| \leq \frac{1}{2}\|\Gamma_0 f''(w_i)\|\|\pounds_f(w_{i+1})\|\|w_{i+1} - w_i\|^2 \\ + \int_0^1 \|\Gamma_0[f''(w^{\vartheta}) - f''(w_i)]\|\|w_{i+1} - w_i\|^2(1-\vartheta)d\vartheta. \end{aligned} \tag{2.14}$$

By Eq. (2.1), we obtain

$$\begin{aligned}\int_0^1 \|\Gamma_0[f''(w^{\vartheta}) - f''(w_i)]\|\|w_{i+1} - w_i\|^2(1-\vartheta)d\vartheta \leq \int_0^1 [g''(\vartheta\|w_{i+1} - w_i\| + \|w_{i+1} - w_0\|) \\ - g''(\|w_{i+1} - w_0\|)]\times \\ \|w_{i+1} - w_i\|^2(1-\vartheta)d\vartheta.\end{aligned}$$

By using above inequality and inductive hypothesis (a_1)–(a_2) in Eq. (2.14), we obtain

$$\begin{aligned}\|\Gamma_0 f(w_{i+1})\| \leq \frac{1}{2}g''(\alpha_i)\pounds_g(\alpha_i)\|w_{i+1} - w_i\|^2 \\ + \int_0^1 [g''(\vartheta\|w_{i+1} - w_i\| + \|w_{i+1} - w_0\|) - g''(\|w_{i+1} - w_0\|)]\times \\ \|w_{i+1} - w_i\|^2(1-\vartheta)d\vartheta.\end{aligned} \tag{2.15}$$

We use Lemma 1.2 (i_2) then, we have

$$\begin{aligned}g''(\vartheta\|w_{i+1} - w_i\| + \|w_i - w_0\|) - \vartheta''(\|w_i - w_0\|) &\leq g''(\vartheta\|w_{i+1} - w_i\| + \alpha_i) \\ &\quad - g''(\alpha_{i+1}) \\ &\leq \Big[g''(\vartheta(\alpha_{i+1} - \alpha_i) + \alpha_i) \\ &\quad - g''(\alpha_i)\Big]\frac{\|w_{i+1} - w_i\|}{\alpha_{i+1} - \alpha_i}.\end{aligned}$$

Then by using above inequality, (2.15) becomes

$$\begin{aligned}\|\Gamma_0 f(w_{i+1})\| &\leq \left(\frac{\|w_{i+1} - w_i\|}{\alpha_{i+1} - \alpha_i}\right)^3\Big[\frac{1}{2}g''(\alpha_i)\pounds_g(\alpha_i)(\alpha_{i+1} - \alpha_i)^2 \\ &\quad + \int_0^1 [g''(\vartheta(\alpha_{i+1} - \alpha_i) + \alpha_i) - g''(\alpha_i)](\alpha_{i+1} - \alpha_i)^2(1-\vartheta)d\vartheta\Big] \\ &\leq g(\alpha_{i+1})\left(\frac{\|w_{i+1} - w_i\|}{\alpha_{i+1} - \alpha_i}\right)^3.\end{aligned}$$

Thus (a_3) holds for $n = i+1$. Finally, for (a_4), we have

$$\begin{aligned}\|w_{i+2}-w_{i+1}\| &\le \left\| I + \frac{1}{2}\pounds_f(w_{i+1})\left(I + \sum_{m\ge 2} 2S_m \pounds_f(w_{i+1})^{m-1}\right)\right\| \times \\ &\quad \left\|\varGamma_{i+1} f'(w_0)\right\| \left\|\varGamma_0 f(w_{i+1})\right\| \\ &\le -\left[1 + \frac{1}{2}\pounds_g(\alpha_{i+1})\left(1 + \sum_{m\ge 2} 2S_m \pounds_g(\alpha_{i+1})^{m-1}\right)\right] \times \\ &\quad \frac{g(\alpha_{i+1})}{g'(\alpha_{i+1})}\left(\frac{\|w_{i+1}-w_i\|}{\alpha_{i+1}-\alpha_i}\right)^3 \\ &= (\alpha_{i+2}-\alpha_{i+1})\left(\frac{\|w_{i+1}-w_i\|}{\alpha_{i+1}-\alpha_i}\right)^3. \end{aligned} \tag{2.16}$$

Hence, the statements hold for all n. The proof is complete □

By using these Lemmas, now we prove Theorem 2.1.

2.1. Proof of Theorem 2.1

Proof . From Lemma 2.7, we can say that the sequence $\{w_n\}$ is well defined. By using Corollary 2.10 and Lemma 2.7 (a_4), we get $\|w_n - w_0\| \le \alpha_n < \bar{\alpha}$, for any $n \in \mathbb{N}$, i.e. $\{w_n\}$ is belongs to $B(w_0, \bar{\alpha})$. By (2.16) and Corollary 2.10, we have

$$\|w_{n+1}-w_n\| \le \alpha_{n+1}-\alpha_n, \qquad n = 0, 1, ...$$

Since $\{\alpha_n\}$ converges to $\bar{\alpha}$, then the above inequality implies that

$$\sum_{n=M}^{\infty} \|w_{n+1}-w_n\| \le \sum_{n=M}^{\infty}(\alpha_{n+1}-\alpha_n) = \bar{\alpha} - \alpha_M < +\infty,$$

for any $M \in \mathbb{N}$. Hence $\{w_n\}$ is a Cauchy sequence in $B(w_0, \bar{\alpha})$ and converge to some $\sigma \in \bar{B}(w_0, \bar{\alpha})$. The above inequality implies that $\|\sigma - w_n\| \le \bar{\alpha} - \alpha_n$ for any $n \in \mathbb{N}$.
Now, we have to prove that $f(\sigma) = 0$. From the first part of Lemma 2.6, it follows that $\{\|f'(w_n)\|\}$ is bounded. Now by Lemma 2.7, we have

$$\|f(w_n)\| \le \|\varGamma_n f'(w_0)\| \|\varGamma_0 f(w_n)\| \|f'(w_n)\| \le (\alpha_{n+1}-\alpha_n)\frac{2}{\pounds_g(\alpha_n)\varLambda_g(\alpha_n)}\|f'(w_n)\|.$$

Due to the fact that $\{\|f'(w_n)\|\}$ is bounded, $\pounds_g(\alpha_n)$ is also bounded from Lemma 2.4 and $\{\alpha_n\}$ and $\varLambda_g(\alpha_n)$ are convergent, we can take limit in the above inequality to get

$$\lim_{n\to\infty} f(w_n) = 0.$$

Since f is continuous in $\bar{B}(w_0, \bar{\alpha})$, $\{w_n\} \subset B(w_0, \bar{\alpha})$ and $\{w_n\}$ converges to σ, we also have

$$\lim_{n\to\infty} f(w_n) = f(\sigma).$$

Thus, theorem is proved. □

Theorem 2.8. *With the assumption of Theorem 2.1, we get the following a priori and a posteriori error estimate:*

(i) $\forall\ n \geq 0$, *we have the a priori estimate:*

$$\|\sigma - w_{n+1}\| \leq (\bar{\alpha} - \alpha_{n+1})\left(\frac{\|\sigma - w_{n+1}\|}{\bar{\alpha} - \alpha_n}\right)^3, \qquad n = 0, 1, ... \tag{2.17}$$

Thus, the sequence $\{w_n\}$ *and* $\{\alpha_n\}$ *generated by MNL method given as* (1.3) *and* (2.3) *converges Q-cubic as follows*

$$\begin{aligned}\|\sigma - w_{n+1}\| \leq &-\frac{1}{g'(\bar{\alpha})}\left[\frac{1}{6}D^{-}g''(\bar{\alpha})\right.\\ &\left.+\frac{1}{8}\left(1 + \sum_{m\geq 2}\left(\frac{1}{2}\right)^{m-2} S_m\right)\frac{g''(\bar{\alpha})}{(\bar{\alpha} - \alpha)}\right]\|\sigma - w_n\|^3 \quad n = 0, 1, ...\end{aligned} \tag{2.18}$$

and

$$\bar{\alpha} - \alpha_{n+1} \leq -\frac{1}{g'(\bar{\alpha})}\left[\frac{1}{6}D^{-}g''(\bar{\alpha}) + \frac{1}{8}\left(1 + \sum_{m\geq 2}\left(\frac{1}{2}\right)^{m-2} S_m\right)\frac{g''(\bar{\alpha})}{(\bar{\alpha} - \alpha)}\right](\bar{\alpha} - \alpha)^3, \tag{2.19}$$

$\alpha \in (0, \bar{\alpha})$, $S_m \in \mathbb{R}^+$ *and* $m \geq 2$.

(ii) $\forall\ n \geq 0$, *we have a posteriori error estimate:*

$$\|\sigma - w_{n+1}\| \leq (\bar{\alpha} - \alpha_{n+1})\left(\frac{\|w_{n+1} - w_n\|}{\alpha_{n+1} - \alpha_n}\right)^3, \qquad n = 0, 1, ...$$

In particularly, conclusion can be drawn that convergence of MNL method to σ *is cubic.*

Proof .

$$\begin{aligned}\sigma - w_{n+1} &= \sigma - w_n + \left[I + \frac{1}{2}\pounds_f(w_n)\left(I + \sum_{m\geq 2} 2S_m \pounds_f(w_n)^{m-1}\right)\right]\mathit{\Gamma}_n f(w_n)\\ &= (\sigma - w_n) + \mathit{\Gamma}_n f(w_n) + \frac{1}{2}\pounds_f(w_n)\mathit{\Gamma}_n f(w_n)\\ &\quad + \left(\sum_{m\geq 2} S_m \pounds_f(w_n)^m\right)\mathit{\Gamma}_n f(w_n)\\ &= \mathit{\Gamma}_n\big[(\sigma - w_n)f'(w_n) + f(w_n)\big] + \frac{1}{2}\pounds_f(w_n)\mathit{\Gamma}_n f(w_n)\\ &\quad + \left(\sum_{m\geq 2} S_m \pounds_f(w_n)^m\right)\mathit{\Gamma}_n f(w_n)\\ &= -\mathit{\Gamma}_n \int_0^1 [f''(w_n^{\vartheta}) - f''(w_n)](\sigma - w_n)^2(1 - \vartheta)d\vartheta\\ &\quad - \frac{1}{2}\pounds_f(w_n)\mathit{\Gamma}_n \int_0^1 f''(w_n^{\vartheta})(\sigma - w_n)^2(1 - \vartheta)d\vartheta\\ &\quad - \mathit{\Gamma}_n\left(\sum_{m\geq 2} S_m \pounds_f(w_n)^m\right)\int_0^1 f''(w_n^{\vartheta})(\sigma - w_n)^2(1 - \vartheta)d\vartheta,\end{aligned}$$

where, $w_n^\vartheta = w_n + \vartheta(\sigma - w_n)$. Now,

$$\begin{aligned}\|\sigma - w_{n+1}\| \leq \; & \|\Gamma_n f'(w_0)\| \times \\ & \left\| \int_0^1 \Gamma_0[f''(w_n^\vartheta) - f''(w_n)](\sigma - w_n)^2(1-\vartheta)d\vartheta \right\| \\ & + \frac{1}{2}\|\|\pounds_f(w_n)\|\|\Gamma_n f'(w_0)\| \times \\ & \left\| \int_0^1 [\Gamma_0 f''(w_n^\vartheta)](\sigma - w_n)^2(1-\vartheta)d\psi \right\| \\ & + \left\| \left(\sum_{m\geq 2} S_m \pounds_f(w_n)^m \right) \right\| \|\Gamma_n f'(w_0)\| \times \\ & \int_0^1 \|[\Gamma_0 f''(w_n^\vartheta)](\sigma - w_n)^2(1-\vartheta)d\vartheta \Big\|. \end{aligned} \tag{2.20}$$

By using Eq. (2.1), we have

$$\begin{aligned}\int_0^1 \Gamma_0[f''(w_n^\vartheta) - f''(w_n)](1-\vartheta)d\vartheta \leq \int_0^1 & \big[g''(\vartheta\|\sigma - w_n\| + \|w_n - w_0\|) \\ & - g''(\|w_n - w_0\|)\big](1-\vartheta)d\vartheta\end{aligned}$$

We use above inequality, Lemma 2.6 and Lemma 2.7 in Eq. (2.20) to get,

$$\begin{aligned}\|\sigma - w_{n+1}\| \leq & -\frac{1}{g'(\alpha_n)} \times \\ & \left[\int_0^1 \big[g''(\vartheta\|\sigma - w_n\| + \|w_n - w_0\|) - g''(\|w_n - w_0\|)\big](1-\vartheta)d\vartheta \|\sigma - w_n\|^2 \right. \\ & + \frac{1}{2}\pounds_g(\alpha_n) \int_0^1 \big[g''(\vartheta\|\sigma - w_n\| + \|w_n - w_0\|)\big](1-\vartheta)d\vartheta\|\sigma - w_n\|^2 \\ & + \left(\sum_{m\geq 2} S_m \pounds_g(\alpha_n)^m \right) \times \\ & \left. \int_0^1 \big[g''(\vartheta\|\sigma - w_n\| + \|w_n - w_0\|)\big](1-\vartheta)d\vartheta\|\sigma - w_n\|^2 \right]. \end{aligned} \tag{2.21}$$

Then, by using Lemma 1.2, we obtain

$$\begin{aligned} g''(\vartheta\|\sigma - w_n\| + \|w_n - w_0\|) - g''(\|w_n - w_0\|) & \leq g''(\vartheta\|\sigma - w_n\| + \alpha_n) - g''(\alpha_n) \\ & \leq [g''(\vartheta(\bar{\alpha} - \alpha_n) + \alpha_n) - g''(\alpha_n)] \frac{\|\sigma - w_n\|}{\bar{\alpha} - \alpha_n}. \end{aligned}$$

We use above inequality and Lemma 2.7 in (2.21) to obtain

$$\begin{aligned}\|\sigma - w_{n+1}\| &\leq -\frac{1}{g'(\alpha_n)}\Bigg[\int_0^1 \big[g''(\vartheta(\bar{\alpha}-\alpha_n)+\alpha_n) - g''(\alpha_n)\big](1-\vartheta)d\vartheta\|\sigma - w_n\|^2 \\ &\quad + \frac{1}{2}\pounds_g(\alpha_n)\int_0^1 g''(\vartheta(\bar{\alpha}-\alpha_n)+\alpha_n)(1-\vartheta)d\vartheta\|\sigma - w_n\|^2 \\ &\quad + \left(\sum_{m\geq 2} S_m \pounds_g(\alpha_n)^m\right) \\ &\quad \times \int_0^1 g''(\vartheta(\bar{\alpha}-\alpha_n)+\alpha_n)(1-\vartheta)d\vartheta\|\sigma - w_n\|^2\Bigg] \\ &\leq -\frac{1}{g'(\alpha_n)}\left[\int_0^1 \big[g''(\vartheta(\bar{\alpha}-\alpha_n)+\alpha_n) - g''(\alpha_n)\big](1-\vartheta)d\vartheta\right]\frac{\|\sigma - w_n\|^3}{\bar{\alpha}-\alpha_n} \\ &\quad - \frac{1}{2}\frac{\pounds_g(\alpha_n)}{g'(\alpha_n)}\left(1 + \sum_{m\geq 2} 2S_m\pounds_g(\alpha_n)^{m-1}\right) \\ &\quad \times \left[\int_0^1 g''(\vartheta(\bar{\alpha}-\alpha_n)+\alpha_n)(1-\vartheta)d\vartheta\right]\frac{\|\sigma - w_n\|^3}{\bar{\alpha}-\alpha_n} \\ &\leq (\bar{\alpha}-\alpha_{n+1})\left(\frac{\|\sigma - w_n\|}{\bar{\alpha}-\alpha_n}\right)^3.\end{aligned}$$

This shows (2.17) holds for all $n \in \mathbb{N}$ and (2.18) follows from Lemma 2.5 (i_2). Now, from Lemma 2.7 (a_4), for all $j \geq 0$ and $n \geq n_0 \geq 0$, we have

$$\|w_{n+j+1} - w_{n+j}\| \leq (\alpha_{n+j+1} - \alpha_{n+j})\left(\frac{\|w_{n_0+1} - w_{n_0}\|}{\alpha_{n_0+1} - \alpha_{n_0}}\right)^{3^{n-n_0+j}}. \tag{2.22}$$

Then, by (2.17),

$$\begin{aligned}\|\sigma - w_{n+1}\| &\leq \sum_{j=0}^{\infty}(\alpha_{n+j+1} - \alpha_{n+j})\left(\frac{\|w_{n_0+1} - w_{n_0}\|}{\alpha_{n_0+1} - \alpha_{n_0}}\right)^{3^{n-n_0+j}} \\ &\leq \sum_{j=0}^{\infty}(\alpha_{n+i+1} - \alpha_{n+i})\left(\frac{\|w_{n_0+1} - w_{n_0}\|}{\alpha_{n_0+1} - \alpha_{n_0}}\right)^{3^{n-n_0}} \\ &\leq (\bar{\alpha} - \alpha_n)\left(\frac{\|w_{n_0+1} - w_{n_0}\|}{\alpha_{n_0+1} - \alpha_{n_0}}\right)^{3^{n-n_0}}\end{aligned}$$

and hence by taking $n = n_0$, the Theorem 2.8 is proved. □

Theorem 2.9. *Under the assumption of Theorem 2.1, the limit σ of $\{w_n\}$ is the unique zero of $f(w) = 0$ in $B(w_0, \rho)$, where ρ is defined as $\rho := sup\{\alpha \in (\bar{\alpha}, \wp) : g(\alpha) \leq 0\}$.*

Proof . We have to show that σ is the unique zero of (1.1) in $\bar{B}(w_0, \bar{\alpha})$. For this, let $\tau \in \bar{B}(w_0, \bar{\alpha})$ be another zero of (1.1). Then $\|\tau - w_0\| \leq \bar{\alpha}$. Now, we will prove by induction that

$$\|\tau - w_n\| \leq \bar{\alpha} - \alpha_n, \qquad n = 0, 1, ... \tag{2.23}$$

For $n = 0$ the above inequality hold trivially, due to $\alpha_0 = 0$. Now we assume that (2.23) is true for $n = i$. Note that Eq. (2.17) implies

$$\|\tau - w_{n+1}\| \leq \frac{\bar{\alpha} - \alpha_{n+1}}{(\bar{\alpha} - \alpha_n)^3}\|\tau - w_n\|^3.$$

By using inductive hypothesis (2.23) to the above inequality, we get (2.23) also hold for $n = i + 1$. Since $\{w_n\}$ converges to σ and $\{\alpha_n\}$ converges $\bar{\alpha}$, from (2.23) we conclude that $\tau = \sigma$. Hence, $\sigma \in \bar{B}(w_0, \bar{\alpha})$ is the unique zero of the operator f.
It remains to prove that the operator f does not have zero in $B(w_0, \rho)\setminus\bar{B}(w_0, \bar{\alpha})$. We prove this by contradiction. For this, we assume that operator f does have zero in $B(w_0, \rho)\setminus\bar{B}(w_0, \bar{\alpha})$, i.e. there exist $\tau \in \Omega \subset W_1$,

$$\bar{\alpha} < \|\tau - w_0\| < \rho, \quad f(\tau) = 0.$$

We will prove that the above assumption cannot hold. Now by the Taylor's series

$$\begin{aligned} f(\tau) &= f(w_0) + f'(w_0)(\tau - w_0) + \frac{1}{2}f''(w_0)(\tau - w_0)^2 \\ &\quad + (1 - \vartheta)\int_0^1 \left[f''(w_n^\vartheta) - f''(w_0)\right](\tau - w_0)^2 d\vartheta, \end{aligned}$$

where, $w_n^\vartheta = w_0 + \vartheta(\tau - w_0)$. Since $f(\tau) = 0$, then

$$\begin{aligned} 0 &\leq \left\|\Gamma_0[f(w_0) + f'(w_0)(\tau - w_0) + \frac{1}{2}f''(w_0)(\tau - w_0)^2]\right\| \\ &\quad + \left\|(1 - \vartheta)\int_0^1 \Gamma_0\left[f''(w_n^\vartheta) - f''(w_0)\right](\tau - w_0)^2 d\vartheta\right\|, \end{aligned} \tag{2.24}$$

Use (2.1) and $g'(0) = -1$ from (I.) in the last term of (2.24) to infer

$$\begin{aligned} &\left\|(1 - \vartheta)\int_0^1 \Gamma_0\left[f''(w_n^\vartheta) - f''(w_0)\right](\tau - w_0)^2 d\vartheta\right\| \\ &\leq \int_0^1 (1 - \vartheta)\left[g''(\vartheta\|\tau - w_0\|) - g''(0)\right]\|\tau - w_0\|^2 \\ &\leq g(\|\tau - w_0\|) - g(0) - g'(0)\|\tau - w_0\| - \frac{1}{2}g''(0)\|\tau - w_0\|^2 \\ &\leq g(\|\tau - w_0\|) - g(0) + \|\tau - w_0\| - \frac{1}{2}g''(0)\|\tau - w_0\|^2. \end{aligned} \tag{2.25}$$

Applying the condition of Eq. (2.2) in the first term of (2.24) to get

$$\begin{aligned} &\left\|\Gamma_0[f(w_0) + f'(w_0)(\tau - w_0) + \frac{1}{2}f''(w_0)(\tau - w_0)^2]\right\| \\ &\geq \|\tau - w_0\| - \|\Gamma_0 f(w_0)\| - \frac{1}{2}\|\Gamma_o f''(w_0)\|\|\tau - w_0\|^2 \\ &\geq \|\tau - w_0\| - g(0) - \frac{1}{2}g''(0)\|\tau - w_0\|^2. \end{aligned} \tag{2.26}$$

Combining (2.25) and (2.26), we obtain that

$$g(\|\tau - w_0\|) \geq 0.$$

By Lemma 2.3, g is strictly convex. Hence g is strictly positive in the interval $(\|\tau - w_0\|, \rho)$. Thus we get $\sigma \leq \|\tau - w_0\|$, which is a contradiction of our assumption. Thus, operator f does not have zeros in $B(w_0, \rho)\setminus\bar{B}(w_0, \bar{\alpha})$ and σ is the unique zero of (1.1). Thus the uniqueness part is proved. Hence the proof is complete. □

Consequently, by using Lemma 2.5, we conclude that

Corollary 2.10. *Let sequence $\{\alpha_n\}$ be defined by* (2.3). *Then $\{\alpha_n\}$ is well defined, strictly increasing and is contained in $(0, \bar{\alpha})$. Furthermore, $\{\alpha_n\}$ satisfies*

$$\bar{\alpha} - \alpha_{n+1} \leq -\frac{1}{g'(\bar{\alpha})}\left[\frac{1}{6}D^{-}g''(\bar{\alpha}) + \frac{1}{8}\left(1 + \sum_{m\geq 2}\left(\frac{1}{2}\right)^{m-2} S_m\right)\frac{g''(\bar{\alpha})}{(\bar{\alpha}-\alpha)}\right](\bar{\alpha}-\alpha)^3$$

$\alpha \in (0, \bar{\alpha})$ $\ S_m \in \mathbb{R}^+$ *and* $m \geq 2$. *Thus, $\{\alpha_n\}$ is Q-cubically converges to $\bar{\alpha}$.*

Thus, all statements about the sequence $\{\alpha_n\}$ in Theorem 2.1 and Theorem 2.8 are true.

3. Special cases with examples

In this section, we will present special cases of Theorem 2.1 and Theorem 2.8 and we present convergence theorem on MNL method under Lipschitz and Smale-type condition. We also provide some examples to show efficiency of our method

3.1. Convergence result under Lipschitz condition

We define majorizing function g by

$$g(\alpha) = \frac{\mu}{6}\alpha^3 + \frac{\varrho}{2}\alpha^2 - \alpha + \beta, \tag{3.1}$$

for positive numbers μ, ϱ, and β. We choose this cubic polynomial as the majorizing function g in Eq. (2.1), then we see that the majorant condition (2.2) and assumptions (I.) and (II.) are satisfied for g.

Theorem 3.1. *Let W_1 and W_2 be $\mathbb{B}$-space and $\Omega \subset W_1$, Ω is a non-empty open convex subset in $\mathbb{B}$-space and $f : \Omega \longrightarrow W_2$ be a twice continuously differentiable nonlinear operator. Take $w_0 \in \Omega$, w_0 is an initial point with $f'(w_0)^{-1} \in \pounds(W_2, W_1)$. With the choice of $g(\alpha)$ from Eq.* (3.1), *the majorant condition* (2.1) *reduces to the Lipschitz condition*

$$\|[\Gamma_0[f''(v) - f''(w)]\| \leq \mu\|v - w\|, \quad w, v \in \Omega \tag{3.2}$$

and

$$\|\Gamma_0 f(w_0)\| \leq \beta \ \ \text{and} \ \ \|\Gamma_0 f''(w_0)\| \leq \varrho. \tag{3.3}$$

Let

$$b := \frac{2(\varrho + 2\sqrt{\varrho^2 + 2\mu})}{3(\varrho + \sqrt{\varrho^2 + 2\mu})^2}. \tag{3.4}$$

Moreover, if $\beta < b$ holds, then the sequence $\{w_n\}$ generated by MNL method (1.3) *for solving $f(w) = 0$ with initial point w_0 are well defined. Sequence $\{w_n\}$ is contained in $B(w_0, \bar{\alpha})$ and converge to a point $\sigma \in \bar{B}(w_0, \bar{\alpha})$, which is the solution of Eq.* (1.1). *By using condition* (III.), *$\bar{\alpha}$ is the minimal positive zero of g in $[0, r_1]$, where $r_1 = \dfrac{(-\varrho + \sqrt{\varrho^2 + 2\mu})}{\mu}$ is the positive root of g'. The limit σ of the sequence $\{w_n\}$ is the unique zero of* (1.1) *in $B(w_0, \bar{\alpha}^*)$, where $\bar{\alpha}^*$ is the root of g in $(r_1, +\infty)$.*

Theorem 3.2. *With the assumption of Theorem 3.1, we get the following a priori and a posteriori error estimate:*

(i) for all $n \geq 0$, we have a priori estimate:

$$\|\sigma - w_{n+1}\| \leq (\bar{\alpha} - \alpha_{n+1})\left(\frac{\|\sigma - w_{n+1}\|}{\bar{\alpha} - \alpha_n}\right)^3. \tag{3.5}$$

Thus, the sequence $\{w_n\}$ generated by MNL method (1.3) converges Q-cubic as follows

$$\|\sigma - w_{n+1}\| \leq \frac{(7\bar{\alpha} - 4\alpha)\mu + 3\varrho + 3\left(\sum_{m\geq 2}\left(\frac{1}{2}\right)^{m-2} S_m\right)(\mu\bar{\alpha} + \varrho)}{24(\bar{\alpha} - \alpha)\left(1 - \varrho\bar{\alpha} - \frac{\mu}{2}\bar{\alpha}\right)}\|\sigma - w_n\|^3$$

(ii) For all $n \geq 0$, we have a posteriori error estimate:

$$\|\sigma - w_{n+1}\| \leq (\bar{\alpha} - \alpha_{n+1})\left(\frac{\|w_{n+1} - w_n\|}{\alpha_{n+1} - \alpha_n}\right)^3. \tag{3.6}$$

In particularly, conclusion can be drawn that convergence of MNL method to σ is Q-cubic.

Example 3.3. *Let $W_1 = C[0,1]$ be the set of continuous functions on interval $[0,1]$. We consider the integral equation $f(w) = 0$, where*

$$f(w)(\varsigma) = w(\varsigma) - \varsigma + \frac{1}{2}\int_0^1 \varsigma\ \cos(w(\vartheta))d\vartheta \tag{3.7}$$

with max norm

$$\|w\| = \max_{\varsigma\in[0,1]} |w(\varsigma)|.$$

The first and second Fréchet derivative of f are

$$f'(w)x(\varsigma) = w(\varsigma) - \frac{1}{2}\int_0^1 \varsigma \sin(w(\vartheta))x(\vartheta)d\vartheta$$

and

$$f''(w)xy(\varsigma) = -\frac{1}{2}\int_0^1 \varsigma \cos(w(\vartheta))xy(\vartheta)d\vartheta$$

respectively. Also,

$$[I - f'(w)]x(\varsigma) = \frac{1}{2}\int_0^1 \varsigma \sin(w(\vartheta))x(\vartheta)d\vartheta$$

and

$$[f''(v) - f''(w)] \leq \frac{\varsigma}{2}\int_0^1 \left[\cos(v(\vartheta)) - \cos(w(\vartheta))\right]d\vartheta.$$

Here, we choose the initial value as $w_0 = w_0(\varsigma) = \varsigma$ *and use max norm, then we have*

$$\|f(w_0)\| \leq \frac{1}{2}\ \sin 1.$$

Since,

$$\|I - f'(w_0)\| \leq \frac{1}{2}\ [\sin 1 - \cos 1] < 1,$$

then by using Banach Lemma [25, 13], $\mathit{\Gamma}_0 = f'(w_0)^{-1}$ *exist. Thus*

$$\|\mathit{\Gamma}_0\| \leq \frac{1}{1 - \|I - f'(w_0)\|} \leq \frac{2}{2 + \cos 1 - \sin 1}.$$

It is to be noted that,

$$\|\mathit{\Gamma}_0 f(w_0)\| \leq \frac{\sin 1}{2 - \sin 1 + \cos 1}, \quad \|\mathit{\Gamma}_0 f''(w_0)\| \leq \frac{\sin 1}{2 - \sin 1 + \cos 1} \tag{3.8}$$

and

$$\big\|f''(v) - f''(w)\big\| = \left\|\frac{\varsigma}{2}\int_0^1 \big[\cos(v(\vartheta)) - \cos(w(\vartheta))\big]d\vartheta\right\| \leq \frac{1}{2}\|v - w\|.$$

Thus,

$$\big\|\mathit{\Gamma}_0[f''(v) - f''(w)]\big\| \leq \frac{1}{2 + \cos 1 - \sin 1}\|v - w\|. \tag{3.9}$$

By comparing Eq. (3.2) *and* (3.3) *from Theorem 3.1 with Eq.* (3.8) *and* (3.9) *respectively, we get*

$$\beta = \varrho = \frac{\sin 1}{2 - \sin 1 + \cos 1} = 0.495323446404751$$

and $\mu = \dfrac{1}{2 + \cos 1 - \sin 1} = 0.588639959484558$. *Also from Eq.* (3.4)

$$b := \frac{2(\varrho + 2\sqrt{\varrho^2 + 2\mu})}{3(\varrho + \sqrt{\varrho^2 + 2\mu})^2} = 0.585678917891157.$$

Here $\beta < b$, *thus the Kantorovich convergence criterion hold. Then the sequence* $\{w_n\}$ *generated by MNL method* (1.3) *converges to the zero of* $f(w)$ *defined by Eq.* (3.7) *with initial point* w_0. *Sequence* $\{w_n\}$ *is contained in* $(\varsigma,\ 0.609569634694877)$, *where* $\varsigma \in [0, 1]$. *By using condition* (III.), 0.609569634694877 *is the minimal positive zero of* g *in* $[0,\ 1.184790612418399]$, *where* $r_1 = \dfrac{(-\varrho + \sqrt{\varrho^2 + 2\mu})}{\mu} = 1.184790612418399$ *is the positive root of* g'. *Thus, the uniqueness and existence solution ball of Eq.*(3.7) *are* $B(\varsigma, 1.709908005327262)$ *and* $\bar{B}(\varsigma, 0.609569634694877)$ *respectively, where* 1.709908005327262 *is the root of* g *in* $(1.184790612418399, +\infty)$.

Example 3.4. *Let* $W_1 = W_2 = C[0, 1]$, *we consider the following nonlinear boundary value problem*

$$w'' + \lambda(w^3 - \eta w^2) = 0 \quad \text{with} \quad w(0) = 0 \ \text{ and } \ w(1) = 1.$$

The above nonlinear boundary value problem can be converted as the integral equation

$$w(\varsigma) = \varsigma + \lambda \int_0^1 Q(\varsigma, \vartheta)(w^3(\vartheta) + \eta w^2(\vartheta))d\vartheta \tag{3.10}$$

where, $Q(\varsigma, \vartheta)$ is the Kernel function defined as

$$Q(\varsigma, \vartheta) = \begin{cases} (1-\varsigma)\vartheta, & \vartheta \leq \varsigma, \\ (1-\vartheta)\varsigma, & \varsigma \leq \vartheta. \end{cases} \tag{3.11}$$

Let $W_1 = W_2 = C[0,1]$ equipped with the maximum norm

$$\|w\| = \max_{\varsigma \in [0,1]} |w(\varsigma)|.$$

For finding the solution of Eq.(3.10)*, it needs to find the solution of $f(w) = 0$, where $f : \Omega \subseteq C[0,1] \longrightarrow C[0,1]$ and $\Omega = \{w \in [0,1] : w(\varsigma) \geq 0,\ \varsigma \in [0,1]\}$ is defined as*

$$f(w)(\varsigma) = w(\varsigma) - \varsigma - \lambda \int_0^1 Q(\varsigma, \vartheta)(w^3(\vartheta) + \eta w^2(\vartheta))d\vartheta. \tag{3.12}$$

Now, the first and second Fréchet derivative of Eq. (3.12) *are*

$$f'(w)x(\varsigma) = w(\varsigma) - \lambda \int_0^1 Q(\varsigma, \vartheta)\ (3w^2(\vartheta) + \eta w(\vartheta)\ d\vartheta$$

and

$$f''(w)xy(\varsigma) = -\lambda \int_0^1 Q(\varsigma, \vartheta)\ (6w(\vartheta) + \eta)xy(\vartheta)\ d\vartheta$$

respectively. Then,

$$[I - f'(w)]x(\varsigma) = -\lambda \int_0^1 Q(\varsigma, \vartheta)\ (3w^2(\vartheta) + \eta w(\vartheta)\ d\vartheta$$

and

$$\begin{aligned} \|f''(v) - f''(w)\| &= \left\| 6\lambda \int_0^1 Q(\varsigma, \vartheta)\ (v(\vartheta) - w(\vartheta))\ d\vartheta \right\| \\ &\leq 6|\lambda| \left\| \int_0^1 Q(\varsigma, \vartheta) d\vartheta \right\| \|v(\vartheta) - w(\vartheta)\|. \end{aligned}$$

Since,

$$\begin{aligned} \left\| \int_0^1 Q(\varsigma, \vartheta) d\vartheta \right\| &= \max_{0 \leq \varsigma \leq 1} \left| \int_0^1 Q(\varsigma, \vartheta) d\vartheta \right| \\ &= \max_{0 \leq \varsigma \leq 1} \left| \int_0^{\varsigma} (1-\varsigma)\vartheta d\vartheta + \int_{\varsigma}^1 \varsigma(1-\vartheta) d\vartheta \right| \\ &= \max_{0 \leq \varsigma \leq 1} \left| \frac{1}{8} - \frac{1}{2}\left(\varsigma - \frac{1}{2}\right)^2 \right| = \frac{1}{8}, \end{aligned}$$

then,

$$\|f''(v) - f''(w)\| \leq \frac{6}{8}|\lambda| \, \|v - w\|.$$

We choose $w_0 = w_0(\varsigma) = \varsigma$ *as the initial approximation, then we have*

$$\|f(w_0)\| \leq \frac{|\lambda|}{8}(1 + \eta).$$

Now,

$$\|I - f'(w_0)\| \leq \frac{|\lambda|}{8} \, (3 + 2\eta).$$

If $2\eta < 5$, *then* $\|I - f'(w_0)\| < 1$, *then by using Banach Lemma [13, 25]* $\varGamma_0 = f'(w_0)^{-1}$ *exist. Thus*

$$\|\varGamma_0\| \leq \frac{1}{1 - \|I - f'(w_0)\|} \leq \frac{8}{8 - |\lambda|(3 + 2\eta)}$$

Also,

$$\|\varGamma_0 f(w_0)\| \leq \frac{|\lambda|(1+\eta)}{8 - |\lambda|(3 + 2\eta)}, \qquad \|\varGamma_0 f''(w_0)\| \leq \frac{|\lambda|(6+\eta)}{8 - |\lambda|(3 + 2\eta)}$$

and,

$$\left\|\varGamma_0 \left[f''(v) - f''(w)\right]\right\| \leq \frac{6|\lambda|}{8 - |\lambda|(3 + 2\eta)} \|v - w\|.$$

Then,

$$\beta := \frac{|\lambda|(1+\eta)}{8 - |\lambda|(3 + 2\eta)}, \quad \varrho := \frac{|\lambda|(6+\eta)}{8 - |\lambda|(3 + 2\eta)} \quad \text{and} \quad \mu := \frac{6|\lambda|}{8 - |\lambda|(3 + 2\eta)}. \tag{3.13}$$

In Table 1 and Table 2, we find the value of ϱ, β, μ *from Eq.* (3.13), b *from Eq.*(3.4) *and* r_1 *from Theorem 3.1, for* $\eta = 0$, $\lambda = 0.25, \ 0.5, \ 0.75, 1$ *and* $\eta = 1$, $\lambda = 0.25, \ 0.5, \ 0.74$. *Here, we see that for all the values of* β *from Table 1 and Table 2,* $\beta < b$. *Therefore, the convergence criterion* $\beta < b$ *holds and by the Theorem 3.1 we conclude that the sequence* $\{w_n\}$ *generated by MNL method* (1.3) *with initial point* w_0 *converges to a zero of operator* f *which is defined by* (3.12). *In both the tables, we also present the existence and uniqueness domain of solution, for respective value of* η *and* λ.

3.2. Convervece result under Smale's condition

In this section, we will present a convergence theorem on MNL method under Smale's condition. Here, we applied the γ-condition in the MNL method (1.3) for convergence result.

Let $w_0 \in \varOmega$, $f : \varOmega \subseteq W_1 \longrightarrow W_2$ is analytic, $\varGamma_0 = f'(w_0)^{-1} \in \pounds(W_1, W_2)$ and f satisfies

$$\left\|\varGamma_0 f^{(k)}(w_0)\right\| \leq k!\gamma^{k-1}, \qquad k \geq 2$$

λ	For η=0 in Eq. (3.13), 0.25	0.5	0.75	1
ϱ	0.1034482...	0.230769...	0.39130434...	0.60000000...
β	0.0344827...	0.07692307...	0.13043478...	0.20000000...
μ	0.0344827...	0.07692307...	0.13043478...	0.200000000...
b	2.993317...	1.6169946...	1.04493404...	0.72127199...
r_1	2.265986...	1.309401...	0.88561808...	0.63299316...
Existence	$\bar{B}(\varsigma, 0.03460809...)$	$\bar{B}(\varsigma, 0.07837774...)$	$\bar{B}(\varsigma, 0.13825957...)$	$\bar{B}(\varsigma, 0.23606797...)$
Uniqueness	$B(\varsigma, 4.06814039...)$	$B(\varsigma, 2.3502617...)$	$B(\varsigma, 1.544540158...)$	$B(\varsigma, 1)$

Table 1: Existence and Uniqueness domain of solution for (3.12)

λ	For η=1 in Eq. (3.13), 0.25	0.5	0.74
ϱ	0.129629629629630	0.318181818181818	0.602325581395349
β	0.074074074074074	0.181818181818182	0.344186046511628
μ	0.037037037037037	0.090909090909091	0.172093023255814
b	2.627912743775339	1.274787453668897	0.731101199888517
r_1	2.685128379379139	1.610317298281767	1.097766959465200
Existence	$\bar{B}(\varsigma, 0.074815160834724)$	$\bar{B}(\varsigma, 0.194527701995009)$	$\bar{B}(\varsigma, 0.583090301569012)$
Uniqueness	$B(\varsigma, 3.683184296822359)$	$B(\varsigma, 1.853235541162442)$	$B(\varsigma, 0.714884978608793)$

Table 2: Existence and Uniqueness domain of solution for (3.12)

where

$$\gamma := \sup_{n>1} \left\| \frac{f'(w_0)^{-1} f^{(n)}(w_0)}{n!} \right\|^{\frac{1}{n-1}}.$$

Define majorizing function, which was introduced by Wang[26] as

$$g(\alpha) = \frac{\gamma\alpha^2}{1-\gamma\alpha} - \alpha + \beta, \quad \gamma > 0, \ 0 \leq \alpha < \frac{1}{\gamma} \tag{3.14}$$

Here, g satisfies assumption (I.) and (II.). Thus, by Theorem 2.1 we provide convergence results under γ condition as follows.

Theorem 3.5. *Let f be a continuously twice Fréchet differentiable nonlinear operator on Ω, with initial point $w_0 \in \Omega$ such that $\Gamma_0 \in \pounds(W_2, W_1)$ exist. With the choice of $g(\alpha)$ given in Eq.* (3.14)*, the majorant condition* (2.1) *reduce to the γ-condition*

$$\|\Gamma_0[f''(v) - f''(w)]\| \leq 2\gamma \left[\frac{1}{(1-\gamma\|v-w\| - \gamma\|w-w_0\|)^3} - \frac{1}{(1-\gamma\|w-w_0\|)^3} \right], \tag{3.15}$$

where, $\|v-w\| + \|w-w_0\| < \frac{1}{\gamma}$ and

$$\|\Gamma_0 f(w_0)\| \leq \beta \quad and \quad \|\Gamma_0 f''(w_0)\| \leq 2\gamma. \tag{3.16}$$

Additionally, if

$$\kappa := \beta\gamma < 3 - 2\sqrt{2}. \tag{3.17}$$

Then, the sequence $\{w_n\}$ generated by method MNL (1.3) *with starting point w_0 is well defined, contained in $B(w_0, \bar{\alpha}^*)$ and converges to $\sigma \in \bar{B}(w_0, \bar{\alpha})$ which is the zero of Eq.* (1.1)*. The limit point σ is the unique zero of* (1.1) *in $B(w_0, \bar{\alpha}^*)$, where $\bar{\alpha}$ and $\bar{\alpha}^*$ given by*

$$\bar{\alpha} = \frac{1+\kappa - \sqrt{(1+\kappa)^2 - 8\kappa}}{4\gamma} \quad and \quad \bar{\alpha}^* = \frac{1+\kappa + \sqrt{(1+\kappa)^2 - 8\kappa}}{4\gamma} \tag{3.18}$$

respectively.

Theorem 3.6. *With the assumption of Theorem 3.5, we get the following a priori and a posteriori error estimate:*

(i) For all n ≥ 0, we have the following a priori estimate:

$$\|\sigma - w_{n+1}\| \leq (\bar{\alpha} - \alpha_{n+1}) \left(\frac{\|\sigma - w_{n+1}\|}{\bar{\alpha} - \alpha_n} \right)^3.$$

Thus, the sequence $\{w_n\}$ generated by MNL (1.3) *converges cubic as follows*

$$\|\sigma - w_{n+1}\| \leq \frac{(3\bar{\alpha} - 4\alpha)\gamma^2 + \gamma + \left(\sum_{m\geq 2} \left(\frac{1}{2}\right)^{m-2} S_m \right)(1-\gamma\bar{\alpha})\gamma}{4(\bar{\alpha}-\alpha)(1-\gamma\bar{\alpha})(\gamma^2\bar{\alpha}^2 - 2\gamma\bar{\alpha})} \|\sigma - w_n\|^3.$$

(ii) For all $n \geq 0$, we have the following a posteriori error estimate:

$$\|\sigma - w_{n+1}\| \leq (\bar{\alpha} - \alpha_{n+1})\left(\frac{\|w_{n+1} - w_n\|}{\alpha_{n+1} - \alpha_n}\right)^3.$$

Example 3.7. *Let $W_1 = W_2 = \mathbb{R}^2$ endowed with max norm $\|.\| = \|.\|_\infty$ and $\Omega = \bar{B}(0,1)$. Here, we define the analytic function $f : W_1 \longrightarrow W_2$ on Ω for $w = (w_1, w_2)^t \in \Omega$ by*

$$f(w) = (x(w), y(w))^T = \left(10w_1 e^{w_2} + 5w_1 w_2,\ 5w_1^2 + \sin w_1 + 10w_2\right)^T \tag{3.19}$$

We assume that $w_0 = (a,b)^T$ be the initial approximation. The first and second Fréchet derivatives of f from Eq. (3.19) are:

$$f'(w) = \begin{bmatrix} 10e^{w_2} + 5w_2 & 10w_1 e^{w_2} + 5w_1 \\ 10w_1 + \cos w_1 & 10 \end{bmatrix}$$

and

$$f''(w) = \left[\begin{array}{cc} 0 & 10e^{w_2} + 5 \\ 10e^{w_2} + 5 & 10w_1 e^{w_2} \\ \hline 10 - \sin w_1 & 0 \\ 0 & 0 \end{array}\right].$$

Now, we find the inverse of $f'(w)$ at the initial point $w_0 = (a,b)^T$

$$\Gamma_0 = f'(w_0)^{-1} = \frac{1}{d}\begin{bmatrix} 10 & -(10ae^b + 5a) \\ -(10a + \cos a) & 10e^b + 5b \end{bmatrix}$$

where,

$$d = det(f'(w_0)) = 100e^b + 50b - (10ae^b + 5a)(10a + \cos a).$$

Thus,

$$\Gamma_0 f(w_0) = \frac{1}{d}\Big(10x(w_0) - (10ae^b + 5a)y(w_0),\ (10e^b + 5b)x(w_0) - (10a + \cos a)y(w_0)\Big)^T$$

and

$$\Gamma_0 f''(w_0) = \frac{1}{d}\left[\begin{array}{cc} -(10ae^b + 5a)(10 - \sin a) & 10(10e^b + 5) \\ 10(10e^b + 5) & 100ae^b \\ \hline (10e^b + 5b)(10 - \sin a) & -(10a + \cos a)(10e^b + 5) \\ -(10a + \cos a)(10e^b + 5) & -(10a + \cos a)(10ae^b) \end{array}\right].$$

Note that,

$$\|\Gamma_0 f(w_0)\| \leq \frac{1}{|d|}\max\Big\{\big|10x(w_0) - (10ae^b + 5a)y(w_0)\big|,$$
$$\big|(10e^b + 5b)x(w_0) - (10a + \cos a)y(w_0)\big|\Big\} \tag{3.20}$$

w_0	β	γ	$\kappa{:=}\beta\gamma$	$\bar{\alpha}$	$\bar{\alpha}^*$
$(0.005, 0.005)^T$	0.004991418...	1.3298437705...	0.006637806...	0.005025226...	0.3734544972...
$(0.025, 0.025)^T$	0.0248019056...	1.3499787325...	0.0334820451...	0.0257276226...	0.3570495353...
$(0.05, 0.05)^T$	0.0492865108...	1.3769060641...	0.0678628956...	0.0535490831...	0.3342271506...
$(0.075, 0.075)^T$	0.0735652726...	1.4058782672...	0.103423818...	0.085142697...	0.3072895083...

Table 3: Estimate value of β, γ, κ, $\bar{\alpha}$, $\bar{\alpha}^*$

and

$$\begin{aligned}\|\varGamma_0 f''(w_0)\| \leq \frac{1}{|d|}\max\Big\{&\big|(10ae^b+5a)(10-\sin a)\big| + \big|10(10e^b+5)\big| + \big|(10e^b+5b)(10-\sin a)\big| \\ &+ \big|(10a+\cos a)(10e^b+5)\big|,\quad \big|10(10e^b+5)\big| + \big|100ae^b\big| + \big|(10a+\cos a)(10e^b+5)\big| \\ &+ \big|(10a+\cos a)(10ae^b)\big|\Big\} \end{aligned}\tag{3.21}$$

By comparing Eq. (3.16) *with* (3.20) *and* (3.21), *we have*

$$\begin{aligned}\beta := \frac{1}{|d|}\max\Big\{&\big|10x(w_0) - (10ae^b+5a)y(w_0)\big|, \\ &\big|(10e^b+5b)x(w_0) - (10a+\cos a)y(w_0)\big|\Big\}\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}\gamma \leq \frac{1}{2|d|}\max\Big\{&\big|(10ae^b+5a)(10-\sin a)\big| + \big|10(10e^b+5)\big| + \big|(10e^b+5b)(10-\sin a)\big| \\ &+ \big|(10a+\cos a)(10e^b+5)\big|,\quad \big|10(10e^b+5)\big| + \big|100ae^b\big| + \big|(10a+\cos a)(10e^b+5)\big| \\ &+ \big|(10a+\cos a)(10ae^b)\big|\Big\}\end{aligned}\tag{3.23}$$

So, we find the value of β, γ, $\bar{\alpha}$ *and* $\bar{\alpha}^*$ *from Eq.* (3.22), (3.23), (3.16) *and* (3.18), *respectively for the initial value* $w_0 = (0.005, 0.005)^T, (0.025, 0.025)^T, (0.05, 0.05)^T,$ $(0.075, 0.075)^T$, *are concluded in Table 3. We also see that in Table 3, the convergence criterion* $\kappa = \beta\gamma < 3 - 2\sqrt{2}$ *is also hold for corresponding initial values. Thus, we conclude that the sequence* $\{w_n\}$ *generated by MNL method* (1.3) *with initial point* w_0, *converges to a zero of* f *defined by* (3.19). *For the initial values* $w_0 = (0.005, 0.005)^T$, $(0.025, 0.025)^T$, $(0.05, 0.05)^T, (0.075, 0.075)^T$, *the corresponding existence and uniqueness domain of solution are concluded in Table 4.*

w_0	Existence	Uniqueness
$(0.005, 0.005)^T$	$\bar{B}(w_0, 0.005025226942780)$	$B(w_0, 0.373454497231585)$
$(0.025, 0.025)^T$	$\bar{B}(w_0, 0.025727622633057)$	$B(w_0, 0.357049535363272)$
$(0.05, 0.05)^T$	$\bar{B}(w_0, 0.053549083157285)$	$B(w_0, 0.334227150625873)$
$(0.075, 0.075)^T$	$\bar{B}(w_0, 0.085142697083026)$	$B(w_0, 0.307289508382676)$

Table 4: Domain of existence and uniqueness of solution for (3.19)

3.3. Convergence result under the Nesterov-Nemirovskii condition

In this subsection, we will present a convergence theorem for MNL method under the Nesterov-Nemirovskii condition.

Let $\Omega \subseteq \mathbb{R}^n$ be an open convex set and $f : \Omega \longrightarrow \mathbb{R}$ be a self-concordant function with parameter $a > 0$. Also, assume that f is a strictly convex function and of family $C^3(\Omega)$ i.e. f is thrice continuously differentiable function in Ω and satisfies the following inequality

$$|f'''(w)[h,h,h]| \leq 2\ a^{-1/2}(f''[h,h])^{3/2}, \quad w \in \Omega, \quad h \in \mathbb{R}^n.$$

Take initial point $w_0 \in \Omega$ such that $f''(w_0)$ is invertible. Define the Hilbert space $W_1 = (\mathbb{R}^n, \langle .,\ . \rangle_{w_0})$ as the Euclidean space $\mathbb{R}^n$ with a new inner product and the associated norm defined as

$$\|w_1\|_{w_0} = \sqrt{\langle w_1, w_1 \rangle_{w_0}} \qquad \forall \ \ w_1 \in \mathbb{R}^n$$

where, $\langle w_1, w_2 \rangle_{w_0} = a^{-1}\langle f''(w_0)w_1, w_2\rangle$ for all $w_1, w_2 \in \mathbb{R}^n$ and some $a > 0$. Now, we define

$$B_\wp(w_0) = \{w \in \mathbb{R}^n : \|w - w_0\|_{w_0} < \wp\} \ \ \textit{and} \ \ B_\wp[w_0] = \{w \in \mathbb{R}^n : \|w - w_0\|_{w_0} \leq \wp\}$$

open and closed balls with radius $\wp > 0$ and center w_0 in W_1 (Dikin's ellipsoid of radius $\wp > 0$ and center w_0), respectively. Let

$$g(\alpha) = \frac{\mu\alpha^2}{(1-\alpha)} - \alpha + \beta, \qquad \mu > 0 \tag{3.24}$$

be the majorant function to f'. Here, the majorant function g satisfied all assumption (I.) - (III.).

Theorem 3.8. *Let $f : \Omega \longrightarrow \mathbb{R}$ a-self-concordant function, $\Omega \subseteq \mathbb{R}^n$ be a convex set, with initial point $w_0 \in \Omega$ and $f''(w_0)$ is non-singular. Let $W_1 = (\mathbb{R}^n, \langle .,\ . \rangle_{w_0})$ be a Hilbert space. With the choice of $g(\alpha)$ from Eq.* (3.24)*, the majorant condition* (2.1) *reduces to*

$$\begin{aligned}\|f''(w_0)^{-1}[f'''(v) - f'''(w)]\| \leq 2\mu\Bigg[&\frac{1}{(1 - \|v-w\| - \|w-w_0\|)^3} \\ &- \frac{1}{(1-\|w-w_0\|)^3}\Bigg] \qquad \mu > 0\end{aligned} \tag{3.25}$$

and initial condition

$$\|f''(w_0)^{-1}f'(w_0)\| \leq \beta \ \ \textit{and} \ \ \|f''(w_0)^{-1}f'''(w_0)\| \leq 2\mu. \tag{3.26}$$

Additionally, if

$$\beta < 3 - 2\sqrt{2}, \tag{3.27}$$

then the sequence $\{w_n\}$ *generated by MNL method to solve* $f'(w) = 0$ *with starting point* w_0

$$w_{n+1} = w_n - \left[I + \frac{1}{2}\pounds_f(w_n)\Lambda_f(w_n)\right] f''(w_n)^{-1} f'(w_n), \quad n \geq 0.$$

where

$$\Lambda_f(w_n) = \left[I + \sum_{m\geq 2} 2S_m \pounds_f(w_n)^{m-1}\right], \qquad S_m \in \mathbb{R}^+, \quad m \geq 2.$$

and $\pounds_f(w_n)$ *is defined by*

$$\pounds_f(w_n) = f''(w_n)^{-1} f'''(w_n) f''(w_n)^{-1} f'(w_n), \qquad w_n \in W_1.$$

is well defined. Also, $\{w_n\}$ *is contained in* $B_{\bar{\alpha}}(w_0) = \{w \in \mathbb{R}^n : \|w - w_0\|_{w_0} < \bar{\alpha}\}$ *and converges to* σ *which is the zero of* $f'(w) = 0$. *The limit* σ *of the sequence* $\{w_n\}$ *is the unique zero of* f *in* $B_{\bar{\alpha}^*}[w_0] = \{w \in \mathbb{R}^n : \|w - w_0\|_{w_0} \leq \bar{\alpha}^*\}$, *where* $\bar{\alpha}$ *and* $\bar{\alpha}^*$ *given as*

$$\bar{\alpha} = \frac{1 + \beta - \sqrt{(1+\beta)^2 - 4(1+\mu)\beta}}{2(1+\mu)} \quad \text{and} \quad \bar{\alpha}^* = \frac{1 + \beta + \sqrt{(1+\beta)^2 - 4(1+\mu)\beta}}{2(1+\mu)} \tag{3.28}$$

respectively.

Theorem 3.9. *With the assumption of Theorem 3.8, we get the following a priori and a posteriori error estimate:*

(i) For all $n \geq 0$, *we have the following a priori estimate:*

$$\|\sigma - w_{n+1}\| \leq (\bar{\alpha} - \alpha_{n+1}) \left(\frac{\|\sigma - w_{n+1}\|}{\bar{\alpha} - \alpha_n} \right)^3.$$

Thus, the sequence $\{w_n\}$ *generated by MNL* (1.3) *converges cubic as follows*

$$\|\sigma - w_{n+1}\| \leq \frac{(3\bar{\alpha} - 4\alpha + 1)\mu + \left(\sum_{m\geq 2} \left(\frac{1}{2}\right)^{m-2} S_m \right)(1 - \bar{\alpha})\mu}{4(\bar{\alpha} - \alpha)(1 - \bar{\alpha})^2[(1 - \bar{\alpha})^2 - \mu(\bar{\alpha}^2 - 2\bar{\alpha})]} \|\sigma - w_n\|^3.$$

(ii) For all $n \geq 0$, *we have the following a posteriori error estimate:*

$$\|\sigma - w_{n+1}\| \leq (\bar{\alpha} - \alpha_{n+1}) \left(\frac{\|w_{n+1} - w_n\|}{\alpha_{n+1} - \alpha_n} \right)^3.$$

4. Conclusion

In this study, the semilocal convergence of MNL method for finding zero of nonlinear operator in $\mathbb{B}$-space is presented using majorizing function. A convergence theorem is established to present a new error estimate based on the twice directional derivative of the majorizing function and dropped the assumption of existence of second root of the majorizing function, still guaranteeing cubic convergence. Three important special convergence analysis based on the Kantorovich-type and Smale-type and Nesterov-Nemirovskii assumption under Lipschitz, γ and optimizing condition respectively are also given. Finally, we considered three numerical examples to show efficiency of our study.

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