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Bessel transforms of Dini-Lipschitz functions on Lebesgue spaces $L_{p,\gamma}(\mathbb{R}^n_+)$

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Abstract

In this paper, we obtain a generalization of Titchmarsh's theorem for the Bessel transform for functions satisfying the (ψ, p)-Bessel Lipschitz condition in the Lebesgue space $L_{p,\gamma}(\mathbb{R}^n_+)$ for $1 < p \leq 2$, $\gamma > 0$.

Keywords: Bessel operator, Bessel generalized translation, Bessel Lipschitz condition.
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1. Introduction

Integral transforms and their inverse transforms play an important role in solving various problems in calculus, mechanics, mathematical physics, and computational mathematics. However, intagral transforms connected with the Bessel functions and Bessel equations on the spaces of generalized functions have been studied intensively, recently. Bessel functions appear in a wide variety of physical problems. When one analyzes the sound vibrations of a drum, the partial differential wave equation (PDE) is solved in cylindrical coordinates. Bessel functions and closely related functions form a rich area of mathematical analysis with many representations, many interesting and useful properties, and many interrelations.

In this paper, we will consider the Bessel Lipschitz classes. As it is well known that if Lipschitz conditions are applied on a function $f(x)$, then these conditions greatly affect the absolute convergence of the Fourier series and behaviour of $F_B f$ Fourier-Bessel transforms of f. In general, if $f(x)$ belongs to a certain function class, then the Lipschitz conditions have bearing as to the dual space to which the Fourier coefficients and Fourier-Bessel transforms of $f(x)$ belong. Younis worked the

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same phenomena for the wider Dini Lipschitz class for some classes of functions (see [11]). Daher, El Quadih, Daher and El Hamma proved an analog Younis (see [11, Theorem 2.5]) in for the Fourier-Bessel transform for functions satisfies the Fourier-Bessel Dini Lipschitz condition in the Lebesgue space $L^2_{\alpha,n}$ (see [4]). El Hamma and Daher proved a generalization of Titchmarsh's theorem for the Bessel transform in the space $L_{2,\gamma}(\mathbb{R}^n_+)$ (see [3]). In this paper we prove this generalization in the space $L_{p,\gamma}(\mathbb{R}^n_+)$, where $1 < p \leq 2$ and $\gamma > 0$.

2. Preliminaries

Let $\mathbb{R}^n$ be n-dimensional Euclidean space, then we have $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Denote $\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) : x_1 > 0, \ldots, x_n > 0\}$. S^n_+ denote the unit sphere on $\mathbb{R}^n_+$, which can be defined as $S^n_+ = \{x \in \mathbb{R}^n_+ : \ |x| = 1\}$.

The Bessel differential operator $B = (B_1, \ldots, B_n)$ is defined by

$$B_i = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i},$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, $\gamma_1 > 0, \ldots, \gamma_n > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_n$, $x^\gamma = x_1^{\gamma_1} \ldots x_n^{\gamma_n}$.

We will denote by $L_{p,\gamma}(\mathbb{R}^n_+)$ the set of all measurable functions f on $\mathbb{R}^n_+$ such that the norm

$$\|f\|_{L_{p,\gamma}} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p x^\gamma dx \right)^{1/p}, \quad 1 \leq p < \infty$$

is finite.

The generalized shift operator T^y is defined by

$$T^y f(x) = C_\gamma \int_0^\pi \ldots \int_0^\pi f\left((x_i, y_i)_{\alpha_i}\right) d\mu(\alpha),$$

where $(x_i, y_i)_{\alpha_i} = (x_i^2 + y_i^2 - 2x_i y_i \cos\alpha_i)^{\frac{1}{2}}$, $1 \leq i \leq n$, $d\mu(\alpha) = \prod_{i=1}^n \sin^{\gamma_i - 1} \alpha_i d\alpha_1 \ldots d\alpha_n$ and

$$C_\gamma = \pi^{-\frac{n}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i + 1}{2}\right) \Gamma^{-1}\left(\frac{\gamma_i}{2}\right),$$

(see [5, 8]). Note that the generalized shift operator is closely connected with Bessel differential operator (see [8]). The singular integral operators generated by generalized shift operator have been studied many mathematicians (see [1, 2, 7]).

For $\gamma \geq 0$, we introduce the Bessel normalized function of the first kind j_γ defined by

$$j_\gamma(z) = \Gamma(\gamma + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \gamma + 1)} \left(\frac{z}{2}\right)^{2n}, \tag{2.1}$$

where Γ is the gamma-function (see [9]). Moreover, from (2.1) we see that

$$\lim_{z \to 0} \frac{j_{\frac{\gamma-1}{2}}(z) - 1}{z^2} \neq 0$$

by consequence, there exist $c > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta >\Rightarrow \left| j_{\frac{\gamma-1}{2}}(z) - 1 \right| \geq c|z|^2. \tag{2.2}$$

The function $u = j_{\frac{\gamma-1}{2}}(z)$ satisfies the differential equation

$$B_{x_i} u(x, y) = B_{y_i} u(x, y)$$

with the initial conditions $u(x, 0) = f(x)$ and $u_y(x, 0) = 0$ is function infinitely differentiable, even, and, moreover entire analytic.

The Fourier-Bessel transform we call the integral transform from [6, 9, 10]

$$F_B f(\xi) = \int_{\mathbb{R}^n_+} f(x) \prod_{i=1}^{n} j_{\frac{\gamma_i-1}{2}}(x_i \xi_i) x^\gamma dx,$$

and its inverse transforms can be given by the formula

$$F_B^{-1} f(x) = C_{n,\gamma} \int_{\mathbb{R}^n_+} F_B f(\xi) \prod_{i=1}^{n} j_{\frac{\gamma_i-1}{2}}(x_i \xi_i) \xi^\gamma d\xi,$$

where $j_\gamma(x) = 2^\gamma \Gamma(\gamma+1) \frac{J_\gamma(x)}{x^\gamma} = \dfrac{\Gamma(\gamma+1)}{\sqrt{\pi}\Gamma(\frac{\gamma+1}{2})} \int\limits_{-1}^{1} (1-u^2)^{\frac{\gamma-1}{2}} \cos(xu) du$, for $\gamma > 0$. $J_\gamma(x)$ being the Bessel function of the first kind.

The following relation connect the Bessel generalized translation and the Bessel transform, we have

$$F_B[T_x^y f(x)] = j_{\frac{\gamma_i-1}{2}}(x_i y_i) F_B[f(x)], \qquad i = 1, \ldots, n. \tag{2.3}$$

We have the Hausdorff-Young inequality

$$\|F_B f\|_{q,\gamma} \leq C \|f\|_{p,\gamma}, \tag{2.4}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and C is a positive constant.

3. Main Result

In this section, we give the main result of this paper. We need first to define (ψ, p)-Bessel Lipschitz class.

Definition 3.1. *A function $f \in L_{p,\gamma}(\mathbb{R}^n_+)$ is said to be in the (ψ, p)-Bessel Lipschitz class, denoted by $Lip(\psi, \gamma, p)$, if*

$$\|T^y f(x) - f(x)\|_{p,\gamma} = O(\psi(y)) \quad as \quad y \to 0,$$

where

1. *$\psi(x)$ is a continuous increasing function on $\mathbb{R}^n_+$,*
2. *$\psi(0) = 0$,*
3. *$\psi(xs) = \psi(x)\psi(s)$ for all $x, s \in \mathbb{R}^n_+$.*

Theorem 3.2. *Let $f(x)$ belong to $Lip(\psi,\gamma,p)$. Then*

$$\int_{\mathbb{R}^n_+} |F_B f(\xi)|^q \xi^\gamma d\xi = C_{n,\gamma}\, O(\psi(r^{-q})), \quad r \to +\infty,$$

where $C_{n,\gamma} = r^n \omega(n,\gamma)$, $\omega(n,\gamma) = \dfrac{\prod_{i=1}^n \Gamma(\frac{\gamma_i+1}{2})}{2^{n-1}\Gamma(\frac{|\gamma|+n}{2})}$.

Proof . Let $f \in Lip(\psi,\gamma,p)$. Then we have

$$\|T^y f(x) - f(x)\|_{p,\gamma} = O(\psi(y)), \quad y \to 0.$$

Now we consider Fourier-Bessel transform of generalized shift operator. We get

$$\begin{aligned} F_B[T^y f(x)](\xi) &= \int_{\mathbb{R}^n_+} T^y f(x) j_{\frac{\gamma-1}{2}}(x\xi) x^\gamma dx \\ &= \int_{\mathbb{R}^n_+} T^y \left[j_{\frac{\gamma-1}{2}}(x\xi)\right] f(x) x^\gamma dx \\ &= \int_{\mathbb{R}^n_+} j_{\frac{\gamma-1}{2}}(x\xi) j_{\frac{\gamma-1}{2}}(y\xi) f(x) x^\gamma dx \\ &= j_{\frac{\gamma-1}{2}}(y\xi) \int_{\mathbb{R}^n_+} f(x) j_{\frac{\gamma-1}{2}}(x\xi) x^\gamma dx \\ &= j_{\frac{\gamma-1}{2}}(y\xi) F_B(f)(\xi), \end{aligned}$$

where $T^y(j_p(\sqrt{\lambda}x)) = j_p(\sqrt{\lambda}y) j_p(\sqrt{\lambda}x)$. From formulas (2.3) and (2.4), we obtain

$$\begin{aligned} \int_{\mathbb{R}^n_+} F_B|T^y f(x) - f(x)|^q x^\gamma dx &= \int_{\mathbb{R}^n_+} |F_B T^y f(x) - F_B f(x)|^q x^\gamma dx \\ &= \int_{\mathbb{R}^n_+} |j_{\frac{\gamma-1}{2}}(\xi y) F_B f(\xi) - F_B f(\xi)|^q \xi^\gamma d\xi \\ &= \int_{\mathbb{R}^n_+} |F_B f(\xi)[1 - j_{\frac{\gamma-1}{2}}(\xi y)]|^q \xi^\gamma d\xi \\ &= \int_{\mathbb{R}^n_+} |1 - j_{\frac{\gamma-1}{2}}(\xi y)|^q |F_B f(\xi)|^q \xi^\gamma d\xi \\ &\le C_q \int_{\mathbb{R}^n_+} |T^y f(x) - f(x)|^q \xi^\gamma d\xi \\ &\le C_q \|T^y f(x) - f(x)\|^q_{p,\gamma}. \end{aligned}$$

From (2.2), we have

$$\begin{aligned}\int_{\mathbb{R}^n_+} |F_Bf(\xi)|^q \xi^\gamma d\xi &= \int_{S^{n-1}_+} \int_0^\infty |F_Bf(r\theta)|^q r^\gamma \theta^\gamma r^{n-1} dr d\theta \\ &= \int_{S^{n-1}_+} \left(\int_0^\infty |F_Bf(r\theta)|^q \theta^\gamma d\theta \right) r^{n+\gamma-1} dr \\ &= \int_{S^{n-1}_+} r^{n+\gamma-1} dr \int_0^\infty |F_Bf(r\theta)|^q \frac{\theta^\gamma}{r^\gamma} d\theta \\ &= \int_{S^{n-1}_+} r^{n-1} dr \int_0^\infty |F_Bf(r\theta)|^q \theta^\gamma d\theta \\ &= r^n \omega(n,\gamma) \int_0^\infty |F_Bf(\xi)|^q \xi^\gamma d\xi,\end{aligned}$$

where $\xi = r\theta$, $d\xi = r^{n-1} dr d\theta$ and $\omega(n,\gamma) = \dfrac{\prod_{i=1}^n \Gamma(\frac{\gamma_i+1}{2})}{2^{n-1}\Gamma\left(\frac{|\gamma|+n}{2}\right)}$.

$$\begin{aligned}\int_{\frac{1}{y}}^{\frac{2}{y}} |F_Bf(\xi)|^q \xi^\gamma d\xi &= C_q \int_{\frac{1}{y}}^{\frac{2}{y}} |1 - j_{\frac{\gamma-1}{2}}(\xi y)|^q |F_Bf(\xi)|^q \xi^\gamma d\xi \\ &\geq\ C_q |y|^{-1} \int_{\frac{1}{y}}^{\frac{2}{y}} |F_Bf(\xi)|^q \xi^\gamma d\xi,\end{aligned}$$

where $0 < y \leq 1$. It follows from the above consideration that there exists a positive constant C such that

$$\int_{\frac{1}{y}}^{\frac{2}{y}} |F_Bf(\xi)|^q \xi^\gamma d\xi \leq C\psi^q(y) = C\psi(y^q).$$

Then

$$\int_\tau^{2\tau} |F_Bf(\xi)|^q \xi^\gamma d\xi \leq C\psi(\tau^{-q}),$$

of course

$$\begin{aligned}\int_\tau^\infty |F_Bf(\xi)|^q \xi^\gamma d\xi &= \left(\int_\tau^{2\tau} + \int_{2\tau}^{2^2\tau} + \ldots + \int_{2^{n-1}\tau}^{2^n\tau} \right) |F_Bf(\xi)|^q \xi^\gamma d\xi \\ &\leq C_q\psi(\tau^{-q}) + C_q\psi\left((2\tau)^{-q}\right) + C_q\psi\left((2^2\tau)^{-q}\right) + \ldots \\ &\leq C_q\psi(\tau^{-q})\left(1 + \psi(2^{-q}) + \psi^2(2^{-q}) + \psi^3(2^{-q}) + \ldots\right).\end{aligned}$$

Therefore, we have

$$\int_\tau^\infty |F_Bf(\xi)|^q \xi^\gamma d\xi \leq C_1\psi(\tau^{-q}),$$

where $C_1 = C_q(1 - \psi(2^{-q}))^{-1}$ since $2^{-q} < 1$. Finally, we get

$$\int_\tau^\infty |F_Bf(\xi)|^q \xi^\gamma d\xi = r^n\omega(n,\gamma)\, O(\psi(\tau^{-q})) \quad \text{as} \quad \tau \to \infty.$$

Thus, the proof is completed. □

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