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Generalized Mittag-Leffler stability of nonlinear fractional regularized Prabhakar differential systems

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Abstract

This work is devoted to study of the stability analysis of generalized fractional nonlinear system including the regularized Prabhakar derivative. We present several criteria for the generalized Mittag-Leffler stability and the asymptotic stability of this system by using the Lyapunov direct method. Further, we provide two test cases to illustrate the effectiveness of results. We apply the numerical method to solve the generalized fractional system with the regularized Prabhakar fractional systems and reveal asymptotic stability behavior of the presented systems by employing numerical simulation.

Keywords: Generalized Mittag-Leffler stability, Asymptotic stability, Regularized Prabhakar derivative, Lyapunov direct method, Numerical simulation.
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1. Introduction

In the recent years, fractional nonlinear differential systems (FNDSs) and assessing their stability have catched a great attention, for example, [12, 13, 14, 26, 27, 36, 41]. However, comparing to that of the integer order systems, the stability analysis of FNDSs is much more complex and only a few works are available. One of the available technique to examine the stability of FNDSs is the Lyapunov direct method (also is called the second method of Lyapunov). The method provides a very effective approach to analyze the stability of nonlinear systems without explicitly solving the differential equations and generalizes the idea that the system is stable if there are some Lyapunov function candidates for the system [4]. This method allows one to evaluate the asymptotic stability

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and the Mittag-Leffler stability for FNDSs. Some authors [27, 26, 40] investigated the asymptotic stability of the following FNDS with the Caputo derivative by using the Lyapunov direct method

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = f(t, x(t)), \quad x(t_0) = x_0, $$

where $0 < \alpha < 1$, $x(t), f(t, x) \in \mathbb{R}^n$ and t represents the time.

In [16, 24, 34], authors generalized the fractional Riemann-Liouville (Caputo) integral and derivative to the fractional Prabhakar integral and derivative with the generalized Mittag-Leffler function (Prabhakar function [34])

$$ E_{\rho,\mu}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)\Gamma(\rho k + \mu)} \frac{z^k}{k!}, \qquad \gamma, \rho, \mu \in \mathbb{C}, \Re(\rho) > 0, $$

in kernel. See more details of the generalized Mittag-Leffler function in [5, 7, 8, 9, 11, 14, 16, 23, 25, 29, 33, 35, 37, 38].

The Prabhakar derivative has fundamental applications in the applied mathematics [3, 8, 9, 10, 11, 12, 14, 16, 33], the time-evolution of polarization processes [16, 17, 21, 33], the fractional Poisson process [16], the fractional Maxwell model in linear viscoelasticity [19], the generalized model of particle deposition in porous media [39] and the generalized reaction-diffusion equations [1]. The great importance for considering the Prabhakar derivative and integral is related to the description of relaxation and response in the anomalous dielectrics of the Havriliak-Negami models [15, 30, 18, 31].

In the present paper, by using the Lyapunov function, we intend to investigate the stability of the following generalized FNDS with the regularized Prabhakar derivative

$$ {}^{C}D_{\rho,\mu,\omega,0+}^{\gamma}x(t) = Ax(t) + f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1} $$

where $\gamma, \mu, \omega \in (0, 1)$, $0 < \rho < 2$, $\rho\gamma < \mu$, $x(t) \in \mathbb{R}^n$ is a state vector, $A \in \mathbb{R}^{n\times n}$ is a constant matrix and $f(t, x) \in \mathbb{R}^n$ with $f(t, 0) = 0$. Our aim is to give several criteria for the generalized Mittag-Leffler stability and the asymptotic stability.

The paper is organized as follows. In Section 2, some definitions and properties of the generalized fractional calculus including the regularized Prabhakar derivative are given. In Section 3, we provide several criteria for the generalized Mittag-Leffler stability and the asymptotic stability by using the Lyapunov direct method in the sense of the regularized Prabhakar derivative. In Section 4, a numerical method is presented for solving the differential equations with the generalized fractional derivative. In order to illustrate the applications of our result, in Section 5, two examples are provided and the numerical values of examples are depicted. In Section 6, the concluding remarks are given.

2. Preliminaries

2.1. The generalized fractional calculus

Definition 2.1. *For $f \in L^1[0, b]$, the Prabhakar integral operator with generalized Mittag-Leffler function in its kernel is defined as follows [16]*

$$ E_{\rho,\mu,\omega,0+}^{\gamma}f(x) = \int_0^x (x-u)^{\mu-1}E_{\rho,\mu}^{\gamma}\big(\omega(x-u)^{\rho}\big)f(u)du, \qquad 0 < x < b \leq \infty, \tag{2.1} $$

where $\rho, \mu, \omega, \gamma \in \mathbb{C}, \Re(\rho), \Re(\mu) > 0$.

Definition 2.2. *For $f \in L^1[0,b]$, the Prabhakar derivative is defined by [16]*

$$D^{\gamma}_{\rho,\mu,\omega,0+}f(x) = \frac{d^m}{dx^m} E^{-\gamma}_{\rho,m-\mu,\omega,0+}f(x), \quad 0 < x < b \leq \infty, \tag{2.2}$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C}, \Re(\rho), \Re(\mu) > 0$. Also, the regularized Caputo counterpart for $f \in AC^m[0,b]$, is presented by

$$\begin{aligned} {}^C D^{\gamma}_{\rho,\mu,\omega,0+}f(x) &= E^{-\gamma}_{\rho,m-\mu,\omega,0+}\frac{d^m}{dx^m}f(x) \\ &= D^{\gamma}_{\rho,\mu,\omega,0+}f(x) - \sum_{k=0}^{m-1} x^{k-\mu}E^{-\gamma}_{\rho,k-\mu+1}(\omega x^{\rho})f^{(k)}(0+). \end{aligned} \tag{2.3}$$

Remark 2.3. *In case $\gamma = 0$, the Prabhakar integral operator (2.1) coincides with the Riemann-Liouville fractional integral of order μ. Thus, the Prabhakar derivative and the regularized Prabhakar derivative (2.2) and (2.3) generalize the Riemann-Liouville and the Caputo fractional derivatives of order μ, respectively.*

2.2. Properties of the regularized Prabhakar derivative

We now establish several important lemmas and theorems of the regularized Prabhakar derivative which will be used for our main results.

Lemma 2.4. *Let ${}^C D^{\gamma}_{\rho,\mu,\omega,0+}x(t) \geq {}^C D^{\gamma}_{\rho,\mu,\omega,0+}y(t)$ and $x(0) = y(0)$, where $\gamma, \mu, \omega \in (0,1)$, $0 < \rho < 2$, $\rho\gamma < \mu$. Then, $x(t) \geq y(t)$ for $t > 0$.*

Proof . It is a straightforward result from ${}^C D^{\gamma}_{\rho,\mu,\omega,0+}x(t) \geq {}^C D^{\gamma}_{\rho,\mu,\omega,0+}y(t)$ that there exists a non-negative function $n(t)$ satisfying

$${}^C D^{\gamma}_{\rho,\mu,\omega,0+}x(t) = n(t) + {}^C D^{\gamma}_{\rho,\mu,\omega,0+}y(t). \tag{2.4}$$

Applying the Laplace transform on the both sides of (2.4) and using the Laplace transform of regularized Prabhakar derivative [10], we obtain

$$\frac{s^{-\rho\gamma+\mu}}{(s^{\rho}-\omega)^{-\gamma}}X(s) - \frac{s^{-\rho\gamma+\mu-1}}{(s^{\rho}-\omega)^{-\gamma}}x(0) = N(s) + \frac{s^{-\rho\gamma+\mu}}{(s^{\rho}-\omega)^{-\gamma}}Y(s) - \frac{s^{-\rho\gamma+\mu-1}}{(s^{\rho}-\omega)^{-\gamma}}y(0).$$

We take into account $x(0) = y(0)$ and use the inverse Laplace transform to get

$$x(t) = E^{\gamma}_{\rho,-\mu,\omega,0+}n(t) + y(t).$$

Since $n(t) \geq 0$, $x(t) \geq y(t)$. ☐

Lemma 2.5. *Assume that $\gamma, \mu, \omega \in (0,1)$, $0 < \rho < 2$, $\rho\gamma < \mu$ and $x(t) \in \mathbb{R}^n$ is a vector of differentiable functions. If a continuous function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ satisfies*

$${}^C D^{\gamma}_{\rho,\mu,\omega,t_0+}V(t,x(t)) \leq -\beta V(t,x(t)), \quad \beta \in \mathbb{R}^+ - \{0\}, \tag{2.5}$$

then

$$V(t,x(t)) \leq V(t_0,x(t_0))\sum_{k=0}^{\infty}(t-t_0)^{\mu k}E^{\gamma k}_{\rho,\mu k+1}(-\beta\omega(t-t_0)^{\rho}).$$

Proof . It is a straightforward result from (2.5) that there exists a nonnegative function $N(t)$ satisfying

$$ {}^C D^{\gamma}_{\rho,\mu,\omega,t_0+} V(t,x(t)) + \beta V(t,x(t)) + N(t) = 0. $$

By applying the Laplace transform on the above equation, we have

$$ s^{\mu}(1-\omega s^{-\rho})^{\gamma} V(s) - s^{\mu-1}(1-\omega s^{-\rho})^{\gamma} V(t_0,x(t_0)) + \beta V(s) + N(s) = 0, $$

where $V(s)$ and $N(s)$ are the Laplace transforms of $V(t,x(t))$ and $N(t)$, respectively. One can easily obtain

$$ \begin{aligned} V(s) &= \frac{s^{-\rho\gamma+\mu-1}(s^{\rho}-\omega)^{\gamma} V(t_0,x(t_0)) - N(s)}{s^{-\rho\gamma+\mu}(s^{\rho}-\omega)^{\gamma}+\beta} \\ &= \frac{s^{-1}V(t_0,x(t_0))}{1+\beta s^{\rho\gamma-\mu}(s^{\rho}-\omega)^{-\gamma}} - \frac{N(s)s^{\rho\gamma-\mu}(s^{\rho}-\omega)^{-\gamma}}{1+\beta s^{\rho\gamma-\mu}(s^{\rho}-\omega)^{-\gamma}} \\ &= V(t_0,x(t_0)) \sum_{k=0}^{\infty} (-\beta)^k \frac{s^{(\rho\gamma-\mu)k-1}}{(s^{\rho}-\omega)^{\gamma k}} - N(s) \sum_{k=0}^{\infty} (-\beta)^k \frac{s^{\rho\gamma(k+1)-\mu(k+1)}}{(s^{\rho}-\omega)^{\gamma(k+1)}}. \end{aligned} \tag{2.6} $$

Taking the inverse Laplace transform of (2.6), we get

$$ \begin{aligned} V(t,x(t)) = V(t_0,x(t_0)) &\sum_{k=0}^{\infty} (t-t_0)^{\mu k} E^{\gamma k}_{\rho,\mu k+1}(-\beta\omega(t-t_0)^{\rho}) \\ - N(t) * &\sum_{k=0}^{\infty} (t-t_0)^{\mu(k+1)-1} E^{\gamma(k+1)}_{\rho,\mu(k+1)}(-\beta\omega(t-t_0)^{\rho}), \end{aligned} $$

where $*$ indicates the Laplace convolution integral given by

$$ (f*g)(t) = \int_0^t f(t-\xi) g(\xi) d\xi. $$

Since both $(t-t_0)^{\mu(k+1)-1}$ and $E^{\gamma(k+1)}_{\rho,\mu(k+1)}(-\beta\omega(t-t_0)^{\rho})$ are nonnegative functions, we deduce

$$ V(t,x(t)) \le V(t_0,x(t_0)) \sum_{k=0}^{\infty} (t-t_0)^{\mu k} E^{\gamma k}_{\rho,\mu k+1}(-\beta\omega(t-t_0)^{\rho}), $$

and the proof is completed. □

Lemma 2.6. *Let $\gamma,\mu,\omega \in (0,1)$, $0<\rho<2$, $\rho\gamma<\mu$ and $x(t)\in\mathbb{R}$ be a differentiable function. Then, for any time instant $t>t_0$, the following inequality holds*

$$ \frac{1}{2} {}^C D^{\gamma}_{\rho,\mu,\omega,t_0+} x^2(t) \le x(t) {}^C D^{\gamma}_{\rho,\mu,\omega,t_0+} x(t). \tag{2.7} $$

Proof . According to the relation (2.3), the inequality (2.7) is equivalent to

$$ \int_{t_0}^{t} (t-u)^{-\mu} E^{-\gamma}_{\rho,1-\mu}\big(\omega(t-u)^{\rho}\big)[x(t)-x(u)]x'(u)du \ge 0. \tag{2.8} $$

Therefore, it is sufficient to show that the inequality (2.8) is true. By letting $z(u) = x(t) - x(u)$, the left-hand side of (2.8) can be written as

$$\int_{t_0}^{t}(t-u)^{-\mu}E^{-\gamma}_{\rho,1-\mu}\big(\omega(t-u)^{\rho}\big)[x(t)-x(u)]x'(u)du$$
$$= -\int_{t_0}^{t}(t-u)^{-\mu}E^{-\gamma}_{\rho,1-\mu}\big(\omega(t-u)^{\rho}\big)z(u)z'(u)du. \tag{2.9}$$

Integrating by parts from (2.9), and then using

$$\left(\frac{d}{dx}\right)^{n}\big[x^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega x^{\rho})\big] = x^{\mu-n-1}E^{\gamma}_{\rho,\mu-n}(\omega x^{\rho}), \quad n\in\mathbb{N},$$

we get

$$\int_{t_0}^{t}(t-u)^{-\mu}E^{-\gamma}_{\rho,1-\mu}\big(\omega(t-u)^{\rho}\big)z(u)z'(u)du$$
$$= -\frac{1}{2}\big[z^2(u)(t-u)^{-\mu}E^{-\gamma}_{\rho,1-\mu}\big(\omega(t-u)^{\rho}\big)\big]|_{u=t} + \frac{1}{2}\big[{z_0}^2(t-t_0)^{-\mu}E^{-\gamma}_{\rho,1-\mu}\big(\omega(t-t_0)^{\rho}\big)\big]$$
$$+\frac{1}{2}\int_{t_0}^{t} z^2(u)(t-u)^{-\mu-1}E^{-\gamma}_{\rho,1-\mu}\big(\omega(t-u)^{\rho}\big)du. \tag{2.10}$$

Now, we consider the following limit for the first term of right-hand side of (2.10) and present

$$\lim_{u\to t}\frac{1}{2}\big[z^2(u)(t-u)^{-\mu}E^{-\gamma}_{\rho,1-\mu}\big(\omega(t-u)^{\rho}\big)\big]$$
$$=\frac{1}{2}\lim_{u\to t}\big[(x(t)^2-2x(t)x(u)+x(t)^2)(t-u)^{-\mu}E^{-\gamma}_{\rho,1-\mu}\big(\omega(t-u)^{\rho}\big)\big].$$

By employing the L'Hopital rule, we conclude

$$\frac{1}{2}\lim_{u\to t}\big[(x(t)^2-2x(t)x(u)+x(t)^2)(t-u)^{-\mu}E^{-\gamma}_{\rho,1-\mu}\big(\omega(t-u)^{\rho}\big)\big]$$
$$=\lim_{u\to t}\big[(x(t)x'(u)-x(u)x'(u))(t-u)^{-\mu-1}E^{-\gamma}_{\rho,-\mu}\big(\omega(t-u)^{\rho}\big)\big]=0.$$

Hence,

$$\frac{1}{2}\big[{z_0}^2(t-t_0)^{-\mu}E^{-\gamma}_{\rho,1-\mu}\big(\omega(t-t_0)^{\rho}\big)\big]+\frac{1}{2}\int_{t_0}^{t} z^2(u)(t-u)^{-\mu-1}E^{-\gamma}_{\rho,1-\mu}\big(\omega(t-u)^{\rho}\big)du \geq 0,$$

and this confirms the inequality (2.8). □

Remark 2.7. *It is clear that if $x(t)\in\mathbb{R}^n$ is a vector of differentiable functions, then for any time instant $t>t_0$*

$$\frac{1}{2}{}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}x^T(t)x(t) \leq x^T(t)\,{}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}x(t). \tag{2.11}$$

This can be deduced by employing Lemma 2.6 and decomposing (2.11) *into a sum of scalar products.*

Theorem 2.8. *[2]. Let $P\in\mathbb{R}^{n\times n}$ be a real symmetric matrix. Then, there exists an orthogonal matrix $B\in\mathbb{R}^{n\times n}$ and a diagonal matrix $\Lambda\in\mathbb{R}^{n\times n}$ such that*

$$P = B\Lambda B^T.$$

Lemma 2.9. *Let $x(t) \in \mathbb{R}^n$ be a vector of differentiable functions and $P \in \mathbb{R}^{n\times n}$ be a constant, square, symmetric and positive definite matrix. Then, for any time instant $t \geq t_0$, the following relationship holds*

$$\frac{1}{2}{}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}(x^T(t)Px(t)) \leq x^T(t)P\,{}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}x(t),$$

where $\gamma, \mu, \omega \in (0,1)$, $0 < \rho < 2$, $\rho\gamma < \mu$.

Proof . The proof is similar to that of Lemma 4 in [6] by employing Theorem 2.8 and Remark 2.7. □

Remark 2.10. *Lemma 2.9 also holds if P is a symmetric and positive semi-definite matrix.*

3. Lyapunov direct method for the stability analysis of generalized FNDS with the regularized Prabhakar derivative

We now present several criteria for the generalized Mittag-Leffler and the asymptotic stability by using the Lyapunov direct method.

Definition 3.1. *x_0 is an equilibrium point of the regularized Prabhakar FNDS* (1.1)*, if and only if $A(x_0) + f(t, x_0) = 0$.*

Definition 3.2. *The regularized Prabhakar FNDS* (1.1) *is stable for any initial value x_0 and $t > 0$, if there exists $\epsilon > 0$ such that $\| x(t) \| < \epsilon$. The system is asymptotically stable if it is stable and* $\lim_{t\to\infty} \| x(t) \| = 0$.

Remark 3.3. *Without loss of generality, we suppose the equilibrium point is $x_0 = 0$. Otherwise, an equilibrium point $\widetilde{x} \neq 0$ can be transformed to the origin by the change of variables $y = x - \widetilde{x}$. Therefore, the regularized Prabhakar derivative of y is given by*

$${}^{C}D^{\gamma}_{\rho,\mu,\omega,0+}y = {}^{C}D^{\gamma}_{\rho,\mu,\omega,0+}(x - \widetilde{x}) = Ax + f(t,x) = A(y + \widetilde{x}) + f(t, y + \widetilde{x}) = By + g(t, y),$$

where $B(0) + g(t, 0) = 0$. For the new variable y, the system has equilibrium at the origin.

Definition 3.4. *The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is the Lipschitz continuous if there exists a constant L such that*

$$\| f(t, x) - f(t, y) \| \leq L \| x - y \|,$$

for all $x, y \in \mathbb{R}^n$.

Lemma 3.5. *((S-procedure)[28]). Let $\Omega_0(z)$ and $\Omega_1(z)$ be two arbitrary quadratic forms over $\mathbb{R}^s$. Then $\Omega_0(z) < 0$ if and only if there exists a scalar $\zeta \geq 0$ such that*

$$\Omega_0(z) - \zeta\Omega_1(z) < 0, \quad \forall z \in \mathbb{R}^s - \{0\},$$

for $\Omega_1(z) \leq 0$.

Definition 3.6. *[22] A continuous function $\alpha : [0, t) \to [0, \infty)$ belongs to class-$\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.*

Definition 3.7. *The trivial solution of* (1.1) *is the generalized Mittag-Leffler stable if*

$$\| x(t) \| \leq \Big[m(x(t_0)) \sum_{k=0}^{\infty} (t-t_0)^{\mu k} E^{\gamma k}_{\rho,\mu k+1}(-\sigma\omega(t-t_0)^{\rho})\Big]^{\alpha},$$

where t_0 is the initial time, $\sigma \geq 0$, $\gamma, \mu, \omega \in (0,1)$, $0 < \rho < 2$, $\rho\gamma < \mu$, $\alpha > 0$. The function $m(x) \geq 0$ with $m(0) = 0$ is the locally Lipschitz with the Lipschitz constant m_0.

Corollary 3.8. *According to the following asymptotic behaviors [30, 32]*

$$t^{\mu-1}E^{\gamma}_{\rho,\mu}(-\lambda t^{\rho}) \sim \lambda^{-\gamma}\frac{t^{\mu-\rho\gamma-1}}{\Gamma(\mu-\rho\gamma)},\ 0<\rho<2,\ 0<\rho\gamma<\mu\leq 1,\ \lambda>0,\ t\to\infty, \tag{3.1}$$

$$E_{\rho}(z) = -\sum_{k=1}^{q}\frac{z^{-k}}{\Gamma(1-\rho k)} + O(|z^{-1-q}|),\ 0<\rho<2,\ \frac{\rho\pi}{2}<\theta<\min\{\pi,\rho\pi\},\ \theta\leq|\arg(z)|\leq\pi,\ z\to\infty, \tag{3.2}$$

the term $\sum_{k=0}^{\infty}(t-t_0)^{\mu k}E^{\gamma k}_{\rho,\mu k+1}(-\sigma\omega(t-t_0)^{\rho})$ tends to zero as $t\to\infty$. Thus, the generalized Mittag-Leffler stability implies the asymptotic stability.

Theorem 3.9. *Let $x = 0$ be an equilibrium point for the system* (1.1) *and $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Also, let $V(t,x(t)) : [0,\infty)\times\mathbb{D}\to\mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x such that*

$$\alpha_1 \| x \|^{a} \leq V(t,x(t)) \leq \alpha_2 \| x \|^{ab}, \tag{3.3}$$

$$^{C}D^{\gamma}_{\rho,\mu,\omega,0+}V(t,x(t)) \leq -\alpha_3 \| x \|^{ab}, \quad t\geq 0, x\in\mathbb{D}, \tag{3.4}$$

where $\gamma,\mu,\omega\in(0,1)$, $0<\rho<2$, $\rho\gamma<\mu$ and $\alpha_1,\alpha_2,\alpha_3,a,b$ are arbitrary positive constants. Then $x = 0$ is stable in the sense of the generalized Mittag-Leffler function. If the assumptions hold globally on $\mathbb{R}^n$, then $x = 0$ is globally stable in the sense of the generalized Mittag-Leffler function.

Proof . Using (3.3) and (3.4), one gets

$$^{C}D^{\gamma}_{\rho,\mu,\omega,0+}V(t,x(t)) \leq -\frac{\alpha_3}{\alpha_2}V(t,x(t)). \tag{3.5}$$

Evidently, (3.5) implies that there exists a non-negative function $N(t)$ such that

$$^{C}D^{\gamma}_{\rho,\mu,\omega,0+}V(t,x(t)) + N(t) + \frac{\alpha_3}{\alpha_2}V(t,x(t)) = 0. \tag{3.6}$$

By applying the Laplace transform operator on the equation (3.6) and taking into account $V(s) = \mathcal{L}\{V(t,x(t))\}$ and $N(s) = \mathcal{L}\{N(t)\}$, we get

$$V(s) = \frac{s^{-\rho\gamma+\mu-1}(s^{\rho}-\omega)^{\gamma}V(0,x(0)) - N(s)}{s^{-\rho\gamma+\mu}(s^{\rho}-\omega)^{\gamma}+\frac{\alpha_3}{\alpha_2}}. \tag{3.7}$$

Obviously, if $x(0) = 0$ then $V(0,x(0)) = 0$ and $x = 0$ is a solution of (1.1). If $x(0) \neq 0$ then $V(0,x(0)) > 0$. By applying the inverse Laplace transform on (3.7), we obtain

$$\begin{aligned} V(t,x(t)) = {} & V(0,x(0))\sum_{k=0}^{\infty} t^{\mu k}E^{\gamma k}_{\rho,\mu k+1}(-\frac{\alpha_3}{\alpha_2}\omega t^{\rho}) \\ & - N(t) * \sum_{k=0}^{\infty} t^{\mu(k+1)-1}E^{\gamma(k+1)}_{\rho,\mu(k+1)}(-\frac{\alpha_3}{\alpha_2}\omega t^{\rho}). \end{aligned}$$

Since both functions $t^{\mu(k+1)-1}$ and $E^{\gamma(k+1)}_{\rho,\mu(k+1)}(-\beta\omega t^{\rho})$ are nonnegative, we have

$$V(t,x(t)) \leq V(0,x(0))\sum_{k=0}^{\infty} t^{\mu k}E^{\gamma k}_{\rho,\mu k+1}(-\beta\omega t^{\rho}). \tag{3.8}$$

Accordingly, the relations (3.3) and (3.8) yield

$$\| x(t) \| \leq [\frac{V(0,x(0))}{\alpha_1}\sum_{k=0}^{\infty} t^{\mu k}E^{\gamma k}_{\rho,\mu k+1}(-\frac{\alpha_3}{\alpha_2}\omega t^{\rho})]^{\frac{1}{a}},$$

for $x(0)\neq 0$, $\frac{V(0,x(0))}{\alpha_1} > 0$. So, $\frac{V(0,x(0))}{\alpha_1} = 0$ holds if and only if $x(0) = 0$ and $\frac{\alpha_3}{\alpha_2}\omega \geq 0$. Hence, from Definition 3.7 we conclude the generalized Mittag-Leffler stability of regularized Prabhakar FNDS (1.1). □

Theorem 3.10. *Let $x = 0$ be an equilibrium point for the regularized Prabhakar FNDS* (1.1) *and suppose that there exists a Lyapunov function $V(t,x(t))$ and class-$\mathcal{K}$ functions $\alpha_i (i = 1,2,3)$ satisfying*

$$\begin{aligned}&\alpha_1(\| x \|) \leq V(t,x(t)) \leq \alpha_2(\| x \|),\\ &{}^{C}D^{\gamma}_{\rho,\mu,\omega,0+}V(t,x(t)) \leq -\alpha_3(\| x \|), \quad t\geq 0, x\in\mathbb{D},\end{aligned}$$

where $\gamma,\mu,\omega\in(0,1)$, $0<\rho<2$, $\rho\gamma<\mu$. Then, the system (1.1) *is asymptotically stable.*

Proof . The proof is similar to that of Theorem 6.2 in [26]. In this sense, we consider Definition 3.1, Lemma 2.4 and Theorem 3.9. □

Now, by using the Lyapunov function, we consider the generalized Mittag-Leffler stability and the asymptotic stability of system (1.1).

For a real matrix A, $\| A \| = \sqrt{\lambda_{\max}(A^TA)}$ denotes the spectral norm of matrix A. In addition, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximal and the minimal eigenvalue of A, respectively. Also, $A > 0$ (or $A < 0$) means the symmetric matrix A is positive definite (or negative definite).

Theorem 3.11. *Let f be the Lipschitz continuous. Then, the trivial solution of regularized Prabhakar FNDS* (1.1) *is generalized Mittag-Leffler stable if there exists a positive definite matrix P such that for all $(t,x)\in\mathbb{R}\times(\mathbb{R}^n-\{0\})$, the following linear matrix inequality holds*

$$[Ax+f(t,x)]^TPx + x^TP[Ax+f(t,x)] < 0. \tag{3.9}$$

Proof . We choose $V(t) = x^T(t)Px(t)$ as a Lyapunov function candidate. By using Lemma 2.9 and the relation (3.9), we get

$$\begin{aligned}{}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}V(t) &\leq 2x^T(t)P\,{}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}x(t)\\ &= [Ax(t)+f(t,x(t))]^TPx(t) + x^T(t)P[Ax(t)+f(t,x(t))] < 0. \end{aligned}\tag{3.10}$$

Since f is the Lipschitz continuous, we have $f^Tf - L^2x^Tx \leq 0$. According to Lemma 3.5, there exists a constant $\zeta > 0$ such that

$$[Ax(t)+f(t,x(t))]^TPx(t) + x^T(t)P[Ax(t)+f(t,x(t))] - \zeta[f^Tf - L^2x^Tx] < 0,$$

or equivalently $(x^T,f^T)\Psi_{\zeta}(x^T,f^T)^T < 0$, where

$$\mathbf{\Psi}_{\zeta} = \begin{pmatrix} A^TP+PA+\zeta L^2A & P\\ P & -\zeta I\end{pmatrix}.$$

It is clear that $\Psi_\zeta < 0$. Setting $\lambda_{\min}(-\Psi_\zeta) = \lambda_0 > 0$, we get

$$\begin{aligned} &[Ax(t) + f(t, x(t))]^T Px(t) + x^T(t)P[Ax(t) + f(t, x(t))] - \zeta[f^T f - L^2 x^T x] \\ &\leq -\lambda_0(\| x \|^2 + \| f(t, x) \|^2) < 0. \end{aligned} \tag{3.11}$$

Since $P > 0$, we have

$$V(t) = x^T(t)Px(t) \leq \lambda_{\max} P \| x \|^2 . \tag{3.12}$$

Accordingly, the relations (3.10), (3.11) and (3.12) yield

$${}^C D^\gamma_{\rho,\mu,\omega,t_0+} V(t) \leq -\lambda_0 \lambda_{\max}(P)^{-1} V(t).$$

Hence from Lemma 2.5, we obtain

$$\begin{aligned} V(t) &\leq V(t_0, x(t_0)) \sum_{k=0}^{\infty} (t - t_0)^{\mu k} E^{\gamma k}_{\rho,\mu k+1}(-\lambda_0 \lambda_{\max}(P)^{-1} \omega(t - t_0)^\rho) \\ &\leq \lambda_{\max}(P) \| x_0 \|^2 \sum_{k=0}^{\infty} (t - t_0)^{\mu k} E^{\gamma k}_{\rho,\mu k+1}(-\lambda_0 \lambda_{\max}(P)^{-1} \omega(t - t_0)^\rho). \end{aligned}$$

Subsequently, we have

$$\lambda_{\min}(P) \| x \|^2 \leq V(t) \leq \lambda_{\max}(P) \| x_0 \|^2 \sum_{k=0}^{\infty} (t - t_0)^{\mu k} E^{\gamma k}_{\rho,\mu k+1}(-\lambda_0 \lambda_{\max}(P)^{-1} \omega(t - t_0)^\rho).$$

Finally

$$\| x \| \leq \sqrt{\lambda_{\min}(P)^{-1} \lambda_{\max}(P)} \| x_0 \| \left(\sum_{k=0}^{\infty} (t - t_0)^{\mu k} E^{\gamma k}_{\rho,\mu k+1}(-\lambda_0 \lambda_{\max}(P)^{-1} \omega(t - t_0)^\rho) \right)^{\frac{1}{2}}.$$

Therefore, according to Definition 3.7, the trivial solution of Prabhakar FNDS (1.1) is stable in the sense of the generalized Mittag-Leffler function. □

Theorem 3.12. *Let f be the Lipschitz continuous. If there exists a positive definite matrix P such that*

$$A^T P + PA + (L^2 + \| P \|^2)I < 0,$$

where L is the Lipschitz constant, then the trivial solution of regularized Prabhakar FNDS (1.1) *is asymptotically stable.*

Proof . We set $V(t) = x^T(t)Px(t)$ as a Lyapunov function candidate. Employing regularized Prabhakar fractional derivative operator ${}^C D^\gamma_{\rho,\mu,\omega,t_0+}$ on the Lyapunov function $V(t) = x^T(t)Px(t)$ and using Lemma 2.9, we get

$$\begin{aligned} {}^C D^\gamma_{\rho,\mu,\omega,t_0+} V(t) &\leq 2x^T(t)P \, {}^C D^\gamma_{\rho,\mu,\omega,t_0+} x(t) \\ &= [Ax(t) + f(t, x(t))]^T Px(t) + x^T(t)P[Ax(t) + f(t, x(t))] \\ &= x^T(t)\big(A^T P + PA\big)x(t) + 2x^T Pf(t, x(t)). \end{aligned}$$

Since f is the Lipschitz continuous, we have

$$\begin{aligned}{}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}V(t) &\le x^T(t)\big(A^TP+PA\big)x(t)+2x^TPf(t,x(t))\\ &\le x^T(t)\big(A^TP+PA\big)x(t)+x^T(t)P^2x(t)+f^T(t,x(t))f(t,x(t))\\ &\le x^T(t)\big(A^TP+PA\big)x(t)+(L^2+\parallel P\parallel^2)\parallel x(t)\parallel^2\\ &\le x^T(t)\Big[A^TP+PA+(L^2+\parallel P\parallel^2)I\Big]x(t)<0.\end{aligned}$$

Therefore, according to Theorem 3.10, the system (1.1) is asymptotically stable and the proof is completed. □

4. Numerical simulation

In [12], Eshaghi et al. proposed a numerical method to solve the regularized Prabhakar FNDS by transforming the original system into a system of ordinary differential equations of first order as follows

$$\begin{aligned}x'(t)=\frac{1}{\Omega}\Bigg[&Ax(t)+f(t,x(t))-\Phi x(t)+Q_2t^{-\mu}x(0)\\ &+\sum_{k=0}^{M}\frac{(-\gamma)_k(\omega t^{\rho})^k}{\Gamma(\rho k-\mu+2)k!}\sum_{p=2}^{M}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)(p-1)!}\frac{\widehat{V}_p(x)(t)}{t^{p+\mu-1}}\Bigg],\end{aligned}$$

where

$$\begin{aligned}&\Omega=(Q_1+R_1)t^{1-\mu},\quad \Phi=(Q_2-R_2)t^{-\mu},\\ &Q_1=\sum_{k=0}^{M}\frac{(-\gamma)_k(\omega t^{\rho})^k}{\Gamma(\rho k-\mu+2)k!},\quad R_1=\sum_{k=0}^{M}\frac{(-\gamma)_k(\omega t^{\rho})^k}{\Gamma(\rho k-\mu+2)k!}\sum_{p=1}^{M}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)p!},\\ &Q_2=\sum_{k=0}^{M}\frac{(-\gamma)_k(\omega t^{\rho})^k}{\Gamma(\rho k-\mu+1)k!},\quad R_2=\sum_{k=0}^{M}\frac{(-\gamma)_k(\omega t^{\rho})^k}{\Gamma(\rho k-\mu+2)k!}\sum_{p=2}^{M}\frac{\Gamma(p-\rho k+\mu-1)}{\Gamma(-\rho k+\mu-1)(p-1)!},\end{aligned}$$

$$\widehat{V}_p(x)(t)=-(p-1)\int_0^t\xi^{p-2}x(\xi)d\xi,\quad p=2,3,\cdots,$$

with the initial condition $x(t_0)=x_0$.
We now apply this numerical method to solve the following systems and find numerical solutions by using the well known fourth order Runge-Kutta method. In these systems, we consider the generalized Mittag-Leffler stability and the asymptotic stability of some regularized Prabhakar FNDSs and depict numerical values of these systems for the different parameters.

Example 4.1. *Let us consider the following regularized Prabhakar FNDS*

$$\begin{cases}{}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}x(t)=-3x(t)+y(t)+\sin x(t),\\ {}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}y(t)=x(t)-2y(t)+\sin y(t),\end{cases}\tag{4.1}$$

with the initial conditions $x(t_0)=x_0, y(t_0)=y_0$ and $\gamma,\mu,\omega\in(0,1)$, $0<\rho<2$, $\rho\gamma<\mu$. We choose $P=I_2$, $V(t)=x^T(t)x(t)$ and $\zeta=1$. The function f is clearly a Lipschitz continuous function with *the Lipschitz constant* $L = 1$ *and*

$$\begin{aligned}{}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}V(t) &\leq 2x^T(t)^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}x(t)\\ &= [Ax(t) + f(t, x(t))]^T x(t) + x^T(t)[Ax(t) + f(t, x(t))]\\ &= -6x^2 + 4xy - 4x^2 + 2x\sin x + 2y\sin y \leq -4\Big(x - \frac{1}{2}y\Big)^2 - y^2.\end{aligned}$$

Thus, ${}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}V(t)$ *is negative definite and the conditions of Theorem 3.11 hold. Therefore, the trivial solution of system* (4.1) *is stable in the sense of the generalized Mittag-Leffler function. Further, the following fact*

$$A^T P + PA + (L^2 + \parallel P \parallel^2)I = \begin{pmatrix} -4 & 2 \\ 2 & -2 \end{pmatrix} < 0,$$

implies that the trivial solution of system (4.1) *is also asymptotically stable according to Theorem 3.12.*
The numerical values of system (4.1) *are presented in Figures. 1 and 2. We consider the fixed parameters* $\gamma = 0.1$, $\rho = 1$, $\mu = 0.75$, $\omega = 0.02$, $h = 0.01$. *Figure 1 is depicted for* $x_0 = -0.02$ *and* $y_0 = 0.04$. *Figure 2 is depicted for different initial conditions.*

Example 4.2. *Consider the following regularized Prabhakar FNDS*

$$\begin{cases} {}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}x(t) = -5x - 10y + \log(\log(10 + x^2)), \\ {}^{C}D^{\gamma}_{\rho,\mu,\omega,t_0+}y(t) = -10x - 5y + \log(\log(10 + y^2)), \end{cases} \tag{4.2}$$

with the initial conditions $x(t_0) = x_0$, $y(t_0) = y_0$ *and* $\gamma, \mu, \omega \in (0, 1)$, $0 < \rho < 2$, $\rho\gamma < \mu$. *We choose* $P = I_2$, $V(t) = x^T(t)x(t)$ *and* $\zeta = 1$. *The function* f *is clearly the Lipschitz continuous with the Lipschitz constant* $L = \frac{1}{\sqrt{10}}$ *and similar to the previous example the conditions of Theorem 3.11 and Theorem 3.12 hold. Therefore, the trivial solution of system* (4.2) *is the generalized Mittag-Leffler stable and hence is asymptotically stable.*
The numerical values of system (4.2) *are given in Figures 3 and 4. We consider the fix set of parameters* $\gamma = 0.6$, $\rho = 1$, $\mu = 0.8$, $\omega = 0.05$, $h = 0.01$. *Figure 3 is presented for* $(x_0, y_0) = (-0.01, 0.06)$. *Figure 4 is presented for the different initial conditions.*

5. Concluding remarks

In this paper, we studied the stability of regularized Prabhakar FNDS (1.1) with respect to the generalized Mittag-Leffler by means of the Lyapunov direct method. In this sense, the asymptotic stability of regularized Prabhakar FNDS was also discussed. Further, we presented two examples for the regularized Prabhakar FNDS to examine the analytical results. The numerical simulations showed the asymptotical stability behaviors of the proposed systems for their equilibrium points along with the convergence behaviors. We should mention that the proposed concept can be extended for many dynamical systems in the applied mathematics and Engineering.

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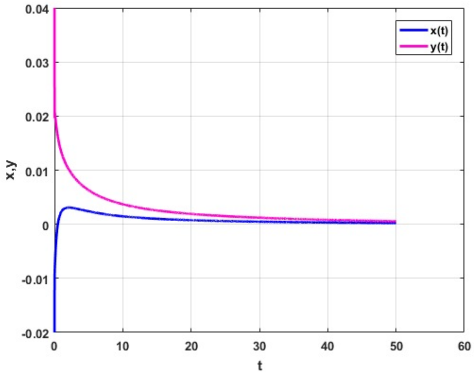

Figure 1: The numerical values of system (4.1) for $\gamma = 0.1$, $\rho = 1$, $\mu = 0.75$, $\omega = 0.02$, $h = 0.01$ and $(x_0, y_0) = (-0.02, 0.04)$.

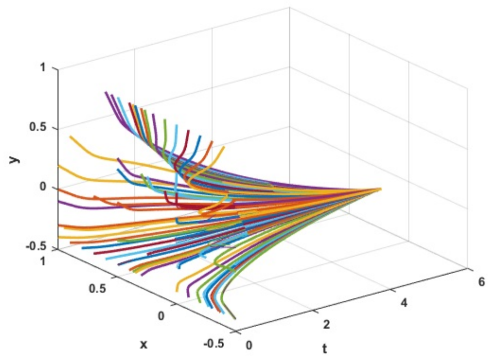

Figure 2: The numerical values of system (4.1) for $\gamma = 0.1$, $\rho = 1$, $\mu = 0.75$, $\omega = 0.02$, $h = 0.01$ and different initial conditions.

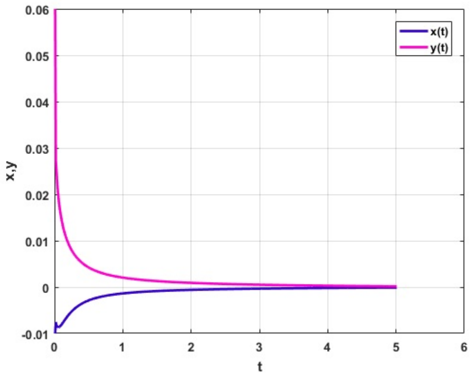

Figure 3: The numerical values of system (4.2) for $\gamma = 0.6$, $\rho = 1$, $\mu = 0.8$, $\omega = 0.05$, $h = 0.01$ and $(x_0, y_0) = (-0.01, 0.06)$.

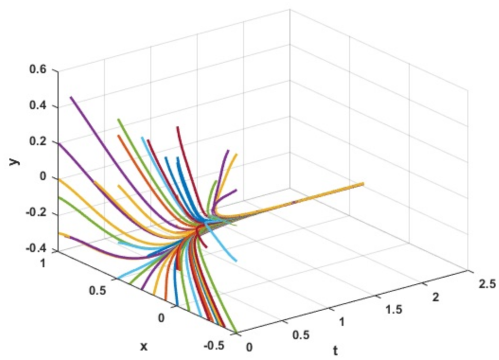

Figure 4: The numerical values of system (4.2) for $\gamma = 0.6$, $\rho = 1$, $\mu = 0.8$, $\omega = 0.05$, $h = 0.01$ and different initial conditions.

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