Abstract

In the paper we obtain sufficient conditions for the existence of common fixed point for a pair of contractive type mappings in bicomplex valued metric spaces.

Keywords: Common fixed point, bicomplex valued metric space, Banach contraction principle, contractive type mapping.

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1. Introduction

Segre [18] made a pioneering attempt in the development of special algebras. He conceptualized commutative generalization of complex numbers as bicomplex numbers, tricomplex numbers, etc. as elements of an infinite set of algebras. Subsequently during the 1930s, other researchers also contributed in this area {cf. [19]-[9]}. But unfortunately the next fifty years failed to witness any advancement in this field. Afterward Price [16] developed the bicomplex algebra and function theory. Recently renewed interest in this subject finds some significant applications in different fields of mathematical sciences as well as other branches of science and technology. An impressive body of work has been developed by a number of researchers. Among them an important work on elementary functions of bicomplex numbers has been done by Luna-Elizarrarás et al. [15].

Recently Beg et al. [4] studied existence of common fixed points for maps on topological vector space valued cone metric spaces. Azam et al. [1] extended it to complex valued metric space and established a common fixed point theorem for a pair of self contracting mappings. Rouzkard & Imdad [10] generalized the result obtained by Azam et al. [1] and they proved another common fixed

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point theorem satisfying some rational inequality in complex valued metric spaces. The Banach contraction principle \cite{3} is a popular and effective tool to solve the existence problems in many branches of mathematical analysis and is an active area of research. The celebrated Banach theorem states that: \"Let \((X,d)\) be a complete metric space and \(T\) be a mapping of \(X\) into itself satisfying \[d(Tx, Ty) \leq kd(x, y), \forall x, y \in X, \text{ where } k \text{ is a constant in } (0, 1)\]. Then \(T\) has a unique fixed point \(x^* \in X^\ast\).\" Choudhury et al. \cite{8} proved some fixed point results in partially ordered complex valued metric spaces. Jebril et al. \cite{12} proved some common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces. This article is in continuation of these works. We further investigate the bicomplex valued metric theorems for rational type expressions. Bhat et al. \cite{6} proved fixed point of mapping satisfying some rational inequality in complex valued metric spaces. The Banach contraction principle \cite{3} is a popular and effective tool to solve the existence problems in many branches of mathematical analysis and is an active area of research. The celebrated Banach theorem states that: \"Let \((X,d)\) be a complete metric space and \(T\) be a mapping of \(X\) into itself satisfying \[d(Tx, Ty) \leq kd(x, y), \forall x, y \in X, \text{ where } k \text{ is a constant in } (0, 1)\]. Then \(T\) has a unique fixed point \(x^* \in X^\ast\).\" Choudhury et al. \cite{8} proved some fixed point results in partially ordered complex valued metric spaces. Jebril et al. \cite{12} proved some common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces. This article is in continuation of these works. We further investigate the bicomplex valued metric spaces and establish fixed point theorems for a pair of contractive type mappings satisfying a rational inequality.

Next we present some basic notions and notations for subsequent use.

We denote the set of real, complex and bicomplex numbers respectively as \(\mathbb{C}_0, \mathbb{C}_1\) and \(\mathbb{C}_2\). Segre \cite{18} defined the bicomplex number as:

\[\xi = a_1 + a_2i_1 + a_3i_2 + a_4i_1i_2,\]

where \(a_1, a_2, a_3, a_4 \in \mathbb{C}_0\), and the independent units \(i_1, i_2\) are such that \(i_1^2 = i_2^2 = -1\) and \(i_1i_2 = i_2i_1\), we denote the set of bicomplex numbers \(\mathbb{C}_2\) is defined as:

\[\mathbb{C}_2 = \{\xi : \xi = a_1 + a_2i_1 + a_3i_2 + a_4i_1i_2, a_1, a_2, a_3, a_4 \in \mathbb{C}_0\},\]

i.e.,

\[\mathbb{C}_2 = \{\xi : \xi = z_1 + i_2z_2, z_1, z_2 \in \mathbb{C}_1\}.\]

where \(z_1 = a_1 + a_2i_1 \in \mathbb{C}_1\) and \(z_2 = a_3 + a_4i_1 \in \mathbb{C}_1\).

If \(\xi = z_1 + i_2z_2\) and \(\eta = w_1 + i_2w_2\) be any two bicomplex numbers then the sum is \(\xi \pm \eta = (z_1 + i_2z_2) \pm (w_1 + i_2w_2) = (z_1 \pm w_1) + i_2(z_2 \pm w_2)\) and the product is \(\xi \eta = (z_1 + i_2z_2)(w_1 + i_2w_2) = (z_1w_1 - z_2w_2) + i_2(z_1w_2 + z_2w_1)\).

There are four idempotent elements in \(\mathbb{C}_2\), they are 0, 1, \(e_1 = \frac{1+i_1i_2}{2}\) and \(e_2 = \frac{1-i_1i_2}{2}\) out of which \(e_1\) and \(e_2\) are nontrivial such that \(e_1 + e_2 = 1\) and \(e_1e_2 = 0\). Every bicomplex number \(z_1 + i_2z_2\) can uniquely be expressed as the combination of \(e_1\) and \(e_2\), namely

\[\xi = z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2.\]

This representation of \(\xi\) is known as the idempotent representation of bicomplex number and the complex coefficients \(\xi_1 = (z_1 - i_1z_2)\) and \(\xi_2 = (z_1 + i_1z_2)\) are known as idempotent components of the bicomplex number \(\xi\).

An element \(\xi = z_1 + i_2z_2 \in \mathbb{C}_2\) is said to be invertible if there exists another element \(\eta\) in \(\mathbb{C}_2\) such that \(\xi\eta = 1\) and \(\eta\) is said to be the inverse (multiplicative) of \(\xi\). Consequently \(\xi\) is said to be the inverse (multiplicative) of \(\eta\). An element which has an inverse in \(\mathbb{C}_2\) is said to be the nonsingular element of \(\mathbb{C}_2\) and an element which does not have an inverse in \(\mathbb{C}_2\) is said to be the singular element of \(\mathbb{C}_2\).

An element \(\xi = z_1 + i_2z_2 \in \mathbb{C}_2\) is nonsingular if and only if \(|z_1^2 + z_2^2| \neq 0\) and singular if and only if \(|z_1^2 + z_2^2| = 0\). The inverse of \(\xi\) is defined as

\[\xi^{-1} = \eta = \frac{z_1 - i_2z_2}{z_1^2 + z_2^2}.\]
Zero is the only element in \( \mathbb{C}_0 \) which does not have multiplicative inverse and in \( \mathbb{C}_1, 0 = 0 + i0 \) is the only element which does not have multiplicative inverse. We denote the set of singular elements of \( \mathbb{C}_0 \) and \( \mathbb{C}_1 \) by \( O_0 \) and \( O_1 \) respectively. But there are more than one element in \( \mathbb{C}_2 \) which do not have multiplicative inverse; we denote this set by \( O_2 \) and clearly \( O_0 = O_1 \subset O_2 \).

A bicomplex number \( \xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 \in \mathbb{C}_2 \) is said to be degenerated if the matrix
\[
\begin{pmatrix}
a_1 & a_2 \\
am_3 & a_4
\end{pmatrix}
\]
is degenerated. In that case \( \xi^{-1} \) exists and it is also degenerated.

The norm \( \| \cdot \| \) of \( \mathbb{C}_2 \) is a positive real valued function and \( \| \cdot \|: \mathbb{C}_2 \to \mathbb{C}_0^+ \) is defined by
\[
\| \xi \| = \| z_1 + i_2 z_2 \| = \left\{ |z_1|^2 + |z_2|^2 \right\}^{1/2} = \left[ \frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2} \right]^{1/2} = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^{1/2},
\]
where \( \xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2 \).

The partial order relation \( \preccurlyeq \) on \( \mathbb{C}_2 \) is defined as:

Let \( \mathbb{C}_2 \) be the set of bicomplex numbers and \( \xi = z_1 + i_2 z_2, \eta = w_1 + i_2 w_2 \in \mathbb{C}_2 \) then \( \xi \preccurlyeq \eta \) if and only if \( z_1 \preccurlyeq w_1 \) and \( z_2 \preccurlyeq w_2 \),
i.e., \( \xi \preccurlyeq \eta \) if one of the following conditions is satisfied:

1. \( z_1 = w_1, \ z_2 = w_2, \)
2. \( z_1 < w_1, \ z_2 = w_2, \)
3. \( z_1 = w_1, \ z_2 < w_2 \text{ and} \)
4. \( z_1 < w_1, \ z_2 < w_2. \)

In particular we can write \( \xi \preccurlyeq \eta \) if \( \xi \preccurlyeq \eta \) and \( \xi \neq \eta \) i.e. one of (2), (3) and (4) is satisfied and we will write \( \xi \prec \eta \) if only (4) is satisfied.

For any two bicomplex numbers \( \xi, \eta \in \mathbb{C}_2 \) we can verify the followings:

- \( \xi \preccurlyeq \eta \Rightarrow \| \xi \| \leq \| \eta \|, \)
- \( \| \xi + \eta \| \leq \| \xi \| + \| \eta \|, \)
- \( \| a \xi \| = a \| \xi \|, \) where \( a \) is a non negative real number,
- \( \| \xi \eta \| \leq \sqrt{2} \| \xi \| \| \eta \| \) and the equality holds only when at least one of \( \xi \) and \( \eta \) is degenerated,
- \( \| \xi^{-1} \| = \| \xi \|^{-1} \) if \( \xi \) is a degenerated bicomplex number with \( 0 < \xi, \)
- \( \| \xi \eta \| = \| \xi \| \| \eta \|, \) if \( \eta \) is a degenerated bicomplex number.

2. Bicomplex valued metric space

In this section we prove two lemmas on bicomplex valued metric spaces which will be needed in the sequel. Choi et al. [11] defined the bicomplex valued metric space as:

**Definition 2.1.** [11] Let \( X \) be a nonempty set. Suppose the mapping \( d: X \times X \to \mathbb{C}_2 \) satisfies the following conditions:

1. \( 0 \preccurlyeq d(x, y) \) for all \( x, y \in X, \)
2. \(d(x, y) = 0\) if and only if \(x = y\),
3. \(d(x, y) = d(y, x)\) for all \(x, y \in X\) and
4. \(d(x, y) \geq d(x, z) + d(z, y)\) for all \(x, y, z \in X\).

Then \((X, d)\) is called the bicomplex valued metric space.

**Definition 2.2.** \([11]\) For a bicomplex valued metric space \((X, d)\)

(i). A sequence \(\{x_n\}\) in \(X\) is said to be a convergent sequence and converges to a point \(x\) if for any \(0 \prec i_2 r \in \mathbb{C}_2\) there is a natural number \(n_0 \in \mathbb{N}\) such that \(d(x_n, x) \prec i_2 r\), for all \(n > n_0\) and we write \(\lim_{n \to \infty} x_n = x\) or \(x_n \to x\) as \(n \to \infty\).

(ii). A sequence \(\{x_n\}\) in \(X\) is said to be a Cauchy sequence in \((X, d)\) if for any \(0 \prec i_2 r \in \mathbb{C}_2\) there is a natural number \(n_0 \in \mathbb{N}\) such that \(d(x_n, x_n + m) \prec i_2 r\), for all \(m, n \in \mathbb{N}\) and \(n > n_0\).

(iii). If every Cauchy sequence in \(X\) is convergent in \(X\) then \((X, d)\) is said to be a complete bicomplex valued metric space.

**Lemma 2.3.** Let \((X, d)\) be a bicomplex valued metric space and \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is convergent and converges to a point \(x\) if and only if \(\lim_{n \to \infty} \|d(x_n, x)\| = 0\).

**Proof.** Let \(\{x_n\}\) is a convergent sequence and converges to a point \(x\) and let \(\epsilon > 0\) be any real number. Suppose

\[
r = \frac{\epsilon}{2} + i_1 \frac{\epsilon}{2} + i_2 \frac{\epsilon}{2} + i_1 i_2 \frac{\epsilon}{2}
\]

Then clearly \(0 \prec i_2 r \in \mathbb{C}_2\) and for this \(r\) there is a natural number \(n_0 \in \mathbb{N}\) such that \(d(x_n, x) \prec i_2 r\) for all \(n > n_0\)

Therefore,

\[
\|d(x_n, x)\| < \|r\| = \epsilon, \quad \forall n > n_0.
\]

And this implies,

\[
\lim_{n \to \infty} \|d(x_n, x)\| = 0
\]

Conversely let \(\lim_{n \to \infty} \|d(x_n, x)\| = 0\). Then for \(0 \prec i_2 r \in \mathbb{C}_2\), there exists a real \(\epsilon > 0\), such that for all \(\xi \in \mathbb{C}_2\)

\[
\|\xi\| < \epsilon \Rightarrow \xi \prec i_2 r.
\]

Then for this \(\epsilon > 0\), there exists a natural number \(n_0 \in \mathbb{N}\) such that

\[
\|d(x_n, x)\| < \epsilon, \quad \forall n > n_0.
\]

Therefore,

\[
d(x_n, x) \prec i_2 r, \quad \forall n > n_0.
\]

Hence \(\{x_n\}\) converges to a point \(x\). \(\square\)

**Lemma 2.4.** Let \((X, d)\) be a bicomplex valued metric space and \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is a Cauchy sequence in \(X\) if and only if \(\lim_{n \to \infty} \|d(x_n, x_{n+m})\| = 0\).

**Proof.** Let \(\{x_n\}\) is a Cauchy sequence in \(X\) and let \(\epsilon > 0\) be any real number. Suppose

\[
r = \frac{\epsilon}{2} + i_1 \frac{\epsilon}{2} + i_2 \frac{\epsilon}{2} + i_1 i_2 \frac{\epsilon}{2},
\]
Then clearly $0 \prec_{i_2} r \in \mathbb{C}_2$ and for this $r$ there is a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec_{i_2} r$ for all $n > n_0$.

Therefore,

$$\|d(x_n, x_{n+m})\| < \|r\| = \epsilon, \quad \forall \ n > n_0.$$ 

And this implies,

$$\lim_{n \to \infty} \|d(x_n, x_{n+m})\| = 0$$

Conversely let $\lim_{n \to \infty} \|d(x_n, x_{n+m})\| = 0$. Then for $0 \prec_{i_2} r \in \mathbb{C}_2$, there exists a real $\epsilon > 0$, such that for all $\xi \in \mathbb{C}_2$

$$\|\xi\| < \epsilon \Rightarrow \xi \prec_{i_2} r.$$ 

Then for this $\epsilon > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that

$$\|d(x_n, x_{n+m})\| < \epsilon, \quad \forall \ n > n_0.$$ 

Therefore,

$$d(x_n, x_{n+m}) \prec_{i_2} r, \quad \forall \ n > n_0.$$ 

Hence $\{x_n\}$ is a Cauchy sequence. □

3. Main results

In this section we prove fixed point theorems on bicomplex valued metric spaces.

**Theorem 3.1.** Let $(X, d)$ be a complete bicomplex valued metric space with degenerated $1 + d(x, y)$ and $\|1 + d(x, y)\| \neq 0$ for all $x, y \in X$ and let $S, T : X \to X$ be mappings satisfying the condition

$$d(Sx, Ty) \prec_{i_2} ad(x, y) + \frac{bd(x, Sx)d(y, Ty)}{1 + d(x, y)}$$

(3.1)

for all $x, y \in X$, where $a, b$ are non-negative real numbers with $a + \sqrt{2}b < 1$. Then $S, T$ have a unique common fixed point.

**Proof.** Let $x_0$ be an arbitrary point in $X$. We construct a sequence $\{x_n\}$ such that

$$x_{2k+1} = Sx_{2k}, \quad x_{2k+2} = Tx_{2k+1}, \quad k = 0, 1, 2, \ldots$$

Then we have

$$d(x_{2k+1}, x_{2k+2}) = d(Sx_{2k}, Tx_{2k+1})$$

$$\succ_{i_2} ad(x_{2k}, x_{2k+1}) + \frac{bd(x_{2k}, Sx_{2k})d(x_{2k+1}, Tx_{2k+1})}{1 + d(x_{2k}, x_{2k+1})}$$

$$\succ_{i_2} ad(x_{2k}, x_{2k+1}) + \frac{bd(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1})}$$

Therefore,

$$\|d(x_{2k+1}, x_{2k+2})\| \leq a \|d(x_{2k}, x_{2k+1})\| + \sqrt{2}b \frac{\|d(x_{2k}, x_{2k+1})\|}{\|1 + d(x_{2k}, x_{2k+1})\|} \|d(x_{2k+1}, x_{2k+2})\|$$
Also \( \|d(x_{2k}, x_{2k+1})\| \leq \|1 + d(x_{2k}, x_{2k+1})\| \).

Thus

\[
\|d(x_{2k+1}, x_{2k+2})\| \leq a\|d(x_{2k}, x_{2k+1})\| + \sqrt{2b}\|d(x_{2k+1}, x_{2k+2})\|
\]

\( i.e., \) \( (1 - \sqrt{2b})\|d(x_{2k+1}, x_{2k+2})\| \leq a\|d(x_{2k}, x_{2k+1})\| \)

\( i.e., \) \( \|d(x_{2k+1}, x_{2k+2})\| \leq \frac{a}{(1 - \sqrt{2b})}\|d(x_{2k}, x_{2k+1})\| \)

Similarly,

\[
d(x_{2k+2}, x_{2k+3}) = d(Tx_{2k+1}, Sx_{2k+2}) = d(Sx_{2k+2}, Tx_{2k+1})
\]

\[
\leq i_2 ad(x_{2k+2}, x_{2k+1}) + \frac{bd(x_{2k+2}, Sx_{2k+2}) d(x_{2k+1}, Tx_{2k+1})}{1 + d(x_{2k+2}, x_{2k+1})}
\]

\[
\leq i_2 ad(x_{2k+2}, x_{2k+1}) + \frac{bd(x_{2k+2}, x_{2k+3}) d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k+2}, x_{2k+1})}
\]

Therefore we obtain that

\[
\|d(x_{2k+2}, x_{2k+3})\| \leq a \|d(x_{2k+2}, x_{2k+1})\|
\]

\[
+ \sqrt{2b} \frac{\|d(x_{2k+1}, x_{2k+2})\|}{1 + d(x_{2k+1}, x_{2k+2})} \|d(x_{2k+2}, x_{2k+3})\|
\]

\( i.e., \) \( \|d(x_{2k+2}, x_{2k+3})\| \leq a \|d(x_{2k+2}, x_{2k+1})\| + \sqrt{2b} \|d(x_{2k+2}, x_{2k+3})\| \)

\( \text{as } \|d(x_{2k+1}, x_{2k+2})\| \leq 1 + d(x_{2k+1}, x_{2k+2}) \)

\( i.e., \) \( (1 - \sqrt{2b})\|d(x_{2k+2}, x_{2k+3})\| \leq a \|d(x_{2k+2}, x_{2k+3})\| \)

\( i.e., \) \( \|d(x_{2k+2}, x_{2k+3})\| \leq \frac{a}{(1 - \sqrt{2b})}\|d(x_{2k+2}, x_{2k+3})\| \)

Suppose \( \alpha = \frac{a}{1 - \sqrt{2b}}. \) Then \( 0 \leq \alpha < 1 \) and

\[
\|d(x_{n+1}, x_{n+2})\| \leq \alpha \|d(x_n, x_{n+1})\|
\]

\[
\leq \alpha^2 \|d(x_{n-1}, x_n)\| \ldots \leq \alpha^{n+1} \|d(x_0, x_1)\| .
\]

Also for any two positive integers \( m, n \) with \( m > n \) we get that

\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m)
\]

Therefore,

\[
\|d(x_n, x_m)\| \leq \|d(x_n, x_{n+1})\| + \|d(x_{n+1}, x_{n+2})\| + \ldots + \|d(x_{m-1}, x_m)\|
\]

\( i.e., \) \( \|d(x_n, x_m)\| \leq [\alpha^n + \alpha^{n+1} + \ldots + \alpha^{m-1}] \|d(x_0, x_1)\| \)

Since, \( 0 \leq \alpha < 1, \) then \( 1 + \alpha + \alpha^2 + \ldots + \alpha^{m-n-1} \leq \frac{1}{1 - \alpha}. \)

Hence

\[
\|d(x_n, x_m)\| \leq \frac{\alpha^n}{1 - \alpha} \|d(x_0, x_1)\|
\]
Again since \( \frac{a}{1-a} \to 0 \) as \( n \to \infty \), then for any \( \varepsilon > 0 \) there exists a positive integer \( n_0 \) such
\[
\|d(x_n, x_m)\| < \varepsilon
\]
for all \( m, n > n_0 \). Hence \( \{x_n\} \) is a Cauchy sequence in \( X \). Also \( X \) is a complete
bicomplex valued metric space. Then there exists \( u \in X \) such that \( \lim_{n \to \infty} x_n = u \).

Now we show that \( u = Su \). If not then there exists an \( 0 \prec \xi \in \mathbb{C}_2 \) such that \( d(u, Su) = \xi \).
Therefore we have
\[
\xi = d(u, Su) \\
\preceq i_2 d(u, x_{2k+2}) + d(x_{2k+2}, Su) \\
\preceq i_2 d(u, x_{2k+2}) + d(Tx_{2k+1}, Su) \\
\preceq i_2 d(u, x_{2k+2}) + ad(x_{2k+1}, u) + \frac{bd(x_{2k+1}, Tx_{2k+1})d(u, Su)}{1 + d(x_{2k+1}, u)} \\
i.e., \quad \xi \preceq i_2 d(u, x_{2k+2}) + ad(x_{2k+1}, u) + \frac{bd(x_{2k+1}, x_{2k+2})\xi}{1 + d(x_{2k+1}, u)}
\]
Hence
\[
\|\xi\| \leq \|d(u, x_{2k+2})\| + a\|d(x_{2k+1}, u)\| + \sqrt{2b}\|d(x_{2k+1}, x_{2k+2})\|\|\xi\| |1 + d(x_{2k+1}, u)|.
\]
Since \( \lim_{n \to \infty} x_n = u \), taking limit on both sides as \( n \to \infty \) we get that \( \|\xi\| \leq 0 \), which is a
contradiction. Therefore \( \|\xi\| = 0 \Rightarrow d(u, Su) = 0 \Rightarrow u = Su \). Similarly, we can show that \( u = Tu \).

Hence \( S \) and \( T \) have a common fixed point.

Now we show that \( S \) and \( T \) have unique common fixed point. If possible suppose \( u^* \in X \) be
another common fixed point of \( S \) and \( T \).

Then
\[
d(u, u^*) = d(Su, Tu^*) \preceq i_2 ad(u, u^*) + \frac{bd(u, Su)d(u^*, Tu^*)}{1 + d(u, u^*)} \\
i.e., \quad \|d(u, u^*)\| \leq a\|d(u, u^*)\| + \sqrt{2b}\|d(u, Su)\|\|d(u^*, Tu^*)\| |1 + d(u, u^*)|
\]
i.e., \( \|d(u, u^*)\| \leq a\|d(u, u^*)\| \\
i.e., \quad \|d(u, u^*)\| = 0 \\
i.e., \quad u = u^*.
\]
This completes the proof of the theorem. \( \square \)

**Corollary 3.2.** Let \((X, d)\) be a complete bicomplex valued metric space with degenerated \(1 + d(x, y)\)
and \(1 + d(x, y)\) \(\neq 0\) for all \( x, y \in X \) and \( S : X \to X \) be any mapping satisfying the condition
\[
d(Sx, Sy) \preceq i_2 ad(x, y) + \frac{bd(x, Sx)d(y, Sy)}{1 + d(x, y)}
\]  
(3.2)
for all \( x, y \in X \), where \( a, b \) are non-negative real numbers with \( a + \sqrt{2b} < 1 \). Then \( S \) has a unique
fixed point.

**Proof.** We can easily prove this result by applying the Theorem 3.1 and taking \( T = S \). \( \square \)
Corollary 3.3. Let \((X, d)\) be a complete bicomplex valued metric space with degenerated \(1 + d(x, y)\) and \(\|1 + d(x, y)\| \neq 0\) for all \(x, y \in X\) and let \(S : X \to X\) be any mapping satisfying the condition
\[
d(S^n x, S^n y) \lesssim_{i_2} a d(x, y) + \frac{bd(S^n x) d(y, S^n y)}{1 + d(x, y)} \tag{3.3}
\]
for all \(x, y \in X\), where \(a, b\) are non-negative real numbers with \(a + \sqrt{2}b < 1\). Then \(S\) has a unique fixed point.

Proof. By Corollary 3.2 there exists a unique point \(u \in X\) such that
\[
Su = u
\]
\[
i.e., S^2 u = S u = u
\]
\[
i.e., S^n u = u.
\]
Therefore,
\[
d(Su, u) = d(SS^n u, S^n u) = d(S^n Su, S^n u) \lesssim_{i_2} a d(Su, u) + \frac{bd(Su, S^n Su) d(u, S^n u)}{1 + d(Su, u)}
\]
i.e., \(d(Su, u) \lesssim_{i_2} ad(Su, u) + \frac{bd(Su, S^n Su) d(u, u)}{1 + d(Su, u)}\)
i.e., \(d(Su, u) \lesssim_{i_2} ad(Su, u)\)
\[
i.e., \|d(Su, u)\| \leq a \|d(Su, u)\|
\]
i.e., \(\|d(Su, u)\| = 0\)
i.e., \(u = u^*\).

This completes the proof of the corollary. \(\square\)

Remark 3.4. The following example ensures the validity of the Corollary 3.3.

Example 3.5. Consider \(X = \{0, \frac{1}{2}, 2\}\), define a bicomplex valued metric \(d : X \times X \to \mathbb{C}_2\) by
\[
d(x, y) = (1 + i_2) |x - y| , \forall x, y \in X.
\]

From the above definition of \(d\) one can easily verify that \((X, d)\) is a bicomplex valued metric space.

Now we consider the mapping \(S : X \to X\) defined by
\[
S(0) = 0, \quad S\left(\frac{1}{2}\right) = 0 \quad \text{and} \quad S(2) = \frac{1}{2}
\]

Let \(a = \frac{1}{3}\) and \(b = \frac{1}{6}\), then clearly \(a + \sqrt{2}b = \frac{1}{3} + \sqrt{2}\frac{1}{6} < 1\). Also the condition (3.2) of the Corollary 3.2 is satisfied, clearly \(x = 0\) is the unique fixed point of \(S\).

Again, notice that \(S^n x = 0, \forall x \in X\). so that \(0 = d(S^n x, S^n y) \lesssim_{i_2} a d(x, y) + \frac{bd(x, S^n x) d(y, S^n y)}{1 + d(x, y)}\), therefore \(S^n x\) satisfies the condition (3.3) of the Corollary 3.3 and clearly \(x = 0\) is the unique fixed point of \(S\).
Theorem 3.6. Let \((X, d)\) be a complete bicomplex valued metric space and let the mappings \(S, T : X \to X\) satisfy the condition
\[
d(Sx, Ty) \lesssim_{i_2} \frac{a [d(x, Sx) d(x, Ty) + d(y, Ty) d(y, Sx)]}{d(x, Ty) + d(y, Sx)}
\]
for all \(x, y \in X\) and if \(\|d(x, Ty) + d(y, Sx)\| \neq 0\) and \(d(x, Ty) + d(y, Sx)\) is degenerated, where 'a' is a non-negative real number with \(0 \leq a < 1\). Then \(S, T\) have a unique common fixed point.

Proof. Let \(x_0\) be an arbitrary point in \(X\). We consider a sequence \(\{x_n\}\) in \(X\) such that
\[
x_{n+1} = Sx_n, \quad \text{and} \quad x_{n+2} = Tx_{n+1} \quad \text{for all} \quad n = 0, 1, 2, \ldots
\]
Then
\[
d(x_{n+1}, x_{n+2}) = d(Sx_n, Tx_{n+1})
\]
\[
\lesssim_{i_2} a [d(x_n, Sx_n) d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_{n+1}) d(x_{n+1}, Sx_n)]
\]
\[
\lesssim_{i_2} a [d(x_n, x_{n+1}) d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2}) d(x_{n+1}, x_{n+1})]
\]
\[
\lesssim_{i_2} ad(x_n, x_{n+1}) d(x_n, x_{n+2})
\]
\[
\lesssim_{i_2} ad(x_n, x_{n+1})
\]
Therefore for all \(n \geq 0\) we get that
\[
d(x_{n+1}, x_{n+2}) \lesssim_{i_2} ad(x_n, x_{n+1}) \lesssim_{i_2} a^2 d(x_{n-1}, x_n) \lesssim_{i_2} a^{n+1} d(x_0, x_1)
\]
Thus for any two positive integers \(m, n\) with \(m > n\) we have
\[
d(x_n, x_m) \lesssim_{i_2} a^m d(x_0, x_1) \quad \text{if} \quad m \leq n,
\]
\[
\lesssim_{i_2} \frac{a^n}{1 - a} d(x_0, x_1).
\]
Since \(0 \leq a < 1\), then \(1 + a + a^2 + \ldots + a^{n-1} \leq \frac{1}{1-a}\).
Hence
\[
\|d(x_n, x_m)\| \leq \frac{a^n}{1 - a} \|d(x_0, x_1)\|.
\]
Again since \(\frac{a^n}{1-a} \to 0\) as \(n \to \infty\), then for any \(\varepsilon > 0\) there exists a positive integer \(n_0\) such \(\|d(x_n, x_m)\| < \varepsilon\), for all \(m, n > n_0\). Hence \(\{x_n\}\) is a Cauchy sequence in \(X\). Also \(X\) is a complete bicomplex valued metric space. Therefore there exists \(u \in X\) such that \(\lim_{n \to \infty} x_n = u\).

Now we show that \(u = Su\). If not then there exists \(0 < i_2 \xi \in \mathbb{C}_2\) such that \(d(u, Su) = \xi\).
Therefore,
\[
\xi = d(u, Su) \\
\preceq i_2 d(u, x_{n+2}) + d(x_{n+2}, Su) \\
\preceq i_2 d(u, x_{n+2}) + d(Su, Tx_{n+1}) \\
\preceq i_2 d(u, x_{n+2}) + \frac{a [d(u, Su) d(u, Tx_{n+1}) + d(x_{n+1}, Tx_{n+1}) d(x_{n+1}, Su)]}{d(u, Tx_{n+1}) + d(x_{n+1}, Su)} \\
\preceq i_2 d(u, x_{n+2}) + \frac{a [\xi d(u, x_{n+2}) + d(x_{n+1}, x_{n+2}) d(x_{n+1}, Su)]}{d(u, x_{n+2}) + d(x_{n+1}, Su)},
\]
which yields that
\[
\|\xi\| \leq \|d(u, x_{n+2})\| + \sqrt{2} a \frac{\|\xi\| \|d(u, x_{n+2})\| + \|d(x_{n+1}, x_{n+2})\| \|d(x_{n+1}, Su)\|}{\|d(u, x_{n+2}) + d(x_{n+1}, Su)\|}
\]
Taking limit as \( n \to \infty \) we get that \( \|\xi\| \leq 0 \), which is a contradiction. Therefore \( \|\xi\| = 0 \Rightarrow \|d(u, Su)\| = 0 \Rightarrow u = Su \). Similarly, we can show that \( u = Tu \).

Hence \( S \) and \( T \) have a common fixed point.

Now we show that \( S \) and \( T \) have unique common fixed point. If possible suppose that \( u^* \in X \) be another common fixed point of \( S \) and \( T \).

Then
\[
d(u, u^*) = d(Su, Tu^*) \preceq i_2 \frac{a [d(u, Su) d(u, Tu^*) + d(u^*, Tu^*) d(u^*, Su)]}{d(u, Tu^*) + d(u^*, Su)}
\]
\[
\|d(u, u^*)\| \leq \sqrt{2} a \frac{\|d(u, Su)\| \|d(u, Tu^*)\| + \|d(u^*, Tu^*)\| \|d(u^*, Su)\|}{\|d(u, Tu^*) + d(u^*, Su)\|}
\]
i.e., \( \|d(u, u^*)\| \leq 0 \)
which is a contradiction, Therefore,

i.e., \( \|d(u, u^*)\| = 0 \)
i.e., \( u = u^* \).

This completes the proof of the theorem. \( \square \)

4. Open problem

In this work we started the study of bicomplex valued metric spaces and established fixed point results for a pair of contractive type mappings satisfying a rational inequality in complete bicomplex valued metric spaces. Ran and Reurings [17] generalized Banach’s contraction principle to partially ordered metric spaces. Beg and Butt [5] further generalized results of Ran and Reurings [17]. Recently Khalehoghli et al. [13, 14] introduced R-metric spaces and obtained a generalization of Banach fixed point theorem. It is an interesting open problem to study the R-bicomplex valued metric spaces and obtain fixed point results on complete R-bicomplex valued metric spaces.

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References


