Rough continuity and rough separation axioms in $G_m$-closure approximation spaces

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Abstract

The theory of general topology view for continuous mappings is general version and is applied for topological graph theory. Separation axioms can be regard as tools for distinguishing objects in information systems. Rough theory is one of map the topology to uncertainty. The aim of this work is to presented graph, continuity, separation properties and rough set to put a new approaches for uncertainty. For the introduce of various levels of approximations, we introduce several levels of continuity and separation axioms on graphs in $G_m$-closure approximation spaces.

AMS Subject Classification: 54C05, 54D10, 05C10

Keywords: Rough sets, $G_m$-closure space, Approximation spaces, lower and upper approximation spaces, Rough continuity, Rough separation axioms.

1. Introduction

Rough sets, presented by author in [10], and introduce the data base property by incomplete and insufficient information. The notations (lower, upper) approximation (written as appxox-) in rough theory, hidden information in intelligent probably disintegrated and presented of decision systems. The closure operator is a tools in many parts of mathematics for example, in algebra theory [2 4], topology theory [7 8] and computer science theory [13 17]. Several works introduce recently for example in structural analysis theory [14 15], in chemistry science [16], and physics science [6]. In this paper we will put a version for the application of topological graph theory.

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2. Preliminaries

We introduce a review of some preliminaries of rough sets [3][10][11] and some preliminaries of $G_m$-closure spaces [1][12][14][15][18].

2.1. Some Preliminaries of Uncertainty

Rough set theory represent as a sub-sets of a universe set in terminology of equivalence classes (written as equival. clas.) of a partition of universe set. The partition introduce a topological space, is said to be approx-space and denoted by $\Omega = (\mathcal{X}, \mathcal{E})$ where $\mathcal{X}$ is said to be the universe set and $\mathcal{E}$ is an equivalent relation [9][11]. The equivalent clas. of $\mathcal{E}$ is said to be the granules, elementary sets, we denote the equivalent clas. containing $x \in \mathcal{X}$ by $\mathcal{E}_x \subseteq \mathcal{X}$. In the approx-space, the operators of the (upper, lower) aprox’s of $A: A \subseteq \mathcal{X}$, then the lower-approx- (resp. the upper-approx-) of $\mathcal{X}$ is define as

$$\mathcal{L}(A) = \{ x \in \mathcal{X} : \mathcal{E}_x \subseteq A \} \text{ (resp. } \mathcal{U}(A) = \{ x \in \mathcal{X} : \mathcal{E}_x \cap A \neq \phi \} \text{)}$$

2.2. Some Preliminaries of $G_m$-Closure Spaces

Closure operators on digraphs are present and many property on the $G_m$-closure spaces are introduce.

Definition 2.1. [12][13] Let $G = (V_G, E_G)$ be a direct graph, $P(V_G)$ be the all direct subgraphs of $G$ and $Cl_G : P(V_G) \rightarrow P(V_G)$ such that $Cl_G(V_H) \subseteq V_G$ is said to be closure subgraph, where $H = (V_H, E_H)$ is a subgraph of $G$ and define as:

$$Cl_G(V_H) = V_H \cup \{ \varpi \in V_G - V_H; (h, \varpi) \in E_G \text{ forall } h \in V_H \}$$

The mapping $Cl_G$ is said to be direct graph closure operator and $(G, \mathcal{F}_G)$ is said to be $G$-closure space (written as $G$-cl-space), such that $\mathcal{F}_G$ is the collection of members of $Cl_G$. Clearly $Cl_G(V_H) = \cap\{V_F; V_F \in \mathcal{F}_G \text{ and } V_H \subseteq V_F \}$. The direct graph interior operator $Int_G : P(V_G) \rightarrow P(V_G)$ defined as $Int_G(V_H) = V_G - Cl_G(V_G - V_H)$, where $H \subseteq G$. Clear that the direct graph interior operator is the dual of direct graph closure operator. A collection of members of $Int_G$ is said to be interior subgraph of $H$ and written as $\mathcal{I}_G$, and have $(G, \mathcal{I}_G)$ is a topological space. Clearly $Int_G(V_H) = \cup\{V_O; V_O \in \mathcal{I}_G \text{ and } V_O \subseteq V_H \}$. Furthermore $Cl_G(V_H) = V_G - Int_G(V_G - V_H)$. A subgraph $H$ of $G$-cl-space $(G, \mathcal{F}_G)$ is said to be closed subgraph if $Cl_G(V_H) = V_H$ and it is said to be open subgraph if its complement is closed subgraph, (i.e., $Cl_G(V_G - V_H) = V_G - V_H$ or $Int_G(V_H) = V_H$).

Example 2.2. Let $G = (V_G, E_G)$ be a digraph such that : $V_G = \{ \varpi_1, \varpi_2, \varpi_3, \varpi_4 \}$, $E_G = \{ (\varpi_1, \varpi_2), (\varpi_1, \varpi_3), (\varpi_2, \varpi_1), (\varpi_2, \varpi_3), (\varpi_3, \varpi_4) \}$.

$$\mathcal{F}_G = \{ V_G, \phi, \{ \varpi_3 \}, \{ \varpi_3, \varpi_4 \}, \{ \varpi_2, \varpi_3 \}, \{ \varpi_3, \varpi_3 \} \}$$

$$\mathcal{I}_G = \{ V_G, \phi, \{ \varpi_4 \}, \{ \varpi_1, \varpi_2 \}, \{ \varpi_1, \varpi_2, \varpi_4 \} \}$$

If we did not get the $Cl_G$-cl-space from step one, we redefine direct graph closure operator as follow :

Definition 2.3. [13][15] Let $G = (V_G, E_G)$ be a direct graph, and $Cl_{G_m} : P(V_G) \rightarrow P(V_G)$ an operator, so we have :

(a) It is said to be $G_m$-cl-operator if $Cl_{G_m} = Cl_G(Cl_{G_m}(Cl_{G_m}..., m-times, \text{ where } H \subseteq G)$

(b) It is said to be $G_m$-topo-cl-operator if $Cl_{G_m+1} = Cl_{G_m}$ for all $H \subseteq G$.

The space $(G, \mathcal{F}_{G_m})$ is said to be $G_m$-cl-space.
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Table 1: According to Example 2.2, \( \text{Cl}_G \) for all subgraph \( H \subseteq G \).

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Example 2.4. Let \( G = (V_G, E_G) \) be a digraph such that: \( V_G = \{ \varpi_1, \varpi_2, \varpi_3, \varpi_4 \} \), \( E_G = \{ (\varpi_1, \varpi_3), (\varpi_2, \varpi_1), (\varpi_2, \varpi_3), (\varpi_3, \varpi_4), (\varpi_4, \varpi_1) \} \).

\( \mathcal{F}_G = \{ V_G, \phi, \{ \varpi_1, \varpi_3, \varpi_4 \} \} \), \( \mathcal{I}_G = \{ V_G, \phi, \{ \varpi_2 \} \} \)

Proposition 2.5. \[ \text{(1)} \] If \( (G, \mathcal{F}_G) \) is a \( G_m \)-cl-space. If \( H, K \subseteq G \); \( H \subseteq K \subseteq G \), then \( \text{Cl}_G(V_H) \subseteq \text{Cl}_G(V_K) \) and \( \text{Int}_G(V_H) \subseteq \text{Int}_G(V_K) \).

Proposition 2.6. \[ \text{(1)} \] If \( (G, \mathcal{F}_G) \) is a \( G_m \)-cl-space. If \( H, K \subseteq G \), then
(a) \( \text{Cl}_G(V_H \cup V_K) = \text{Cl}_G(V_H) \cup \text{Cl}_G(V_K) \).
(b) \( \text{Int}_G(V_H \cap V_K) = \text{Int}_G(V_H) \cap \text{Int}_G(V_K) \).
(c) \( \text{Cl}_G(V_H \cap V_K) \subseteq \text{Cl}_G(V_H) \cap \text{Cl}_G(V_K) \), and
(d) \( \text{Int}_G(V_H) \cup \text{Int}_G(V_K) \subseteq \text{Int}_G(V_H \cup V_K) \).
In $G_m$-cl-space $(G, \mathcal{F}_{G_m})$ the direct subgraph $H \subseteq G$ is said to be [14]:

(a) Regular open (written as $r$-osg) if $V_H = \text{Int}_{G_m}(\text{Cl}_{G_m}(V_H))$.
(b) Semi-open (written as $s$-osg) if $V_H \subseteq \text{Cl}_{G_m}(\text{Int}_{G_m}(V_H))$.
(c) Pre-open (written as $p$-osg) if $V_H \subseteq \text{Int}_{G_m}(\text{Cl}_{G_m}(V_H))$.
(d) $\gamma$-open (written as $\gamma$-osg) if $V_H \subseteq \text{Cl}_{G_m}(\text{Int}_{G_m}(V_H)) \cup \text{Int}_{G_m}(\text{Cl}_{G_m}(V_H))$.
(e) $\alpha$-open (written as $\alpha$-osg) if $V_H \subseteq \text{Int}_{G_m}(\text{Cl}_{G_m}(V_H))$.
(f) $\beta$-open (written as $\beta$-osg) if $V_H \subseteq \text{Cl}_{G_m}(\text{Int}_{G_m}(\text{Cl}_{G_m}(V_H)))$.

The complement of above $j$-osg is said to be $j$-closed subgraph (written as $j$-csg) and the collection of all $j$-osg’s of $(G, \mathcal{F}_{G_m})$ is written as $j-O_{G_m}(G)$ where $j = r, s, p, \gamma, \alpha, \beta$. Also, all of $j-O_{G_m}(G)$ are bigger than $\mathcal{F}_{G_m}$ and closed under union property where $j = r, s, p, \gamma, \alpha, \beta$. The collection of all $j$-csg’s of $(G, \mathcal{F}_{G_m})$ is written as $j-C_{G_m}(G)$ where $j = r, s, p, \gamma, \alpha, \beta$. The $j$-closure (resp. $j$-interior) of $H \subseteq G$ in a $G_m$-cl-space $(G, \mathcal{F}_{G_m})$ is written as $Cl^j_{G_m}(V_H)$ (resp. $\text{Int}^j_{G_m}(V_H)$) and defined by

$$Cl^j_{G_m}(V_H) = \cap \{V_F; V_F \text{is } j \text{-csg and } V_H \subseteq V_F\}$$

$$\text{resp. } \text{Int}^j_{G_m}(V_H) = V_G - Cl^j_{G_m}(V_G - V_H) \text{where } j = r, s, p, \gamma, \alpha, \beta.$$

Proposition 2.7. [14] If $(G, \mathcal{F}_{G_m})$ is $G_m$-cl-space, we have the following statements.
(a) $r-O_{G_m}(G) \subseteq \mathcal{F}_{G_m} \subseteq s-O_{G_m}(G) \subseteq \gamma-O_{G_m}(G) \subseteq \beta-O_{G_m}(G)$.
(b) $\alpha-O_{G_m}(G) \subseteq p-O_{G_m}(G) \subseteq \gamma-O_{G_m}(G)$.

3. Generalization of Pawlak Approximation Spaces

The approx-space $G_m = (G, Cl_{G_m})$ with $Cl_{G_m}$ on $G$ is $G_m$-cl-space $(G, \mathcal{F}_{G_m})$; $\mathcal{F}_{G_m}$ is the $G_m$-cl-space to $G_m$. So we have:

**Definition 3.1.** If $G_m = (G, Cl_{G_m})$ is an approx-space; $G$ is a nonempty universe direct graph, $Cl_{G_m}$ is define on $G_m$, and $\mathcal{F}_{G_m}$ is the $G_m$-cl-space to $G_m$. Then $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is said to be a $G_m$-cl-approx-space.

We present the definitions of lower (resp. near lower) and upper (resp. near upper) approx’s in a $G_m$-cl-approx-space $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$.

**Definition 3.2.** If $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $G_m$-cl-approx-space and $H \subseteq G$. The lower approx (resp. the upper approx- ) of $H$ is denoted by $\mathcal{L}(V_H)$ (resp. $\mathcal{U}(V_H)$ ) and is defined by

$$\mathcal{L}(V_H) = \text{Int}_{G_m}(\text{Cl}_{G_m}(V_H))$$

and $\mathcal{U}(V_H) = \text{Cl}_{G_m}(V_H)$. 

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**Table 2: According to Example 2.4 ClG and ClG2 for all subgraph H ⊆ G**

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<tr>
<th>$V_H$</th>
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</table>
**Definition 3.3.** If $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $G_m$-cl-approx-space and $H \subseteq G$. The near lower approx-
"written as $j$-lower-approx-
(resp. near upper approx-
"written as $j$-upper-approx-
") of $H$ is denoted by $\mathcal{L}^j(V_H)$ (resp. $\mathcal{U}^j(V_H)$) and is defined by

$$\mathcal{L}^j(V_H) = Int_{G_m}^j(V_H)$$

(resp. $\mathcal{U}^j(V_H) = Cl_{G_m}^j(V_H)$), where $j = r, s, p, \gamma, \alpha, \beta$.

**Proposition 3.4.** If $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $G_m$-cl-approx-space and $H \subseteq G$, then

$\mathcal{L}(V(H)) \subseteq \mathcal{L}^j(V(H)) \subseteq V_H \subseteq \mathcal{U}^j(V(H)) \subseteq \mathcal{U}(V_H)$, for all $j \in \{s, p, \gamma, \alpha, \beta\}$.

**Proof.** The proofs are similar for the five cases; So, we will only prove the case $j = S$: Now,

$$\mathcal{U}(V(H)) = Cl_{G_m}(V_H)$$

$$= \cap\{V_F; V_F \in \mathcal{F}_{G_m} \text{ and } V_H \subseteq V_F\}$$

$$\supseteq \cap\{V_F; V_F \in s - C_{G_m} \text{ and } V_H \subseteq V_F\} \text{ since } \mathcal{F}_{G_m} \subseteq s - C_{G_m}(G)$$

$$= Cl_{G_m}^s(V_H) = \mathcal{U}^s(V_H) \supseteq V_H$$

(3.1)

$$\mathcal{L}(V(H)) = Int_{G_m}(V_H)$$

$$= V_G - Cl_{G_m}(V_G - V_H) \subseteq V_G - Cl_{G_m}^s(V_G - V_H) \text{ since } \mathcal{F}_{G_m} \subseteq s - O_{G_m}(G)$$

$$= Int_{G_m}^s(V_H) = \mathcal{L}^s(V_H) \subseteq V_H$$

(3.2)

From (3.1) and (3.2) we get

$$\mathcal{L}(V(H)) \subseteq \mathcal{L}^s(V_H) \subseteq V_H \subseteq \mathcal{U}^s(V_H) \subseteq \mathcal{U}(V_H) \square$$

**Proposition 3.5.** If $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $G_m$-cl-approx-space and $H \subseteq G$, then the following holds for $j = s, p, \gamma, \alpha, \beta$.

(a) $\mathcal{L}(V(H)) \subseteq \mathcal{L}^\alpha(V_H) \subseteq \mathcal{L}^s(V_H) \subseteq \mathcal{L}^\gamma(V_H)$.

(b) $\mathcal{L}^\alpha(V(H)) \subseteq \mathcal{L}^p(V_H) \subseteq \mathcal{L}^\gamma(V_H)$.

**Proof.** By Proposition (3.1), we have $\mathcal{L}(V(H)) \subseteq \mathcal{L}^\alpha(V_H)$. To prove $\mathcal{L}^\alpha(V(H)) \subseteq \mathcal{L}^s(V_H)$. Now,

$$\mathcal{L}^\alpha(V_H) = Int_{G_m}^\alpha(V_H) = V_G - Cl_{G_m}^\alpha(V_G - V_H)$$

$$\subseteq V_G - Cl_{G_m}^\alpha(V_G - V_H) \text{ since } \alpha_{G_m}(G) \subseteq s - O_{G_m}(G) \text{ Thus }$$

$$\mathcal{L}^\alpha(V_H) = Int_{G_m}^\alpha(V_H) \subseteq Int_{G_m}^s(V_H) = \mathcal{L}^s(V_H). \square$$

The prove of the other cases are similarly.

4. Rough Continuous Mappings in $G_m$-Closure Approximation Spaces

The main goal of this part is to give one of the $G_m$-topological applications that represented by the concept of rough continuous mappings. This notion has a great importance in the theory of rough set, since this type of continuity can make different approximation spaces to be related to each others. The following definition introduces rough continuous mappings between two $G_m$-cl-approx-space’s.

**Definition 4.1.** Let $\mathcal{G}_m^1 = (G^1, Cl_{G_m}^1, \mathcal{F}_{G_m}^1)$ and $\mathcal{G}_m^2 = (G^2, Cl_{G_m}^2, \mathcal{F}_{G_m}^2)$ be two $G_m$-cl-approx-

spaces. Then a mapping $f: \mathcal{G}_m^1 \rightarrow \mathcal{G}_m^2$ is said to be rough continuous if $f^{-1}(\mathcal{L}_2(V_H)) \subseteq \mathcal{L}_1(f^{-1}(V_H))$ for every subgraph $H$ in $G$. 
Example 4.2. Let $\mathcal{G}^1_m = (G^1, Cl^1_{G_m}, \tau^1_{G_m})$ and $\mathcal{G}^2_m = (G^2, Cl^2_{G_m}, \tau^2_{G_m})$ be two $G_m$-cl-approx-spaces's where:

$G^1 = (V_{G^1}, E_{G^1}); V_{G^1} = \{ \omega_1, \omega_2, \omega_3 \}, E_{G^1} = \{ (\omega_1, \omega_2), (\omega_1, \omega_3), (\omega_2, \omega_3) \}$,

$G^2 = (V_{G^2}, E_{G^2}); V_{G^2} = \{ u_1, u_2, u_3 \}, E_{G^2} = \{ (u_1, u_2), (u_1, u_3), (u_2, u_3) \}$.

Define a mapping $f : \mathcal{G}^1_m \rightarrow \mathcal{G}^2_m$ such that $f(\omega_1) = f(\omega_2) = u_2, f(\omega_3) = u_3$.

Hence $f$ is rough continuous since $f^{-1}(L_2(V_H)) \subseteq L_1(f^{-1}(V_H))$ for every subgraph $H$ in $\mathcal{G}^2$.

Example 4.3. Let $\mathcal{G}^1_m = (G^1, Cl^1_{G_m}, \tau^1_{G_m})$ and $\mathcal{G}^2_m = (G^2, Cl^2_{G_m}, \tau^2_{G_m})$ be two $G_m$-cl-approx-spaces where:

$G^1 = (V_{G^1}, E_{G^1}); V_{G^1} = \{ \omega_1, \omega_2, \omega_3 \}, E_{G^1} = \{ (\omega_1, \omega_2), (\omega_1, \omega_3), (\omega_2, \omega_3) \}$,

$G^2 = (V_{G^2}, E_{G^2}); V_{G^2} = \{ u_1, u_2, u_3 \}, E_{G^2} = \{ (u_2, u_3), (u_3, u_2) \}$.

Define a mapping $f : \mathcal{G}^1_m \rightarrow \mathcal{G}^2_m$ such that $f(\omega_1) = u_1, f(\omega_2) = u_2, f(\omega_3) = u_3$.

Let $H = (V_H, E_H); V_H = \{ u_2, u_3 \}, E_H = \{ (u_2, u_3), (u_3, u_2) \}$ be a subgraph of $G^2$. Then, $f^{-1}(L_2(V_H)) = f^{-1}\{ u_2, u_3 \} \subseteq \{ \omega_2, \omega_3 \}$, but $L_1(f^{-1}(V_H)) = L_1(\{ \omega_2, \omega_3 \}) = \phi$.

Hence there exists a subgraph $H$ of $G^2$ such that $f^{-1}(L_2(V_H))$ is not subset of $L_1(f^{-1}(V_H))$. Thus $f$ is not a rough continuous mapping.
Definition 4.4. Let $\mathcal{G}_m^1 = (G^1, Cl_1^{G_m^1}, \mathcal{F}_1^{G_m^1})$ and $\mathcal{G}_m^2 = (G^2, Cl_2^{G_m^2}, \mathcal{F}_2^{G_m^2})$ be two $G_m$-cl-approx-spaces. Then a mapping $f : \mathcal{G}_m^1 \to \mathcal{G}_m^2$ is said to be continuous if the inverse image of each open graph in $G^2$ is open in $G^1$.

Example 4.5. Let $\mathcal{G}_m^1 = (G^1, Cl_1^{G_m^1}, \mathcal{F}_1^{G_m^1})$ and $\mathcal{G}_m^2 = (G^2, Cl_2^{G_m^2}, \mathcal{F}_2^{G_m^2})$ be two $G_m$-cl-approx-spaces:

$G^1 = (V_{G^1}, E_{G^1}); V_{G^1} = \{ \varpi_1, \varpi_2, \varpi_3 \}, E_{G^1} = \{ (\varpi_1, \varpi_2), (\varpi_1, \varpi_3), (\varpi_2, \varpi_3) \}$,

$G^2 = (V_{G^2}, E_{G^2}); V_{G^2} = \{ u_1, u_2, u_3 \}, E_{G^2} = \{ (u_2, u_3), (u_3, u_2) \}$.

Figure 5: Graphs $G^1$ and $G^2$ in Exam. 4.5

Define a mapping $f : \mathcal{G}_m^1 \to \mathcal{G}_m^2$ such that $f(\varpi_1) = u_2, f(\varpi_2) = f(\varpi_3) = u_3$.

Then $f$ is continuous, since $f^{-1}(H) \in \mathcal{F}_G^1$ for all $H \in \mathcal{F}_G^2$.

Example 4.6. In example 4.5 a mapping $f$ is not continuous, since $H = (V_H, E_H); V_H = \{ u_2, u_3 \}, E_H = \{ (u_2, u_3), (u_3, u_2) \}$ is an open subgraph in $G^2$ but $f^{-1}(H) = (V_{f^{-1}(H)}, E_{f^{-1}(H)}); V_{f^{-1}(H)} = \{ \varpi_2, \varpi_3 \}, E_{f^{-1}(H)} = \{ (\varpi_2, \varpi_3) \}$ is not open in $G^1$.

Theorem 4.7. Let $\mathcal{G}_m^1 = (G^1, Cl_1^{G_m^1}, \mathcal{F}_1^{G_m^1})$ and $\mathcal{G}_m^2 = (G^2, Cl_2^{G_m^2}, \mathcal{F}_2^{G_m^2})$ be two $G_m$-cl-approx-space's. Then $f : \mathcal{G}_m^1 \to \mathcal{G}_m^2$ is a rough continuous mapping if and only if $f$ is continuous.

Proof. $(\Rightarrow)$ Let $f$ be a rough continuous mapping and $H \subseteq G^2$ be an open graph, hence $L_2(V_H) = Int_{G^m}(V_H) = V_H$. Thus $f^{-1}(V_H) = f^{-1}(L_2(V_H) \subseteq L_1(f^{-1}(V_H)))$, since $f$ is a rough continuous mapping.

But $L_1(f^{-1}(V_H)) = Int_{G^m}(f^{-1}(V_H))$. Then $f^{-1}(V_H) \subseteq Int_{G^m}(f^{-1}(V_H))$ and hence $f^{-1}(V_H) = Int_{G^m}(f^{-1}(V_H))$. Thus $f^{-1}(V_H)$ is an open graph in $G^1$.

Therefore $f$ is continuous mapping.

$(\Leftarrow)$ Let $f$ be a continuous mapping and $H \subseteq G^2$. Then $f^{-1}(L_2(V_H) \subseteq f^{-1}(V_H)$, since $L_2(V_H) \subseteq V_H$.

Thus $L_1(f^{-1}(L_2(V_H))) \subseteq L_1(f^{-1}(V_H))$ (4.1.1)

But $f^{-1}(L_2(V_H)) = f^{-1}(Int_{G^m}(V_H)) \in \mathcal{F}_G^1$, since $Int_{G^m}(V_H) \in \mathcal{F}_G^2$ and $f$ is continuous. Hence $f^{-1}(L_2(V_H)) \subseteq Int_{G^m}(f^{-1}(L_2(V_H))) = L_1(f^{-1}(L_2(V_H)))$ and then from (4.1.1) we get $f^{-1}(L_2(V_H)) \subseteq L_1(f^{-1}(L_2(V_H))) \subseteq L_1(f^{-1}(V_H))$. Therefore $f$ is a rough continuous mapping. □
5. Near Rough Continuous Mappings in $G_m$-Closure Approximation Spaces

Near rough (written as $j$-rough) continuous mappings represent different levels of continuity; $j = r, s, p, \gamma, \alpha, \beta$. In this section we present the concepts of $j$-rough continuous mappings between two $G_m$-cl-approx-spaces.

**Definition 5.1.** Let $G^1 = (G^1, Cl^1_{G_m}, \mathcal{F}^1_{G_m})$ and $G^2 = (G^2, Cl^2_{G_m}, \mathcal{F}^2_{G_m})$ be two $G_m$-cl-approx-spaces. A mapping $f : G^1_m \rightarrow G^2_m$ is said to be near rough (written as $j$-rough) continuous for all $j = r, s, p, \gamma, \alpha, \beta$ if $f^{-1} \subseteq \mathcal{L}_2(V_H) \subseteq \mathcal{L}_1(f^{-1}(V_H))$ for every subgraph $H$ in $G^2_m$.

**Example 5.2.** Let $G^1 = (G^1, Cl^1_{G_m}, \mathcal{F}^1_{G_m})$ and $G^2 = (G^2, Cl^2_{G_m}, \mathcal{F}^2_{G_m})$ be two $G_m$-cl-approx-spaces where:

$G^1 = (G_{G^1}, E_{G^1}); V_{G^1} = \{ \omega_1, \omega_2, \omega_3, \omega_4 \}, E_{G^1} = \{ (\omega_2, \omega_3), (\omega_3, \omega_4), (\omega_4, \omega_4) \}$,

$G^2 = (G_{G^2}, E_{G^2}); V_{G^2} = \{ u_1, u_2, u_3, u_4 \}, E_{G^2} = \{ (u_1, u_2), (u_1, u_3), (u_2, u_1), (u_2, u_3), (u_2, u_3) \}$.

![Graphs G^1 and G^2 in Exam. 5.2](image)

$\mathcal{F}^1_{G^1} = \{ V_{G^1}, \phi, \{ \omega_1 \}, \{ \omega_2, \omega_3, \omega_4 \} \}, \mathcal{F}^2_{G^1} = \{ V_{G^1}, \phi, \{ \omega_1 \}, \{ \omega_2, \omega_3, \omega_4 \} \}$ and

$\mathcal{F}^1_{G^2} = \{ V_{G^2}, \phi, \{ u_3 \}, \{ u_3, u_4 \}, \{ u_1, u_2, u_3 \} \}, \mathcal{F}^2_{G^2} = \{ V_{G^2}, \phi, \{ u_4 \}, \{ u_1, u_2 \}, \{ u_1, u_2, u_3 \} \}$.

Hence $p - O_{G^2}(G^1) = \phi$ the power set of vertices of $G^1$.

Define a mapping $f : G^1_m \rightarrow G^2_m$ such that $f(\omega_1) = u_1, f(\omega_2) = u_2, f(\omega_3) = u_3, f(\omega_4) = u_4$.

Thus $f$ is $p$-rough continuous since $f^{-1}(\mathcal{L}_2(V_H)) \subseteq \mathcal{L}_1(f^{-1}(V_H))$ for every subgraph $H$ in $G^2_m$ as illustrated in Table (3).

**Example 5.3.** In example (5.1), we get $s - O_{G^2}(G^1) = \{ V_{G^1}, \phi, \{ \omega_1 \}, \{ \omega_2, \omega_3, \omega_4 \} \}.

Let $H = (V_H, E_H); V_H = \{ u_1, u_2 \}, E_H = \{ (u_1, u_2), (u_2, u_1) \}$ be a subgraph of $G^2$. Then $f^{-1}(\mathcal{L}_2(V_H)) = f^{-1}(\{ u_1, u_2 \}) = \{ \omega_1, \omega_2 \}$, but $\mathcal{L}_1(f^{-1}(V_H)) = \mathcal{L}_1(\{ \omega_1, \omega_2 \}) = \{ \omega_1 \}$. Hence there exists a subgraph $H$ of $G^2$ such that $f^{-1}(\mathcal{L}_2(V_H))$ is not subset of $\mathcal{L}_1(f^{-1}(V_H))$. Thus $f$ is not a s-rough continuous mapping.

**Proposition 5.4.** The implication between rough continuity and $j$-rough continuity for all $j = r, s, p, \gamma, \alpha, \beta$ are given by the following diagram.

\[
r\text{-rough continuous} \quad \downarrow \\
\alpha\text{-rough continuous} \Rightarrow s\text{-rough continuous}
\]
Table 3: $\mathcal{L}_2(V_H), f^{-1}(V_H), \mathcal{L}_1^f(f^{-1}(V_H))$ and $f^{-1}(\mathcal{L}_2(V_H))$ for every subgraph $H$ in $\mathcal{G}_1$, where $H, G^2, \mathcal{L}_1, \mathcal{L}_2$ and $f$ are given in Example 5.2.

<table>
<thead>
<tr>
<th>$H \subseteq G^2$</th>
<th>$\mathcal{L}_2(V_H)$</th>
<th>$f^{-1}(V_H)$</th>
<th>$\mathcal{L}_1^f(f^{-1}(V_H))$</th>
<th>$f^{-1}(\mathcal{L}_2(V_H))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{G^2}$</td>
<td>$V_{G^2}$</td>
<td>$V_{G^1}$</td>
<td>$V_{G^1}$</td>
<td>$V_{G^1}$</td>
</tr>
<tr>
<td>${ u_1 }$</td>
<td>$\phi$</td>
<td>$\omega_1$</td>
<td>$\omega_1$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${ u_2 }$</td>
<td>$\phi$</td>
<td>$\omega_2$</td>
<td>$\omega_2$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${ u_3 }$</td>
<td>$\phi$</td>
<td>$\omega_3$</td>
<td>$\omega_3$</td>
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<tr>
<td>${ u_4 }$</td>
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<td>${ u_1, u_2 }$</td>
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<td>$\phi$</td>
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<tr>
<td>${ u_1, u_4 }$</td>
<td>$\phi$</td>
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<tr>
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<td>$\phi$</td>
<td>$\omega_2, \omega_3$</td>
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<td>${ u_2, u_4 }$</td>
<td>$\phi$</td>
<td>$\omega_2, \omega_4$</td>
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<td>$\omega_2, \omega_4$</td>
</tr>
<tr>
<td>${ u_3, u_4 }$</td>
<td>$\phi$</td>
<td>$\omega_3, \omega_4$</td>
<td>$\omega_3, \omega_4$</td>
<td>$\omega_3, \omega_4$</td>
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<tr>
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<td>$\omega_1, \omega_2, \omega_4$</td>
<td>$\omega_1, \omega_2, \omega_4$</td>
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<td>$\omega_1, \omega_2, \omega_3, \omega_4$</td>
</tr>
</tbody>
</table>

Definition 6.1. If $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $G_m$-cl-approx-space. Then $\mathcal{G}_m$ is said to be a rough $\mathcal{G}_m$ space (written as $\mathcal{G}_m$-space), if for every two distinct vertices $\varpi, u \in G$, find a subgraph $H \subseteq G$ ; either $\varpi \in L(V_H), u \in V_G-\mathcal{L}(V_H)$ or $u \in \mathcal{L}(V_H), \varpi \in V_G-\mathcal{L}(V_H)$.

6. Rough Separation Axioms in $G_m$-Closure Approximation Spaces

The main goal of this section is to give one of the $G_m$-topological applications that represented by the concept of rough separations axioms. Two different objects of the universe can be belong to the same category and then they are indiscernible in view of the available information which making the imprecise and uncertainty about data. The main purpose of separation axioms is to make vertices and graphs of spaces topologically distinguishable that is a very useful in the information systems to extracting the given data.

Example 5.5. In example (5.1), we get $\alpha-O_{G^2}(G^1) = \{ V_{G^1}, \phi, \{ \omega_1 \}, \{ \omega_2, \omega_3, \omega_4 \} \}$. Let $H = (V_H, E_H); V_H = \{ u_4 \}, E_H = \phi$, be a subgraph of $G^2$. Then $f^{-1}(\mathcal{L}_2(V_H)) = f^{-1}(\{ u_4 \}) = \{ \omega_4 \}$, but $\mathcal{L}_1^f(f^{-1}(V_H)) = \mathcal{L}_1^f(\{ \omega_4 \}) = \phi$. Hence there exists a subgraph $H$ of $G^2$ such that $f^{-1}(\mathcal{L}_2(V_H))$ is not subset of $\mathcal{L}_1^f(f^{-1}(V_H))$. Thus $f$ is not a $\alpha$-rough continuous mapping, but it is $p$-rough continuous.
Theorem 6.2. If $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $G_m$-cl-approx-space. Then $\mathcal{G}_m$ is a $\mathcal{G}_{m_0}$-space if and only if $\mathcal{U}(\{v\}) \neq \mathcal{U}(\{w\})$ for every two distinct vertices $v, w \in G$.

Proof. $(\Rightarrow)$ If $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $\mathcal{G}_{m_0}$-space. Then for every two distinct vertices $v, w \in G$ find a subgraph $H \subseteq G ; v \in \mathcal{L}(H) = Int_{G_m}(H)$ and $w \notin \mathcal{L}(H) = Int_{G_m}(H)$. Thus $v \notin [Int_{G_m}(V_H)]^c$ and $[Int_{G_m}(V_H)]^c$ is closed graph. Hence $v \notin \cup \{K; V_K \subseteq [Int_{G_m}(V_H)]^c, K \in \mathcal{F}_{G_m}, v \in \mathcal{L}(H) = Int_{G_m}(H)\}$. Thus $v \notin \cup \{u\}$, but $v \in Cl_{G_m}(v) = \mathcal{U}(v)$. Therefore $\mathcal{U}(\{v\}) \neq \mathcal{U}(\{w\})$.

$(\Leftarrow)$ Let $\mathcal{U}(\{v\}) \neq \mathcal{U}(\{w\})$ for every two distinct vertices $v, w \in G$. Then find $v \in G \cap w \notin \mathcal{U}(\{v\})$. Now, suppose thus $v \in \mathcal{U}(\{u\})$, then $\mathcal{U}(\{v\}) \subseteq \mathcal{U}(\{u\})$, since $v \in \mathcal{U}(\{v\})$ and $w \in \mathcal{U}(\{u\})$, which is a contradiction. Hence $v \notin \mathcal{U}(\{u\})$ and then $v \notin [\mathcal{U}(\{v\})]^c = \mathcal{L}(\{v\})^c$. But $u \notin [\mathcal{U}(\{v\})]^c = \mathcal{L}(\{u\})^c$, hence for every two distinct vertices $v, u \in G$ find a subgraph $H = \{u\}$ of $G ; v \in \mathcal{L}(V_H) = \mathcal{L}(\{u\})$ and $u \notin \mathcal{L}(V_H) = \mathcal{L}(\{u\})$ (i.e. $u \in V_G - \mathcal{L}(V_H)$). Therefore $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $\mathcal{G}_{m_0}$-space. □

Definition 6.3. If $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $G_m$-cl-approx-space. Then $\mathcal{G}_m$ is said to be a rough $\mathcal{G}_m$-space (written as $\mathcal{G}_{m_1}$-space), if for every two distinct vertices $v, w \in G$, find two subgraph $H$ and $K$ of $G ; v \in \mathcal{L}(V_H), u \notin \mathcal{L}(V_H)$ and $u \in \mathcal{L}(V_K), w \notin \mathcal{L}(V_K)$.

Theorem 6.4. If $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $G_m$-cl-approx-space. Then $\mathcal{G}_m$ is a $\mathcal{G}_{m_1}$-space if and only if $\mathcal{U}(\{v\}) \neq \mathcal{U}(\{w\})$ for every $v \in G$.

Proof. $(\Rightarrow)$ If $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $\mathcal{G}_{m_1}$-space. Then for every two distinct vertices $v, u \in G$ there exists two subgraph $H$ and $K$ of $G$ such that $v \in \mathcal{L}(V_H), u \notin \mathcal{L}(V_H)$ and $u \in \mathcal{L}(V_K), w \notin \mathcal{L}(V_K)$. Clearly $v \in \{v\}^c$ and $u \notin \{v\}^c$, thus for all $u \in \{v\}^c$ find a subgraph $K_u$ of $G ; u \in \mathcal{L}(V_{K_u}) = Int_{G_m}(V_{K_u}) \subseteq \{v\}^c$ and thus $\{v\}^c = \cup u \in \{v\}^c \mathcal{L}(V_{K_u}) = \cup u \in \{v\}^c Int_{G_m}(V_{K_u})$. Hence $\{v\}^c \subseteq \mathcal{F}_{G_m}, that is \{v\} \in \mathcal{F}_{G_m}$. Thus $\{v\} = \mathcal{G}_m(\{v\}) = \mathcal{U}(\{v\})$. Therefore $\{v\} = \mathcal{U}(\{v\})$ for every $v \in G$.

$(\Leftarrow)$ Let $\{v\} = \mathcal{U}(\{v\})$ for every $v \in G$. Then $\{v\} \in \mathcal{F}_{G_m}$ and $\{v\}^c \in \mathcal{F}_{G_m}$ for every $v \in G$. Thus for every two distinct vertices $v, u \in G$, we get $v \in \{v\}^c, u \notin \{v\}^c$ and $u \in \{v\}^c, v \notin \{v\}^c$ such that $\{v\}^c, \{u\}^c \subseteq \mathcal{F}_{G_m}$. Since $\mathcal{L}(\{v\})^c = \{v\}^c$ and $\mathcal{L}(\{u\})^c = \{u\}^c$, then for every two distinct vertices $v, u \in G$ there exists two subgraphs $H = \{u\}^c$ and $K = \{v\}^c$ of $G$ such that $v \in \mathcal{L}(V_H), u \notin \mathcal{L}(V_K)$ and $u \in \mathcal{L}(V_K), w \notin \mathcal{L}(V_K)$. Therefore $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $\mathcal{G}_{m_1}$-space. □

Theorem 6.5. If $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $G_m$-cl-approx-space. If $\mathcal{G}_m$ is a $\mathcal{G}_{m_1}$-space and $\{v\} = \mathcal{L}(\{v\})$ for all $v \in G$, then for all subgraph $H$ of $G$, $H$ is an exact graph and $\mathcal{F}_{G_m}$ is the discrete topology.

Proof. If $\mathcal{G}_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $\mathcal{G}_{m_1}$-space. Then by Theorem (6.2), for all $v \in G, \{v\} = \mathcal{U}(\{v\}) = Cl_{G_m}(\{v\})$, hence $\{v\} \in \mathcal{F}_{G_m} and \{v\}^c \in \mathcal{F}_{G_m}$ (6.5.1)

But it is given that for all $u \in G, \{v\} = \mathcal{L}(\{v\}) = Int_{G_m}(\{v\})$. Hence $\{v\} \in \mathcal{F}_{G_m} and \{v\}^c \in \mathcal{F}_{G_m}$ (6.5.2)

Thus for all $v \in G, \{v\}$ is exist graph (i.e. $\mathcal{L}(\{v\}) = \{v\} = \mathcal{U}(\{v\})$).

Let $H$ be any subgraph of $G$, then $V_H = \cup u \in V_H \{v\} and V_G - V_H = \cup u \in V_G - V_H \{v\}$

Hence $V_H, V_G - V_H \in \mathcal{F}_{G_m}$ since by (6.5.3) $V_H$ and $V_G - V_H$ are union of open graphs. By taking the complement of (6.5.3) we get $V_G - V_H = \cap \cup u \in V_H \{v\} and V_G - (V_G - V_H) = V_H = \cup u \in V_G - V_H \{u\}$.

Then $V_G - V_H, V_H \in \mathcal{F}_{G_m}$ since $V_G - V_H, V_H$ are intersection of closed graphs. Therefore $H$ is an exact graph for every subgraph $H$ of $G$ and thus $\mathcal{F}_{G_m}$ is the discrete topology. □
Theorem 7.2. If \(D = (G, Cl_G, F_G)\) is a \(G_m\)-cl-approx-space. Then \(D_m\) is said to be a rough \(D_m\) space (written as \(D_m\)-space), if for every two distinct vertices \(u, v \in G\), there exists two subgraph \(H\) and \(K\) of \(G\) such that \(v \in L(H), u \in L(V)\) and \(L(V) \cap L(V) = \phi\).

**Remark 6.7.** The implication between \(D_m\)-spaces, \(D_m\)-spaces and \(D_m\)-spaces are given in the following diagram.

\[D_m - spaces \Rightarrow D_m - spaces \Rightarrow D_m - spaces.\]

In general, the converse of **Remark 6.7** is not true as an example:

**Example 6.8.** If \(D = (G, Cl_G, F_G)\) is a \(G_m\)-cl-approx-spaces;

\[G = (V_G, E_G); V_G = \{ \varpi_1, \varpi_2, \varpi_3 \}, E_G = \{ (\varpi_2, \varpi_3) \}, \]

\[F_G = \{ V_G, \emptyset, \{ \varpi_1 \}, \{ \varpi_3 \}, \{ \varpi_2, \varpi_3 \} \}, \]

\[G_G = \{ V_G, \emptyset, \{ \varpi_1 \}, \{ \varpi_2 \}, \{ \varpi_2, \varpi_3 \} \}. \]

7. Near Rough Separation Axioms in \(G_m\)-Closure Approximation Spaces

The concepts of near approximation have an important role in separation axioms. By using these concepts we can construct many several separation axioms. The following definition introduces some new separation axioms.

**Definition 7.1.** If \(D = (G, Cl_G, F_G)\) is a \(G_m\)-cl-approx-space. Then for all \(j = r, s, p, \gamma, \alpha, \beta\), \(D_m\) is said to be near rough \(D_m\)-space (written as \(D_m\)-space), if for every two distinct vertices \(v, u \in G\), find a subgraph \(H \subseteq G\); either \(v \in L^j(H), u \in V_G - L^j(H)\) or \(u \in L^j(H), v \in V_G - L^j(H)\).

**Theorem 7.2.** If \(D = (G, Cl_G, F_G)\) is a \(G_m\)-cl-approx-space. Then \(D_m\) is a \(D_m\)-space for all \(j = r, s, p, \gamma, \alpha, \beta\) if \(\mathcal{W}^j(\{ v \}) \neq \mathcal{W}^j(\{ u \})\) for every two distinct vertices \(v, u \in G\).

**Proof.** For \(j = \beta\): Now,

\(\Rightarrow\) If \(D = (G, Cl_G, F_G)\) is a \(D_m\)-space. Then for every two distinct vertices \(v, u \in G\) find a subgraph \(H \subseteq G\); either \(v \in L^j(H), u \in V_G - L^j(H)\) or \(u \in L^j(H), v \in V_G - L^j(H)\). Thus \(v \notin [\text{Int}^j_{G_m}(V)]^c, u \notin [\text{Int}^j_{G_m}(V)]^c\) and \([\text{Int}^j_{G_m}(V)]^c \subseteq \text{cl}^j_{G_m}(G)\). Hence \(v \notin \{ K; V_K \subseteq [\text{Int}^j_{G_m}(V)]^c, K \in \text{cl}^j_{G_m}(G), \{ u \} \subseteq H \} = \text{cl}^j_{G_m}(\{ u \}) = \mathcal{W}^j(\{ u \})\). Thus \(v \notin \mathcal{W}^j(\{ u \})\), but \(v \in \text{cl}^j_{G_m}(\{ u \}) = \mathcal{W}^j(\{ v \})\). Therefore \(\mathcal{W}^j(\{ v \}) \neq \mathcal{W}^j(\{ u \})\).

\(\Leftarrow\) Let \(\mathcal{W}^j(\{ v \}) \neq \mathcal{W}^j(\{ u \})\) for every two distinct vertices \(v, u \in G\). Then find \(w \in G\);
Let \( \mathcal{G}_m = (G, Cl_{G_m}, F_{G_m}) \) be a \( G_m \)-cl-approx-space. Then for all \( j = r, s, p, \gamma, \alpha, \beta \), \( \mathcal{G}_m \) is said to be near rough \( \mathcal{G}_m \)-space (written as \( \mathcal{G}_m \)-space), if for every two distinct vertices \( \varpi, u \in G \), there exists two subgraph \( H \) and \( K \) of \( G \) such that \( \varpi \in L^j(V_H), u \notin L^j(V_H) \) and \( u \in L^j(V_K), \varpi \notin L^j(V_K) \).

Theorem 7.4. If \( \mathcal{G}_m = (G, Cl_{G_m}, F_{G_m}) \) is a \( G_m \)-cl-approx-space. Then \( G_m \) is a \( \mathcal{G}_m \)-space for all \( j \in \{ r, s, p, \gamma, \alpha, \beta \} \) if \( \{ \varpi \} = \mathcal{W}^j(\{ v \}) \) for every \( \varpi \in G \).

Proof. For \( j = \alpha \): Now,

\[
\Rightarrow \text{ If } \mathcal{G}_m = (G, Cl_{G_m}, F_{G_m}) \text{ is a } \mathcal{G}_m \text{-space. Then for every two distinct vertices } \varpi, u \in G \text{ find two subgraph } H \text{ and } K \text{ of } G \text{ such that } \varpi \in L^\alpha(V_H), u \notin L^\alpha(V_H) \text{ and } u \in L^\alpha(V_K), \varpi \notin L^\alpha(V_K). \text{ Clearly } \alpha \in \{ u \}^c \text{ and } u \in \{ v \}^c, \text{ thus for all } u \in \{ \varpi \}^c \text{ find a subgraph } K_u \text{ of } G \text{ such that } \varpi \in L^\alpha(V_{K_u}) = Int_{G_m}(V_{K_u}) \subseteq \{ \varpi \}^c \text{ and thus } \{ \varpi \}^c = \bigcup_{u \in \{ \varpi \}} L^\alpha(V_{K_u}) = \bigcup_{u \in \{ \varpi \}} \varpi \text{ is an } \alpha \text{-open graph, that is } \{ \varpi \} \text{ is an } \alpha \text{-closed graph. Thus } \{ \varpi \} \neq \varpi \in G \text{. Therefore } \{ \varpi \} = \mathcal{W}^\alpha(\{ v \}) \text{ for every } \varpi \in G. \\
\Leftarrow \text{ Let } \{ \varpi \} = \mathcal{W}^\alpha(\{ v \}) \text{ for every } \varpi \in G. \text{ Then } \{ \varpi \} \text{ is an } \alpha \text{-closed graph and } \{ \varpi \}^c \text{ is an } \alpha \text{-open graph for all } \varpi \in G. \text{ Thus for every distinct vertices } \varpi, u \in G, \text{ we get } \varpi \in \{ u \}^c, u \notin \{ u \}^c \text{ and } u \in \{ \varpi \}^c, u \notin \{ \varpi \}^c \text{ such that } \{ \varpi \}^c, \{ u \}^c \text{ are } \alpha \text{-open graphs. Since } L^\alpha(\{ \varpi \}^c) = \{ \varpi \}^c \text{ and } L^\alpha(\{ u \}^c) = \{ u \}^c \text{, then for every two distinct vertices } \varpi, u \in G \text{ there exists two subgraph } H = \{ u \}^c \text{ and } K = \{ \varpi \}^c \text{ such that } \varpi \in L^\alpha(V_H), u \notin L^\alpha(V_H) \text{ and } u \in L^\alpha(V_K), \varpi \notin L^\alpha(V_K). \text{ Therefore } \mathcal{G}_m = (G, Cl_{G_m}, F_{G_m}) \text{ is a } \mathcal{G}_m \text{-space.} \]

The proofs of the other cases are similar.

Theorem 7.5. If \( \mathcal{G}_m = (G, Cl_{G_m}, F_{G_m}) \) is a \( G_m \)-cl-approx-space. If \( G_m \) is a \( \mathcal{G}_m \)-space and \( \{ \varpi \} = L^p(\{ \varpi \}) \) for all \( \varpi \in G \) and \( j = s, p, \gamma, \alpha, \beta \), then for every subgraph \( H \) of \( G \), \( H \) is a \( j \)-exact graph and the family of all \( j \)-open graphs is the discrete topology.

Proof. For \( j = p \): Now, let \( \mathcal{G}_m = (G, Cl_{G_m}, F_{G_m}) \) be a \( \mathcal{G}_m \)-space. Then by Theorem 7.4, for all \( \varpi \in G, \{ \varpi \} = \mathcal{W}^p(\{ \varpi \}) = Cl_{G_m}(\{ \varpi \}), \text{ hence } \{ \varpi \} \in p - C_{G_m}(G) \text{ and } \{ \varpi \}^c \in p - O_{G_m}(G) \text{. Therefore } \{ \varpi \} \in p - C_{G_m}(G) \text{ and } \{ \varpi \}^c \in p - O_{G_m}(G) \text{. Hence } \{ \varpi \} \in p - C_{G_m}(G) \text{ and } \{ \varpi \}^c \in p - C_{G_m}(G) \text{. Thus for all } \varpi \in G, \{ \varpi \} \text{ is a } p \text{-exact graph (i.e. } L^p(\{ \varpi \}) = \{ \varpi \} = \mathcal{W}^p(\{ \varpi \}). \text{ Let } H \text{ be any subgraph of } G, \text{ then } V_H = \bigcup_{\varpi \in V_H} \{ \varpi \} \text{ and } V(G) - V_H = \bigcup_{u \in V_G - V_H} \{ u \} \text{. Hence } V_H, V_G - V_H \in p - O_{G_m}(G) \text{ since by (7.3) } V_H \text{ and } V - G - V_H \text{ are union of } p \text{-open graphs. By taking the complement of (7.3), we get } \}
If \( G - V_H = \cap_{u \in V_H} \{ u \}^c \) and \( V_G - (V_G - V_H) = V_H = \cap_{u \in V_G - V_H} \{ u \}^c \).

Then \( V_G - V_H, V_H \in p - C_{G_m}(G) \) since \( V_G - V_H, V_H \) are intersection of \( p \)-closed graphs. Therefore \( H \) is a \( p \)-exact graph for every subgraph \( H \) of \( G \) and thus the family of all \( p \)-open graphs is the discrete topology. \( \square \)

The proofs of the other cases are similar.

**Definition 7.6.** If \( G_m = (G, C_{G_m}, \mathcal{F}_{G_m}) \) is a \( G_m \)-cl-approx-space. Then for all \( j = r, s, p, \gamma, \alpha, \beta \), \( G_m \) is said to be a near rough \( G_m \)-space (written as \( \mathcal{G}_m \)-space), if for every two distinct vertices \( v, u \) in \( G \), there exists two subgraph \( H \) and \( K \) of \( G \) such that \( v \in \mathcal{L}^{j}(V_H), u \in \mathcal{L}^{j}(V_K) \) and \( \mathcal{L}^{j}(V_H) \cap \mathcal{L}^{j}(V_K) = \phi \).

**Remark 7.7.** The implication between \( \mathcal{G}_m \)-spaces, \( \mathcal{G}_m \)-spaces and \( \mathcal{G}_m \)-spaces for all \( j = r, s, p, \gamma, \alpha, \beta \) are given in the diagram.

\[
\mathcal{G}_m \text{-spaces} \Rightarrow \mathcal{G}_m \text{-spaces} \Rightarrow \mathcal{G}_m \text{-spaces}.
\]

In general, the converse of Remark 7.7 is not true. Example 7.8 illustrated that the converse of Remark 7.7 is not true if \( j = p \).

**Example 7.8.** If \( G_m = (G, C_{G_m}, \mathcal{F}_{G_m}) \) is a \( G_m \)-cl-approx-space in example 2.2

\[ \mathcal{F}_{G_1} = \{ V_G, \phi, \{ \varnothing_3 \}, \{ \varnothing_3, \varnothing_4 \}, \{ \varnothing_1, \varnothing_2, \varnothing_3 \} \} \]

\[ \mathcal{F}_{G_1} = \{ V_G, \phi, \{ \varnothing_4 \}, \{ \varnothing_1, \varnothing_2 \}, \{ \varnothing_1, \varnothing_2, \varnothing_3 \} \} \]

Then \( p - O_{G_1}(G) = \{ V_G, \phi, \{ \varnothing_1 \}, \{ \varnothing_1, \varnothing_2 \}, \{ \varnothing_1, \varnothing_2, \varnothing_3 \} \} \). Hence \( G_m = (G, C_{G_m}, \mathcal{F}_{G_m}) \) is a \( \mathcal{G}_m \)-space, but it is not a \( \mathcal{G}_m \)-space.

**Theorem 7.9.** The implications between \( \mathcal{G}_m \)-spaces and \( \mathcal{G}_m \)-spaces for all \( i \in \{ 0, 1, 2 \} \) and \( j \in \{ r, s, p, \gamma, \alpha, \beta \} \) are given by the following diagram.

\[
\mathcal{G}_m \text{-spaces} \Rightarrow \mathcal{G}_m \text{-spaces} \Rightarrow \mathcal{G}_m \text{-spaces}.
\]

**Proof.** Using Remark 7.6 (resp. Remark 7.7), it is clear that

\( \mathcal{G}_m \)-spaces \( \Rightarrow \mathcal{G}_m \)-spaces.

(Resp. \( \mathcal{G}_m \)-spaces \( \Rightarrow \mathcal{G}_m \)-spaces. for all \( j \in \{ s, p, \gamma, \alpha, \beta \} \)).

Now, we shall prove that \( \mathcal{G}_m \)-spaces \( \Rightarrow \mathcal{G}_m \)-spaces for all \( j \in \{ s, p, \gamma, \alpha, \beta \} \). Let \( G_m = (G, C_{G_m}, \mathcal{F}_{G_m}) \) be a \( \mathcal{G}_m \)-space. Then for every two distinct vertices \( v, u \) in \( G \) find a subgraph \( H \) of \( G \) such that \( v \in \mathcal{L}(V_H) = \text{Int} \mathcal{G}_m(V_H) \) and \( u \notin \mathcal{L}(V_H) = \text{Int} \mathcal{G}_m(V_H) \). But \( \text{Int} \mathcal{G}_m(V_H) \in \mathcal{G}_m \), then \( \text{Int} \mathcal{G}_m(V_H) \) is a \( j \)-open graph for all \( j \in \{ s, p, \gamma, \alpha, \beta \} \). Hence \( \text{Int} \mathcal{G}_m(V_H) = \mathcal{L} \mathcal{G}_m(V_H) = \mathcal{L}(V_H) \). Thus for every two distinct vertices \( v, u \) in \( G \) find a subgraph \( H \) of \( G \) such that \( v \in \mathcal{L}(V_H) = \text{Int} \mathcal{G}_m(V_H) \) and \( u \notin \mathcal{L}(V_H) = \text{Int} \mathcal{G}_m(V_H) \). Therefore \( G_m = (G, C_{G_m}, \mathcal{F}_{G_m}) \) is a \( \mathcal{G}_m \)-space for all \( j \in \{ s, p, \gamma, \alpha, \beta \} \).

Similarly we can show that

\( \mathcal{G}_m \)-spaces \( \Rightarrow \mathcal{G}_m \)-spaces \( \Rightarrow \mathcal{G}_m \)-spaces. for all \( j \in \{ s, p, \gamma, \alpha, \beta \} \) \( \square \)

In general, the converse of Theorem 7.9 is not true, as illustrated by the following example.

**Example 7.10.** Using the same \( G_m \)-cl-approx-space \( G_m = (G, C_{G_m}, \mathcal{F}_{G_m}) \) which is given in example (7.8), we get

\[ \mathcal{F}_{G_1} = \{ V_G, \phi, \{ \varnothing_3 \}, \{ \varnothing_3, \varnothing_4 \}, \{ \varnothing_1, \varnothing_2, \varnothing_3 \} \} \]

\[ \mathcal{F}_{G_1} = \{ V_G, \phi, \{ \varnothing_4 \}, \{ \varnothing_1, \varnothing_2 \}, \{ \varnothing_1, \varnothing_2, \varnothing_3 \} \} \]

\( \mathcal{F}_{G_1} = \{ V_G, \phi, \{ \varnothing_1 \}, \{ \varnothing_1, \varnothing_2 \}, \{ \varnothing_1, \varnothing_2, \varnothing_3 \} \} \) and \( \mathcal{F}_{G_1} = \{ V_G, \phi, \{ \varnothing_4 \}, \{ \varnothing_1, \varnothing_2 \}, \{ \varnothing_1, \varnothing_2, \varnothing_3 \} \} \).

Hence \( G_m = (G, C_{G_m}, \mathcal{F}_{G_m}) \) is a \( \mathcal{G}_m \)-space, but it is not a \( \mathcal{G}_m \)-space.
Theorem 7.11. The implications between $\mathcal{G}_{mi}^i$-spaces for all $i \in \{0, 1, 2\}$ and $j \in \{s, p, \gamma, \alpha, \beta\}$ are given by the following diagram.

$$
\mathcal{G}_{mi}^\alpha \text{-spaces} \Rightarrow \mathcal{G}_{mi}^S \text{-spaces}.
\downarrow
\downarrow

\mathcal{G}_{mi}^p \text{-spaces} \Rightarrow \mathcal{G}_{mi}^\gamma \text{-spaces} \Rightarrow \mathcal{G}_{mi}^\beta \text{-spaces}.
$$

Proof. Using Proposition 5.4, the proof is similar to Theorem 7.9. □

In general, the converse of Theorem 7.11 is not true, as illustrated by the following example.

Example 7.12. Using the same $G_m$-cl-approx-space $G_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ which is given in example 7.8, we get

$$
s - O_{G_1}(G) = \{V_G, \emptyset, \{\varnothing 1, \varnothing 2\}, \{\varnothing 3, \varnothing 4\}, \{\varnothing 1, \varnothing 2, \varnothing 3\}, \{\varnothing 1, \varnothing 2, \varnothing 4\}\},
\beta - O_{G_1}(G) = \{V_G, \emptyset, \{\varnothing 1\}, \{\varnothing 2\}, \{\varnothing 4\}, \{\varnothing 1, \varnothing 2\}, \{\varnothing 1, \varnothing 3\}, \{\varnothing 1, \varnothing 4\}, \{\varnothing 2, \varnothing 3\}, \{\varnothing 2, \varnothing 4\}, \{\varnothing 3, \varnothing 4\}, \{\varnothing 1, \varnothing 3, \varnothing 4\}, \{\varnothing 2, \varnothing 3, \varnothing 4\}\}.
$$

Hence $G_m = (G, Cl_{G_m}, \mathcal{F}_{G_m})$ is a $\mathcal{G}_{40}$-space, but it is not a $\mathcal{G}_{10}$-space.

8. Conclusions

The continuity in the $G_m$-cl-approx-spaces is useful, since it connected two different $G_m$-cl-approx-spaces and this helps to make a comparison between the $G_m$-cl-approx-spaces. Also, the separation axioms which introduce in this paper considered as tools for separate the vertices under certain condition.

References


