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Ulam-Hyers stability for fuzzy delay differential equation

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Abstract

In this paper, we aim to study the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of the fuzzy delay differential equation under some suitable conditions by the fixed point technique and successive approximation method. Moreover, we provide two illustrative examples of application of our results.

Keywords: Ulam-Hyers stability, fuzzy differential equations, generalized differentiability, successive approximation method, fixed point theory.
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1. Introduction

The study on fuzzy differential equations (FDEs) has been rapidly advancing in recent years. FDEs have exited considerable interests in both mathematics and engineering areas. A lot of research have been published to consider the qualitative theory of fuzzy differential equations (see papers [13, 14, 12, 15] and therein). Qiu et al.[16, 17] introduced the Lyapunov-like functions stability theory for fuzzy differential equations in the quotient space of fuzzy numbers. Some sufficient criteria for the stability, uniformly stability and exponentially stability of the trivial solution of the fuzzy differential equations were obtained by using the differential inequalities and the comparison principle for Lyapunov-like functions. In [18], authors solved the initial value problem for fuzzy differential equations provided that the involved mappings are continuous, of uniformly bounded variation, and were bounded functions. They established a variety of comparison results for the solutions of fuzzy differential equations in the quotient space of fuzzy numbers.

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In recent years, the Ulam stability problem of many types of differential equations have been studied. Especially, Ulam stability of FDEs. Such as Ren [11] studied the Hyers-Ulam stability of the Hermite fuzzy differential equation associated with the inhomogeneous Hermite fuzzy differential equation under some suitable conditions. The fixed point method has been successfully used to study the Ulam stability of fuzzy differential equations by [5, 4, 3]. In [5, 4], Shen considered the Ulam stability of the first order linear (partial) fuzzy differential equations under generalized differentiability. In [3], authors studied the Ulam stability of fuzzy differential equations by using the fixed point technique. Ulam stability of this problem requires various prerequisites under different types of differentiability.

Motivative by the studies of Allahviranloo et al.[13, 14, 12, 15], Khastan [6], Lupulescu [8], Shen et al. [5, 4, 3],... in the field topic fuzzy differential equations and Ulam stability under uncertain environment. In this paper, we make a connection between the Ulam stability and fuzzy delay differential equation. We aim to study the Ulam-Hyers stability problem of the fuzzy delay differential equations under some suitable conditions, by the fixed point technique and successive approximation method.

We organize the present work as follows. In Section 2, preliminaries and notations are presented. In section 3, we establish Ulam-Hyers and Ulam-Hyers-Rassias stability results for fuzzy delay differential equations via fixed point theory and successive approximation method. Section 4, we provide two illustrative examples.

2. Preliminaries

In this section, we present some basic definitions, theorems and lemmas, which are require throughout this paper.

Definition 2.1. *(see Diaz and Margolis [2])* A function $d : \mathbb{X} \times \mathbb{X} \to [0, +\infty)$ is called a generalized metric on X if and only if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y, z \in \mathbb{X}$;

(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in \mathbb{X}$.

Theorem 2.2. *(see Diaz and Margolis [2])* Let $d : \mathbb{X} \times \mathbb{X} \to [0, +\infty)$ be a generalized metric on $\mathbb{X}$ and $(\mathbb{X}, d)$ is a generalized complete metric space. Assume that $T : \mathbb{X} \to \mathbb{X}$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer n such that $d\big(T^{n+1}x, T^n x\big) < \infty$ for some $x \in \mathbb{X}$, then the followings are true:

(i) the sequece $\{T^n x\}$ converges to a fixed point x^* of T;

(ii) x^* is the unique fixed point of T in

$$\mathbb{X}^* = \big\{y \in \mathbb{X} \,|\, d\big(T^n x, y\big) < \infty\big\};$$

(iii) if $y \in \mathbb{X}^*$, then we have

$$d(y, x^*) \le \frac{1}{1 - L} d\big(Ty, y\big).$$

Denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy sets $u : \mathbb{R} \to [0,1]$ with the following properties: (i) u is normal, i.e., there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$; (ii) u is fuzzy convex, that is, $u(\lambda x + (1-\lambda)x) \geq \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}$, $u \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in [0,1]$; (iii) u is upper semi-continuous; (iv) $cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact, where cl denotes the closure of a set.

Usually, the set $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers and it is easy to see that $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. For $\alpha \in (0,1]$, we denote $[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$ and $[u]^0 = \{x \in \mathbb{R} : u(x) > 0\}$. Then it follows from the conditions (i)-(iv) that the α-level set $[u]^\alpha$ is a non-empty compact interval for all $\alpha \in [0,1]$ and each $u \in \mathbb{R}_{\mathcal{F}}$. For any $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, the addition $u + v$ and scalar multiplication λu can be defined, levelwise, by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[\lambda u] = \lambda[u]^\alpha$ for all $\alpha \in [0,1]$.

The supremum metric between u and v is defined by

$$\begin{aligned} &D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}, \\ &D(u,v) = \sup_{\alpha \in [0,1]} d_H\big([u]^\alpha, [v]^\alpha\big) = \sup_{\alpha \in [0,1]} \max\big\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\big\}. \end{aligned}$$

It is easy to see that $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space. It is well known that the supremum metric has the properties as follows:

(D1) $D(u+w, v+w) = D(u,v)$ for any $u, v, w \in \mathbb{R}_{\mathcal{F}}$;

(D2) $D(\lambda u, \lambda v) = \lambda D(u,v)$ for any $\lambda \in \mathbb{R}_+$, $u, v \in \mathbb{R}_{\mathcal{F}}$;

(D3) $D(u+v, w+e) \leq D(u,w) + D(v,e)$ for any $u, v, w, e \in \mathbb{R}_{\mathcal{F}}$.

Definition 2.3. *(see [7])*Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v + w$, then w is called the H-difference of u and v, and it is denoted by $u \ominus v$.

Throughout this pape, the symbol " $\ominus$ " always stands for the H-difference. In general, $u \ominus v = u + (-1)v$.

Definition 2.4. *(see [7])* Let $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$ and $t_0 \in (a,b)$. We say f is generalized differential at t_0, if there exists an element $D_H^g f(t_0) \in \mathbb{R}_{\mathcal{F}}$, such that

(1) for all $h > 0$ sufficiently small, there exists $f(t_0+h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0-h)$ and then limits (in metric D)

$$\lim_{h\to 0} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h\to 0} \frac{f(t_0) \ominus f(t_0-h)}{h} = D_H^g f(t_0).$$

(2) for all $h > 0$ sufficiently small, there exists $f(t_0) \ominus f(t_0+h)$, $f(t_0-h) \ominus f(t_0)$ and then limits (in metric D)

$$\lim_{h\to 0} \frac{f(t_0) \ominus f(t_0+h)}{-h} = \lim_{h\to 0} \frac{f(t_0-h) \ominus f(t_0)}{-h} = D_H^g f(t_0).$$

Note to that Bede and Gal ([7]) considered four cases in the definition of derivative. In this paper, we consider only the two first cases of Definition 5 in [7]. In the other cases, the derivative reduces to a crisp element i.e. $D_H^g f(t_0) \in \mathbb{R}$.

Theorem 2.5. *(see [7]) Let $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$ and denote $[f(t)]^\alpha = [f_1^\alpha(t), f_2^\alpha(t)]$ for each $\alpha \in [0,1]$, $t \in (a,b)$.*

(i) If f is (1)-differentiable at all $t \in (a,b)$, then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are differentiable functions and we have

$$\big[D_H^g f(t)\big]^\alpha = \big[(f_1^\alpha(t))', (f_2^\alpha(t))'\big]$$

(ii) If f is (2)-differentiable at all $t \in (a,b)$, then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are differentiable functions and we have

$$\big[D_H^g f(t)\big]^\alpha = \big[(f_2^\alpha(t))', (f_1^\alpha(t))'\big]$$

In Theorem 2.5, we see that if f is (1)-differentiable, then it is not (2)-differentiable and vice versa.

Theorem 2.6. *(see [7]) Let $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$ be a differentiable on (a,b) and assume that derivative $D_H^g f$ is integrable over (a,b). For each $t \in (a,b)$, we have*

(i) If f is (1)-differentiable, then

$$f(t) = f(a) + \int_a^t D_H^g f(s)ds;$$

(ii) If f is (2)-differentiable, then

$$f(t) = f(a) \ominus (-1) \int_a^t D_H^g f(s)ds.$$

Lemma 2.7. *Let $\phi : J \to [0,+\infty)$ be a continuous function. We define the set*

$$\mathbb{X} := \big\{x : J \to \mathbb{R}_{\mathcal{F}} \,|\, x \text{ is continuous function on } J\big\}$$

equipped with the metric

$$d(x,y) = \inf\big\{\eta \in [0,+\infty) \cup \{+\infty\} \,|\, D\big(x(t), y(t\big) \le \eta\varphi(t),\ \forall t \in J\big\}.$$

Then, $(\mathbb{X}, d)$ is a complete generalized metric space.

Proof . The proof of this lemma can found in Shen et al. [3]. □

Let $J := [0,T]$ (with $T > 0$) be a compact interval of $\mathbb{R}$. We denote by

$$C(J, \mathbb{R}_{\mathcal{F}}) = \big\{u \,|\, u : J \to \mathbb{R}_{\mathcal{F}} \text{ is a countinuous functions on } J\big\}.$$

On the space $C(J, \mathbb{R}_{\mathcal{F}})$, we consider the supremum metric as follows:

$$D(u,v) = \sup_{t \in J} D[u(t), v(t)].$$

For a non-negative real number σ, we denote by $C_\sigma = C([-\sigma,0], \mathbb{R}_{\mathcal{F}})$ and the following metric (see [8])

$$D_\sigma(u,v) = \sup_{t \in [-\sigma,0]} D[u(t), v(t)].$$

In next section, we consider the fuzzy delay differential equation as follows:

$$\begin{cases} D_H^g u(t) = f(t, u_t), & t \in J, \\ u(t) = \phi(t), & t \in [-\sigma, 0], \end{cases} \tag{2.1}$$

where the symbol D_H^g is generalized Hukuhara derivative, the mapping $f : J \times \mathbb{R}_{\mathcal{F}} \times C_\sigma \to \mathbb{R}_{\mathcal{F}}$ is continuous on J and $\phi : [-\sigma, 0] \to \mathbb{R}$ is a continuous function on $[-\sigma, 0]$.

Definition 2.8. *(see [6, 8])* We say that a mapping $u : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$,is continuous function on $[-\sigma, T]$, is solution to the problem (2.1) if $u(t) = \phi(t)$ for $t \in [-\sigma, 0]$, u is generalized Hukuhara differentiable on $[0, T]$ and $D^g_H u(t) = f(t, u_t)$ for $t \in J$.

Lemma 2.9. *(see [6, 8]) Let $u : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ be a continuous function on $[-\sigma, T]$. Problem (2.1) is equivalent to one of the following fuzzy functional integral equations:*

$$u(t) = \begin{cases} \phi(t), & t \in [-\sigma, 0], \\ \phi(0) + \int_0^t f(s, u_s) ds, & t \in [0, T]. \end{cases} \tag{2.2}$$

if u is (1)-differentiable.

$$u(t) = \begin{cases} \phi(t), & t \in [-\sigma, 0], \\ \phi(0) \ominus (-1) \int_0^t f(s, u_s) ds, & t \in [0, T]. \end{cases} \tag{2.3}$$

if u is (2)-differentiable.

3. Main results

In this section, we establish Ulam-Hyers-Rassias stability results of the problem 2.1 via the fixed point technique and Ulam-Hyers stability results of the problem 2.1 by using successive approximation method.

Definition 3.1. We say that the problem (2.1) is Ulam-Hyers stable if there exists a real number $K_f > 0$ such that for $\epsilon > 0$ and for each $v \in C^1([-\sigma, T], \mathbb{R}_{\mathcal{F}})$ to the problem

$$D\big[D^g_H v(t), f(t, v_t)\big] \le \epsilon$$

there exists a solution to the problem (2.1) with

$$D[v(t), u(t)] \le K_f \epsilon$$

for all $t \in [-\sigma, T]$. We call K_f a Hyers-Ulam stability constant of (2.1).

Remark 3.2. *If $u_0 = v_0$, then we say that the problem (2.1) has the Hypers-Ulam stability with initial condition.*

Definition 3.3. We say that the problem (2.1) is Ulam-Hyers-Rassias stable if there exists a real number $C_f > 0$ such that for $\epsilon > 0$ and for each solution $v \in C^1([-\sigma, T], \mathbb{R}_{\mathcal{F}})$ to the problem

$$D\big[D^g_H v(t), f(t, v_t)\big] \le \varphi(t)$$

there exists a solution to the problem (2.1) with

$$D[v(t), u(t)] \le C_f \varphi(t)$$

for all $t \in [-\sigma, T]$. We call C_f a Hyers-Ulam-Rassias stability constant of (2.1).

Theorem 3.4. *Let $f : J \to \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ be a continuous function on J and the function f satisfies the following conditions: (i) there exists a constant $L > 0$ such that $D\big[f(t,u), f(t,v)\big] \leq LD_\sigma(u,v)$ for each $(t,u),(t,v) \in J \times C_\sigma$; (ii) there exists a constant $C > 0$ such that $0 < LC < 1$. Let $\varphi : J \to (0,+\infty)$ be a continuous function and increasing on J with*

$$\int_0^t \varphi(s)ds \leq C\varphi(t), \quad \textit{for each} \quad t \in J. \tag{3.1}$$

If a continuously (1)-differentiable function $u : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ satisfies the following inequality

$$D\big[D_H^g u(t), f(t,u_t)\big] \leq \varphi(t) \tag{3.2}$$

for any $t \in J$, then there exists a unique (1)-solution $\tilde{u} : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ of (2.1) such that $\tilde{u}(t) = \phi(t)$ for $t \in [-\sigma, 0]$ and

$$\tilde{u}(t) = \phi(0) + \int_0^t f(s, \tilde{u}_s)ds$$

for any $t \in J$ and

$$D\big[u(t), \tilde{u}(t)\big] \leq \frac{C}{1-LC}\varphi(t) \tag{3.3}$$

for any $t \in J$.

Proof . The general structure of this proof is similar the proof of Theorem 3.5. □

Theorem 3.5. *Suppose that f and φ satisfy all the conditions as in Theorem 3.4. Let $u : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ be a continuous and the H-difference $\phi(0) \ominus (-1)\int_0^t f(s,u_s)ds$ exists on J for $\phi \in C([-\sigma,0],\mathbb{R})$. If u is (2)-differentiable and satisfies the inequality (3.2) for any $t \in J$, then there exists a unique (2)-solution $\tilde{u} : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ of (2.1) such that $\tilde{u}(t) = \phi(t)$ for $t \in [-\sigma, 0]$ and*

$$\tilde{u}(t) = \phi(0) \ominus (-1)\int_0^t f(s, \tilde{u}_s)ds \tag{3.4}$$

for any $t \in J$. Moreover, we have

$$D\big[u(t), \tilde{u}(t)\big] \leq \frac{C}{1-LC}\varphi(t) \tag{3.5}$$

for any $t \in J$.

Proof . Let us define

$$\mathbb{X} = \big\{v : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}} \,|\, v(t) = \phi(t) \text{ for } t \in [-\sigma, 0] \text{ and } v \text{ is continuous on } J\big\}$$

equipped with the metric

$$d(v,w) = \inf\big\{C \in [0,+\infty) \cup \{+\infty\} \,|\, D\big[v(t), w(t)\big] \leq C\varphi(t), \ \forall t \in J\big\}.$$

By Lemma 2.7 in [3], it is easy to see that $(\mathbb{X}, d)$ is also a complete generalized metric space.

Let $\mathcal{Q} : \mathbb{X} \to \mathbb{X}$ be defined by

$$\big(\mathcal{Q}v\big)(t) = \begin{cases} \phi(t), & \text{for } t \in [-\sigma, 0], \\ \phi(0) \ominus (-1) \int_0^t f(s, v_s)ds, & \text{for } t \in [0, T]. \end{cases} \tag{3.6}$$

for all $v \in X$. Base on Khastan et.al. by [6], We see that $\mathcal{Q}v$ is (2)-differentiable and so $\mathcal{Q}v \in X$.

Now, we shall prove that the operator $\mathcal{Q}$ is strict contractive on $\mathbb{X}$. For any $v, w \in \mathbb{X}$ and let $C_{v,w} \in [0, +\infty) \cup \{+\infty\}$ be a arbitrary constant with $d(v, w) \leq C_{v,w}$, that is, by the definition of d we have

$$D\big[v(t), w(t)\big] \leq C_{v,w}\varphi(t) \tag{3.7}$$

for any $t \in J$. Since f satisfies a Lipschitz condition and by the inequality (3.7), we obtain

$$\begin{aligned} D\big[\big(\mathcal{Q}v\big)(t), \big(\mathcal{Q}w\big)(t)\big] &= D\bigg[\phi(0) \ominus (-1) \int_0^t f(s, v_s)ds, \phi(0) \ominus (-1) \int_0^t f(s, w_s)ds\bigg] \\ &\leq \int_0^t D\big[f(s, v_s), f(s, w_s)\big]ds \leq L \int_0^t D\big[v_s, w_s\big]ds \\ &\leq L \int_0^t \sup_{\theta\in[-\sigma,0]} D\big[v(\theta), w(\theta)\big]ds + L \int_0^t \sup_{\theta\in[0,s]} D\big[v(\theta), w(\theta)\big]ds \\ &\leq LC_{v,w} \int_0^t \sup_{\theta\in[0,s]} \varphi(\theta)ds \leq CLC_{v,w}\varphi(t) \end{aligned}$$

for any $t \in J$. By the definition of d, we have

$$d\Big(\big(\mathcal{Q}v\big)(t), \big(\mathcal{Q}w\big)(t)\Big) \leq CLC_{v,w}\varphi(t)$$

for any $t \in J$. Hence, we can conclude that

$$d\Big(\big(\mathcal{Q}v\big)(t), \big(\mathcal{Q}w\big)(t)\Big) \leq CLd(v, w)$$

for any $t \in J$. By assumption (ii), we infer that the operator $\mathcal{Q}$ is a strict contractive on $\mathbb{X}$.

For arbitrary $\tilde{w} \in \mathbb{X}$, there exists a constant $0 < C < +\infty$ such that

$$D\big[\big(\mathcal{Q}\tilde{w}\big)(t), \tilde{w}(t)\big] = D\bigg[\phi(0) \ominus (-1) \int_0^t f(s, \tilde{w}_s)ds, \tilde{w}(t)\bigg] \leq C\varphi(t)$$

for any $t \in J$, since $f(t, \tilde{w}_t)$ and $\tilde{w}(t)$ are bounded on J, and $min_{t\in J}\varphi(t) > 0$. Thus, by definition of d, we imply that

$$d\big(\mathcal{Q}\tilde{w}, \tilde{w}\big) \leq C < +\infty.$$

So, by Theorem 2.2, there exists a continuous function $\tilde{u}$ on J such that $J^n\tilde{w} \to \tilde{u}$ as $n \to +\infty$ in the space $(\mathbb{X}, d)$ and $J^n\tilde{u} = \tilde{u}$.

Moreover, we have $\mathbb{X} = \big\{w \in X \,|\, d\big(\tilde{w}, w\big) < +\infty\big\}$.Indeed, for any $w \in \mathbb{X}$, there exists a constant $0 < C < + < \infty$ such that

$$D\big[\tilde{w}(t), w(t)\big] \leq C\varphi(t),$$

since $\tilde{w}$ and w are bounded on J and $\min_{t\in J}\varphi(t) > 0$. It follows form the preceding inequality that

$$d\big(\tilde{w}, w\big) < +\infty$$

for all $w \in \mathbb{X}$, that is, $\mathbb{X} = \big\{w \in X \,|\, d\big(\tilde{w}, w\big) < +\infty\big\}$

By (iii) of Theorem 2.2, we infer that $\tilde{w}$ is a unique fixed point of J in X. It is obvious that $\tilde{w}$ is a unique fuzzy function in $\mathbb{X}$ which satisfies the equality $J\tilde{w} = \tilde{w}$.

On the other hand, it follows from (3.1) and (3.2) that

$$\begin{aligned} D\left[u(t), \phi(0) \ominus (-1)\int_0^t f(s, u_s)ds\right] &= D\left[\phi(0) \ominus u(t), (-1)\int_0^t f(s, u_s)ds\right] \\ &= D\left[\int_0^t D_H^g u(t), (-1)\int_0^t f(s, u_s)ds\right] \\ &\le \int_0^t D\big[D_H^g u(t), f(t, u_t)\big] \le \int_0^t \varphi(s)ds \le C\varphi(t) \end{aligned}$$

for any $t \in J$, which implies that

$$d\big(u, \mathcal{Q}u\big) \le C. \tag{3.8}$$

Finally, Theorem 2.2 together with inequation (3.8) implies that

$$d\big(u, \tilde{u}\big) \le \frac{1}{1 - LC} d\big(u, \mathcal{Q}u\big) \le \frac{C}{1 - LC}\varphi(t)$$

for any $t \in J$, which means that inequality (3.5) holds true for all $t \in J$. $\square$

Corollary 3.6. *Assume that $f : J \to \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ be a continuous function on J satisfies the following conditions: (i) there exists a constant $L > 0$ such that $D\big[f(t, u), f(t, v)\big] \le LD_\sigma(u, v)$ for each $(t, u), (t, v) \in J \times C_\sigma$; (ii) there exists a constant $C > 0$ such that $0 < LT < 1$. If a continuously (1)-differentiable (or (2)-differentiable) function $u : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ satisfies the following inequality*

$$D\big[D_H^g u(t), f(t, u_t)\big] \le \epsilon \tag{3.9}$$

for any $t \in J$, then there exists a unique (1)-solution (or (2)-solution) $\hat{u} : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ of (2.1), where $\hat{u}$ is define as in Theorem 3.5. Moreover, we have

$$D\big[\hat{u}(t), u(t)\big] \le \frac{T}{1 - LT}\epsilon \tag{3.10}$$

for any $t \in J$.

Next, we consider the following inequation

$$D\big[D_H^g v(t), f(t, v_t)\big] \le \epsilon \quad \text{for} \quad v \in \mathbb{R}_{\mathcal{F}}. \tag{3.11}$$

Definition 3.7. We say that

(a) A function $v \in \mathbb{C}^1([-\sigma, T], \mathbb{R}_{\mathcal{F}})$ is a (1)-solution of the problem (3.11) if and only if there exists a function $\delta_1 \in C([-\sigma, T], \mathbb{R}_{\mathcal{F}})$ such that

(i) $D[\delta_1(t), \hat{0}] \leq \epsilon$ for any $t \in J$;

(ii) $D_H^g v(t) = f(t, v_t) + \delta_1(t)$ for any $t \in J$.

(b) A function $v \in \mathbb{C}^1([-\sigma, T], \mathbb{R}_{\mathcal{F}})$ is a (2)-solution of the problem (3.11) if and only if there exists a function $\delta_2 \in C([-\sigma, T], \mathbb{R}_{\mathcal{F}})$ such that

(i) $D[\delta_2(t), \hat{0}] \leq \epsilon$ for any $t \in J$;

(ii) $D_H^g v(t) = f(t, v_t) + \delta_2(t)$ for any $t \in J$.

Theorem 3.8. *Assume that*

(i) the function $f : J \times C_\sigma \to \mathbb{R}_{\mathcal{F}}$ is Lipschitz with respect to the second variable, that is, there exists a non-negative real number L such that

$$D\big[f(t,u), f(t,v)\big] \leq L D_\sigma(u,v)$$

for each $(t,u), (t,v) \in J \times C_\sigma$;

(ii) for each $\epsilon > 0$, if the function $v : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ satisfies

$$D\big[D_H^g v(t), f(t, v_t)\big] \leq \epsilon$$

for $t \in J$, then there exists a unique solution $u : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ of (2.1) with initial condition $u(0) = v(0) = \phi(0)$ and

$$D\big[v(t), u(t)\big] \leq \frac{e^{LT-1}}{L}\epsilon,$$

for $t \in [-\sigma, T]$.

Proof . Without loss of generality, in the proof of this theorem, we only consider that the functions u, v are (2)-differentiable on $[-\sigma, T]$ and assume that the Hukuhara differences exist. The proof of another cases are similar.

By assumption (ii), we have for each $\epsilon > 0$, let $v : [-\sigma, T] \times \mathbb{R}_{\mathcal{F}}$ is a (2)-solution of inequality

$$D\big[D_H^g v(t), f(t, v_t)\big] \leq \epsilon, \quad \text{for all } t \in J. \tag{3.12}$$

Then there exists a function $\delta_2 \in C([-\sigma, T], \mathbb{R}_{\mathcal{F}})$ such that

$$D[\delta_2(t), \hat{0}] \leq \epsilon \quad \text{for all } t \in J. \tag{3.13}$$

and

$$D_H^g v(t) = f(t, v_t) + \delta_2(t), \quad \text{for all } t \in J. \tag{3.14}$$

If $v(t)$ satisfies the equation (3.14) then in view of Lemma 2.9 it satisfies equivalent fuzzy delay integral equation

$$v(t) = \begin{cases} \phi(t), & t \in [-\sigma, 0], \\ \phi(0) \ominus (-1)\int_0^t f(s, v_s)ds \ominus (-1)\int_0^t \delta_2(s)ds, & t \in [0, T]. \end{cases} \tag{3.15}$$

Define

$$u^0(t) = v(t) \quad \text{for } t \in [-\sigma, T]$$

It is easy to see that $u^0 \in C([-\sigma, T], \mathbb{R}_{\mathcal{F}})$ and hence $(t, x_t^0) \in J \times C([-\sigma, 0], \mathbb{R}_{\mathcal{F}})$. Further, we construct a sequence of the continuous functions $u^m(t)$, $m = 1, 2, 3, \ldots$ as follows:

$$u^{m+1}(t) = \begin{cases} v(t), & \text{for } t \in [-\sigma, 0], \\ v(0) \ominus (-1) \int_0^t f(s, u_s^m) ds, & \text{for } t \in [0, T]. \end{cases} \tag{3.16}$$

Hence $(t, u_t^0) \in J \times C([-\sigma, 0], \mathbb{R}_{\mathcal{F}})$, assuming that $(t, u_t^m) \in J \times C([-\sigma, 0], \mathbb{R}_{\mathcal{F}})$ and proceeding recursively, we obtain $(t, u_t^{m+1}) \in J \times C([-\sigma, 0], \mathbb{R}_{\mathcal{F}})$ for any $m = 1, 2, \ldots$. So, this sequence functions is well-defined.

Next, we prove that

$$D\big[u^m(t), u^{m-1}(t)\big] \leq \frac{\epsilon}{L} \frac{(Lt)^m}{m!}, \quad \text{for all } t \in J, m = 1, 2, \ldots. \tag{3.17}$$

Using the inequalities (3.13),(3.14) and for all $t \in J$, we have

$$D\big[u^1(t), u^0(t)\big] \leq \int_0^t D\big[f(s, u_s^0), D_H^{g,2} v(s)\big] ds \leq \int_0^t D[\delta_2(s), \hat{0}] ds \leq t\epsilon$$

for all $t \in J$. Therefore,

$$D\big[u^1(t), u^0(t)\big] \leq t\epsilon$$

which proves the inequality (3.17) for $m = 1$.

Assume that the inequality (3.17) holds for any $m = n$ with $n = 1, 2, \ldots$. We shall prove the inequality (3.17) holds for $m = n + 1$ with $n = 1, 2, \ldots$. For any $t \in J$ and by assumption (i), we obtain

$$\begin{aligned} D\big[u^{n+1}(t), u^n(t)\big] &\leq \int_0^t D\big[f(s, u_s^n), f(s, u_s^{n-1})\big] ds \\ &\leq L \int_0^t D_\sigma\big[u_s^n, u_s^{n-1}\big] ds \\ &\leq L \int_0^t \sup_{r \in [-\sigma, 0]} D\big[u^n(s+r), u^{n-1}(s+r)\big] ds \\ &\leq L \int_0^t \sup_{\theta \in [s-\sigma, s]} D\big[u^n(\theta), u^{n-1}(\theta)\big] ds. \end{aligned} \tag{3.18}$$

Since the inequality (3.17) holds for $m = n$, we have

$$D\big[u^n(t), u^{n-1}(t)\big] \leq \frac{\epsilon}{L} \frac{(Lt)^n}{n!} \tag{3.19}$$

for $t \in J$.

Combining the inequalities (3.18), (3.19), we get

$$D\big[u^{n+1}(t), u^n(t)\big] \leq \frac{\epsilon}{L} \frac{(Lt)^{n+1}}{(n+1)!} \tag{3.20}$$

for any $t \in J$, that is, the inequality (3.17) holds for $m = n+1$. By mathematical induction, the proof of the inequality (3.17) is completed.

Notice that for $t \in J$ and using the inequality (3.17), we obtain

$$\sum_{m=1}^{\infty} D\big[u^m(t), u^n(t)\big] \le \frac{\epsilon}{L} \sum_{m=1}^{\infty} \frac{(Lt)^m}{m!}. \tag{3.21}$$

Hence, the sequence $\{u^m(t)\}$ is uniformly convergent on J. It follows that there exists a continuous function $u : J \to \mathbb{R}_{\mathcal{F}}$ such that

$$\sup_{t\in J} D\big[u^m(t), u(t)\big] \to 0 \tag{3.22}$$

as $m \to \infty$. From assumption (i), we obtain

$$D\big[f(t, u_t^m), f(t, u_t)\big] \le L D_\sigma(u^m, u) \le L \sup_{t\in J} D\big[u^m(t), u(t)\big].$$

Together with (3.22), we deduce that $D\big[f(t, u_t^m), f(t, u_t)\big]$ converges uniformly to 0 as $m \to \infty$ for any $t \in J$.

Notice also that

$$\begin{aligned} D\bigg[\int_0^t f(s, u_s^m)ds, \int_0^t f(s, u_s)ds,\bigg] &\le \int_0^t D\big[f(s, u_s^m), f(s, u_s)\big]ds \\ &\le L \int_0^t \sup_{s\in J} D\big[u^m(s), u(s)\big]ds, \end{aligned}$$

and combine with (3.22), we infer that

$$\int_0^t f(s, u_s^m)ds \to \int_0^t f(s, u_s)ds$$

as $m \to \infty$ for $t \in J$.

For any $t \in J$, letting $m \to \infty$ on both sides of

$$u^{m+1}(t) = v(0) \ominus (-1) \int_0^t f(s, u_s^m)ds$$

we have

$$u(t) = v(0) \ominus (-1) \int_0^t f(s, u_s)ds. \tag{3.23}$$

Therefore, the function u is a (2)-solution of the problem (2.1) on $[-\sigma, T]$. Furthermore, from (3.21) and (3.23), we have

$$D\big[v(t), u(t)\big] \le \frac{e^{LT-1}}{L}\epsilon,$$

for $t \in [-\sigma, T]$. This proves that (2.1) is Ulam-Hyers stable.

Finally, we shall prove the uniqueness of the problem (2.1) on $[-\sigma, T]$. Assume that $\tilde{u} : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ is a another (2)-solution of the problem (2.1) with the initial conditions $v(0)$. For any $t \in J$, we have

$$D\big[u(t), \tilde{u}(t)\big] \leq \int_0^t D\big[f(s, u_s), f(s, \tilde{u}_s)\big]ds \leq L\int_0^t D_\sigma\big(u_s, \tilde{u}_s\big)ds$$
$$\leq L\int_0^t \sup_{\theta\in[s-\sigma,s]} D\big[u(\theta), \tilde{u}(\theta)\big]ds.$$

Put $\xi(s) = \sup_{\theta\in[s-\sigma,s]} D\big[u(\theta), \tilde{u}(\theta)\big]$ for any $s \in [0, t]$, we have

$$\xi(t) \leq L\int_0^t \xi(s)ds.$$

Applying Bellman-Gronwall lemma, we obtain $\xi(t) = 0$ for any $t \in J$. This completes the proof. □

4. Applications

To illustrate the applicability of our main results, we consider the following examples corresponding to the fuzzy version of Malthusian model of population and a tumor growth model.

Now, we shall notice that $u(t)$ is neither the unique nor necessarily the best approximate element without the initial condition.

Example 4.1. Consider the following fuzzy time-delay Malthusian model

$$D_H^g u(t) = u(t-1), \quad t \in [0, 1]. \tag{4.1}$$

Then $v(t) = (-1, 0, 1) \in \mathbb{R}_{\mathcal{F}}$ for $t \in [-1, 1]$ satisfies $D\big[D_H^g v(t), v(t-1)\big] \leq 1$ for $t \in [0, 1]$. Indeed, we have

$$D[D_H^g v(t), v(t-1)] = \sup_{\alpha\in[0,1]} d_H\big([D_H^g v(t)]^\alpha, [v(t-1)]^\alpha\big)$$
$$= \sup_{\alpha\in[0,1]} \max\big\{|\big(\underline{v}^\alpha(t)\big)' - \underline{v}^\alpha(t-1)|; |\big(\overline{v}^\alpha(t)\big)' - \overline{v}^\alpha(t-1)|\big\} \leq 1,$$

where $v(t-1) = (-1, 0, 1) \in \mathbb{R}_{\mathcal{F}}$ for $t \in [0, 1]$.

In the case u is (1)-differentiable. Using the method of steps, we construct a solution

$$[u(t)]^\alpha = \begin{cases} [\alpha-1, 1-\alpha], & \text{for } t \in [-1, 0], \\ [\alpha-1, 1-\alpha](1+t), & \text{for } t \in [0, 1]. \end{cases}$$

By Theorem 3.8, we have

$$D[v(t), u(t)] = \sup_{\alpha\in[0,1]} \max\big\{|\underline{v}^\alpha(t) - \underline{u}^\alpha(t)|; |\overline{v}^\alpha(t) - \overline{u}^\alpha(t)|\big\}$$
$$= \sup_{\alpha\in[0,1]} \max\big\{|(1-\alpha)t|; |(\alpha-1)t|\big\} \leq 1 \text{ for any } t \in [-1, 1].$$

But we can find a function fuzzy $u_1(t)$ defined by

$$[u_1(t)]^\alpha = \begin{cases} \frac{2}{3}[\alpha-1, 1-\alpha], & \text{for } t \in [-1, 0], \\ \frac{2}{3}[\alpha-1, 1-\alpha](1+t), & \text{for } t \in [0, 1] \end{cases}$$

which is also a solution of (4.1) and it is easy to see that

$$D[v(t), u_1(t)] = \sup_{\alpha\in[0,1]} \max\left\{\frac{1}{3}|(1-\alpha)t|; |\frac{1}{3}(\alpha-1)t|\right\} \leq \frac{1}{3}$$

for any $t \in [-1, 1]$. That is, $u_1(t)$ is a better approximation element.

For case u is (2)-differentiable, our calculations are similar as case a above.

Example 4.2. Consider the following fuzzy Ehrlich ascites tumor model

$$D^g_H u(t) = ru(t-1)\big(1 - u(t-1)\big),\ \forall t \in [0, T] := J, \tag{4.2}$$

where $r > 0$ is the net reproduction of the rate tumor.

Let B be a bounded subset of the C_σ. Assume that $D_\sigma(u, \hat{0}) \leq \tilde{K}$ for any $u \in B$ with $\tilde{K} > 0$. We choose the constant $L = r(1 + 2\tilde{K}) > 0$ and $C \geq 1 - e^{-t}$ for all $t \in J$. Let the function $\varphi : J \to (0, +\infty)$ defined by $\varphi(t) = \epsilon e^t$ for all $t \in J$.

Assume that the function $v : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ is a continuous and (2)-differentiable on J and satisfies

$$D\big[D^g_H v(t), f(t, v_t))\big] \leq \epsilon e^t.$$

Then there exists a unique $u : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ is a continuous function such that

$$D\big(v(t), u(t)\big) \leq \frac{C\epsilon e^t}{1 + LC}$$

for all $t \in J$.

Indeed, we set $f(t, u_t) = ru(t-1)\big(1 - u(t-1)\big)$ and by Khastan et.al. [6], we have

$$\begin{aligned} D[f(t, u_s), f(t, v_s)] &= rD\big[u(t-1)\big(1 - u(t-1), v(t-1)\big(1 - v(t-1)\ \big] \\ &\leq r\Big(1 + \sup_{\alpha\in[0,1]} |\big(\underline{v}(t-1)| + |\overline{u}(t-1)|\big)\Big) D_\sigma(u, v) \\ &\leq r(1 + 2\tilde{K}) D_\sigma(u, v) = LD_\sigma(u, v) \end{aligned}$$

for any $t \in [-\sigma, T]$, $u, v \in B$.

Moreover, for $t \in J$ we can infer that

$$\int_0^t \varphi(s)ds = \int_0^t \epsilon e^s ds = \epsilon e^t - \epsilon \leq C\epsilon e^t = C\varphi(t)$$

for any $t \in J$.

We see that the equation (4.2) satisfies all conditions of Theorem 3.8. Therefore, Equations (4.2) has a Ulam-Hyers-Rassias stability in the sense (2)-differentiable so there exists a unique $v : [-\sigma, T] \to \mathbb{R}_{\mathcal{F}}$ such that

$$D\big(v(t), u(t)\big) \leq \frac{C\epsilon e^t}{1 + LC}$$

for all $t \in J$.

In particular, if we choose $\varphi(t) = \epsilon$, then we have

$$D\big(v(t), u(t)\big) \leq \frac{C\epsilon}{1 + LC}$$

for all $t \in J$. That is, the equation (4.2) has a Ulam-Hyers stability in the sense (2)-differentiable.

5. Conlusion

In this paper the Ulam stability and Ulam-Hyers-Rassias stability of the fuzzy functional differential equations via the fixed point technique and successive approximation method are studied. Moreover, we provide two illustrative examples. In future work, we will study Ulam stability problem of fuzzy delay differential equations in the quotient space of fuzzy numbers, introduced by [17].

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