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The Awareness Effect of The Dynamical Behavior of *SIS* Epidemic Model with Crowley-Martin Incidence Rate and Holling Type III Treatment Function

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Abstract

This article deals with the dynamical behaviors for a biological model of epidemic diseases with holling type III treatment function. A Crowley-Martin formula to transmission of disease with coverage media programs effect on the population are introduced and investigated. Through some basic analyses, an explicit formula for the basic reproduction number of the model is calculated, and some results such as the stability analysis and instability of all equilibrium points for the model are established. The local bifurcation occurs near all equilibrium points for the model under some special cases that are studied. The numerical simulations are executed to confirm the theoretical results.

Keywords: Infection Diseases, Treatment Function, Awareness Programs, Local Bifurcation, Crowley-Martin formula.

1. Introduction

Since the early twentieth century, researchers have designed many epidemiological models that play a fundamental role in understanding the spread of infectious diseases and drawing appropriate and emergency planning to control their spread and reduce risks by developing appropriate policies.

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Therefore, mathematical models have become important and essential tools in facing the challenges facing public health [7, 13, 5, 11]. In recent years, there has been an evolution in mathematical models that have become concerned with studying important factors that help reduce the spread of diseases and control them such as (treatment, vaccination, disease transmission mechanism and awareness programs), [6, 18].

The vaccine rate is defined a product that stimulates a person's immune system to produce immunity to a specific disease, protecting the person from that disease and the treatment rate is defined a crucial part to reduce the spread of epidemic diseases. While, The incidence rate or transmission rate of disease is defined as the number of infected individuals per unit time as (hours, days, weeks, month or year). The definition of awareness programs is program designed to increase awareness of a diseases or anything for more inform [7-14]. Moreover, there are more than one formula for both incidence rate and treatment such as Kumar et al. [8], proposed SEIR epidemic model with nonlinear incidence and treatment rates. Kumar and Nilam introduced SIR epidemic model with delay involving Crowley-Martin type incidence rate [9]. Yang and Wei studied and analyzed mathematical model for epidemic disease with Crowley-Martin incidence rate and treatment effects[20]. Adnani et al. studied stability analysis of SIR epidemic model with specific nonlinear incidence rate [1]. Dubey et al. [4], suggested SIR model with nonlinear incidence rate and treatment function.

In this study, we are aiming to analyze an epidemic diseases dynamical model of SIS type with a Crowley-Martin incidence rate and holling type III treatment function. This incidence function will cover a variety of incidence functions presented in all the studies cited previously. Another important feature of our model is the fact that we include also awareness effect by the media coverage to reduce and control of the diseases spread such as [15, 12]. On the other hand, we discuss the changes in dynamic behavior or so-called bifurcation that received great attention due to they have been observed in the incidence of many infectious diseases. Hence, The rest of this work is outlined as follows. In the next section, we will discuss the well posed of the proposeded model by confirming the existence, positivity and boundedness of solutions. In Section 3, we present an analysis of the model such as calculate the basic reproduction number of this model and equilibrium points. In Sections 4,5, we prove the stability analysis of the all equilibrium points. In Section 6, we discuss the local bifurcation accrue near all the equilibrium points. Finally, numerical simulations are given in Section 7, to confirm the analytic results and the comparison with the numerical results.

2. Model formulation and basic properties

2.1. Model formulation

We develop and formulate a biological mathematical model by nonlinear ordinary differential equations in this section. This model describes of an infectious disease of *SIS* type in the population. To understand the dynamical behavior of the proposed model we assume that the population is divided into three compartments are: the all susceptible individuals and unawareness of the disease $S(t)$, the individuals who have awareness of the disease $S_a(t)$, the all infected individuals $I(t)$, the media programs for awareness denoted by $M(t)$. The way disease transmission is direct contact through nonlinear function. Finally, the infected individuals can treat them with a type III function.

All the above hypotheses can be written as below

$$\begin{aligned}
\frac{dS}{dt} &= \psi - \gamma SM - \frac{\beta SI}{(1+\alpha S)(1+\theta I)} + \frac{a_1 I^2}{1+b_1 I^2} - \mu S,\\
\frac{dS_a}{dt} &= \gamma SM - \mu S_a,\\
\frac{dI}{dt} &= \frac{\beta SI}{(1+\alpha S)(1+\theta I)} - \frac{a_1 I^2}{1+b_1 I^2} - \mu I,\\
\frac{dM}{dt} &= \rho S - \sigma M,
\end{aligned} \tag{2.1}$$

with the initial conditions S (0) > 0; S_a (0) > 0; I (0) ≥ 0 ,$M(0) > 0$.

All the parameters in the propose model are positive. The birth rate in the population is ψ, the transmission rate of disease is β with α is a measure of inhibition effect, such as preventive measure taken by susceptible individuals and θ is a measure of inhibition effect such as treatment with respect to infective, the term $\frac{a_1 I^2}{1+b_1 I^2}$ represented to treatment function such that a_1 is treatment rate and b_1 is limitation rate in treatment availability, the death rate is μ, the media rate represented by γ, the implementation rate of media campaigns is denoted by ρ and diminishing rate is represented by θ .

2.2. Boundedness

Theorem 2.1. *The uniformly bounded of the any solutions are discussion in the following.*

Proof . *Let* $(S(t), S_a(t), I(t), M(t))$ *is the solution of the model* (2.1) *with positive initial condition* $(S(0), S_a(0), I(0), M(0))$, *we assume that*

$$N(t) = S(t) + S_a(t) + I(t),$$

by the derivative of $N(t)$ *along the solution of the model* (2.1), *this gives*

$$\begin{aligned}
\frac{dN}{dt} &= \psi - \mu(S + S_a + I),\\
\frac{dN}{dt} &\leq -\mu N.
\end{aligned} \tag{2.2}$$

which implies that

$$\lim_{t\to\infty} supN(t) \leq \frac{\psi}{\mu}. \tag{2.3}$$

while, the last equation of system (2.1) *it follows that*

$$\begin{aligned}
\frac{dM}{dt} &\leq \rho S - \sigma M,\\
M(t) &\leq \frac{\rho}{\sigma}.
\end{aligned} \tag{2.4}$$

By similar way we get:

$$M(t) \leq \frac{\rho\psi}{\sigma\mu},\ as\ t \to \infty. \tag{2.5}$$

We obtain that, the solution of model (2.1) *are confined in the following region*

$$\Omega = \left\{ (S, S_a, I, M)?R_+^4 : N \leq \frac{\psi}{\mu}, 0 \leq M \leq \frac{\rho\psi}{\sigma\mu} \right\}. \tag{2.6}$$

Clearly, we have any solutions of model (2.1) *are uniformly bounded.* □

3. The Number of Equilibria Points

obviously, the aware susceptible S_a is related with variables $S(t)$ and $M(t)$ only. Hence for find values of, $S(t)$ and $M(t)$, the calculate value of S_a can be found simply by solving the model (2.1). In fact, we can determine the value of S_a by the following equation

$$S_a = \frac{\gamma \check{S} M}{1+\mu}. \tag{3.1}$$

Consequently, we can reduced system (2.1) and rewrite it to the following system

$$\begin{aligned}
\frac{dS}{dt} &= \psi - \gamma SM - \frac{\beta SI}{(1+\alpha S)(1+\theta I)} + \frac{a_1 I^2}{1+b_1 I^2} - \mu S, \\
\frac{dI}{dt} &= \frac{\beta SI}{(1+\alpha S)(1+\theta I)} - \frac{a_1 I^2}{1+b_1 I^2} - \mu I, \\
\frac{dM}{dt} &= \rho S - \sigma M.
\end{aligned} \tag{3.2}$$

Now, we can computing the reproduction number for the given system (3.2) and denoted by R_0, such that

$$R_0 = \frac{\beta S_0}{\mu(1+\alpha S_0)}. \tag{3.3}$$

Therefore, system (3.2) has at most two biologically feasible points, namely, $E_i = (S_i, I_i, M_i), i = 0, 1$. The existence conditions for each of these equilibrium points are discussed in following:

- The first equilibrium point is exist when $I = 0$, and called disease free steady state which denoted by $E_0 = (S_0, 0, M_0)$, where

$$M_0 = \frac{\rho}{\sigma} S_0. \tag{3.4}$$

As well as, S_0 is a positive through the following quadratic equation

$$A_1 S_0^2 + A_2 S_0 + A_3 = 0. \tag{3.5}$$

Such that

$$\begin{aligned}
S_0 &= \frac{-(A_2 + \sqrt{A_2^2 - 4A_1 A_3})}{2A_1}, \\
A_1 &= \frac{-\gamma \rho}{\sigma}, \\
A_2 &= -\mu, \\
A_3 &= \psi.
\end{aligned} \tag{3.6}$$

- The endemic steady state, denoted by $E_1 = (S_1, I_1, M_1)$ such that

$$M_1 = \frac{\rho}{\sigma} S_1. \tag{3.7}$$

While (S_1, I_1) represents a positive intersection point of the following two isocline:

$$\begin{aligned}
f(S,I) = {} & r_1 S^3 + r_2 S^3 I^3 + r_3 S^3 I^2 + r_4 S^3 I + r_5 S^2 + r_6 S^2 I^3 + r_7 S^2 I^2 + r_8 S^2 I \\
& + r_9 S + r_{10} S I^3 + r_{11} S I^2 + r_{12} S I + r_{13} I^3 + r_{14} I^2 + r_{15} I + r_{16} = 0
\end{aligned} \tag{3.8}$$

$$g(S,I) = q_1SI^3 + q_2I^3 + q_3SI^2 + q_4I^2 + q_5SI + q_6I + q_7S + q_8 = 0. \tag{3.9}$$

Here,

$$\begin{array}{lll}
r_1 = -\gamma\rho\alpha, & r_2 = -\gamma\rho\alpha\theta b_1, & r_3 = -\gamma\rho\alpha b_1, \\
r_4 = -\gamma\rho\alpha\theta, & r_5 = -(\gamma\rho + \mu\sigma\alpha), & r_6 = -(\gamma\rho\theta b_1 + \mu\sigma\alpha\theta b_1), \\
r_7 = -(\gamma\rho b_1 + \mu\sigma\alpha b_1), & r_8 = -(\gamma\rho\theta + \mu\sigma\alpha\theta), & r_9 = (\psi\sigma\alpha - \mu\sigma), \\
r_{10} = (\psi\sigma\alpha\theta b_1 - \beta\sigma b_1 + a_1\sigma\alpha\theta - \mu\sigma\theta b_1) & r_{11} = (\psi\sigma\theta b_1 + a_1\sigma\alpha - \mu\sigma b_1), & r_{12} = (\psi\sigma\alpha\theta - \beta\sigma - \mu\sigma\theta), \\
r_{13} = (\psi\sigma\theta b_1 + a_1\sigma\theta), & r_{14} = (\psi\sigma b_1 + a_1\sigma), & r_{15} = \psi\sigma\theta, \\
r_{16} = \psi\sigma, & q_1 = -\mu\alpha\theta b_1, & q_2 = -\mu b_2, \\
q_3 = (\beta b_1 - a_1\alpha\theta - \mu\alpha b_1), & q_4 = -(a_1\theta + \mu b_1), & q_5 = -(a_1\alpha + \mu\alpha\theta), \\
q_6 = -(a_1 - \mu\theta), & q_7 = (\beta - \mu\alpha), & q_8 = -\mu
\end{array}$$

It easy, when $I \to 0$, we get the equations (3.8) and (3.9) becomes as follows

$$f(S) = r_1S^3 + r_5S^2 + r_9S + r_{16} = 0, \tag{3.10}$$
$$g(S) = q_7S + q_8 = 0. \tag{3.11}$$

Obviously, by using Descartes rule equation (3.10) has a unique positive root $\hat{S}$. However, equation (3.11), has a positive root $\tilde{S} = \frac{-q_8}{q_7}$. Then, the equations (3.8) and (3.9) have a unique positive root and the endemic equilibrium point E_1 exists if the following conditions are satisfy

$$\begin{aligned}
&\alpha < min\left\{\frac{\mu}{\psi}, \frac{\beta}{\mu}\right\}, \\
&\hat{S} < \tilde{S} \\
&\frac{\partial I}{\partial S} = -\frac{\frac{\partial f}{\partial S}}{\frac{\partial f}{\partial I}} > 0, \\
&\frac{\partial I}{\partial S} = -\frac{\frac{\partial g}{\partial S}}{\frac{\partial g}{\partial I}} < 0.
\end{aligned} \tag{3.12}$$

4. Local dynamical behavior

In this section, the stability analysis investigation of model (3.2) about $E_i, i = 0,1$ are studied in the following theorems.

Theorem 4.1. *The disease free equilibrium point E_0 of the system* (3.2) *is locally stable under $R_0 < 1$.*

Proof . *From the linearization method of system* (3.2) *about E_0 we have*

$$J(E_0) = \begin{bmatrix} -\gamma M_0 - \mu & \frac{-\beta S_0}{(1+\alpha S_0)} & -\gamma S_0 \\ 0 & \frac{\beta S_0}{(1+\alpha S_0)} - \mu & 0 \\ \rho & 0 & -\sigma \end{bmatrix} \tag{4.1}$$

Therefore the characteristic equation is

$$\left[\left(\frac{\beta S_0}{(1+\alpha S_0)} - \mu\right) - \lambda\right]\left[\lambda^2 + A_1\lambda + A_2\right] = 0. \tag{4.2}$$

Here,

$$A_1 = \gamma M_0 + \mu + \sigma, \qquad A_2 = (\gamma M_0 + \mu)\sigma + \gamma\rho S_0. \tag{4.3}$$

Consequently the eigenvalues of equation (4.2) *can be written in below*

$$\left.\begin{aligned} \lambda_I &= \frac{\beta S_0}{(1+\alpha S_0)} - \mu\ , \\ \lambda_s &= -\frac{A_1}{2} + \frac{1}{2}\sqrt{A_1^2 - 4A_2}\ , \\ \lambda_\mu &= -\frac{A_1}{2} - \frac{1}{2}\sqrt{A_1^2 - 4A_2}\ , \end{aligned}\right\} \tag{4.4}$$

Therefore, all the eigenvalues have negative real part and hence the disease-free equilibrium point is locally stable in case $R_0 < 1$. □

Theorem 4.2. *The local stability about* E_1 *of system* (3.2) *is guarantee when* $R_0 > 1$ *and the following sufficient conditions*

$$\frac{\beta s_1}{(1+\alpha s_1)(1+\theta I_1)^2} < \frac{\{2a_1 I_1\}}{(1+b_1 I_1^2)^2}, \tag{4.5}$$

$$\left[\frac{-\beta s_1}{(1+\alpha s_1)(1+\theta I_1)^2} + \frac{\{2a_1 I_1\}}{(1+b_1 I_1^2)^2}\right] \left[\frac{\beta I_1}{(1+\theta I_1)(1+\alpha s_1)^2}\right] < \rho\gamma s_1. \tag{4.6}$$

Proof . *From the linearization method of system* (3.2) *about* E_1 *we have*

$$J(E_1) = \begin{bmatrix} -\gamma M_1 - \frac{\beta I_1}{(1+\theta I_1)(1+\alpha s_1)^2} - \mu & \frac{-\beta s_1}{(1+\alpha s_1)(1+\theta I_1)^2} + \frac{\{2a_1 I_1\}}{(1+b_1 I_1^2)^2} & -\gamma s_1 \\ \frac{\beta I_1}{(1+\theta I_1)(1+\alpha s_1)^2} & \frac{\beta s_1}{(1+\alpha s_1)(1+\theta I_1)^2} - \frac{\{2a_1 I_1\}}{(1+b_1 I_1^2)^2} - \mu & 0 \\ \rho & 0 & -\sigma \end{bmatrix} \tag{4.7}$$

The characteristic equation is given by;

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0. \tag{4.8}$$

Here

$$\begin{aligned} B_1 &= -(b_{11} + b_{22} + b_{33}), \\ B_2 &= b_{11}b_{22} - b_{12}b_{21} + b_{11}b_{33} - b_{13}b_{31} + b_{22}b_{33}, \\ B_3 &= -b_{11}b_{22}b_{33} + b_{13}b_{22}b_{31} + b_{12}b_{21}b_{33}, \\ \Delta &= -b_{11}b_{22}\left[b_{11} + b_{22}\right] - b_{11}b_{33}\left[b_{11} + b_{33}\right] - b_{22}b_{33}\left[b_{22} + b_{33}\right] \\ &\quad - 3b_{11}b_{22}b_{33} + \left[b_{11} + b_{22} + b_{33}\right]\left[b_{12}b_{21} + b_{13}b_{31}\right]. \end{aligned}$$

Now, if the conditions of Routh-Hurwitz method ($B_i(i = 1, 3) > 0$ *and* $\Delta = B_1B_2 - B_3 > 0$*)are satisfied. Then, we get the all eigenvalues of* $J(E_1)$ *have negative real part roots. Consequently, the it is locally stable under the conditions* (4.5) *and* (4.6). □

5. Global dynamical behavior

In this section, the region of global dynamical behavoir of all equilibria points of system (3.2) is studied in below theorems.

Theorem 5.1. *The E_0 is a globally asymptotically stable when $R_0 < 1$, and provided that the conditions in below*

$$\left(\frac{\rho}{M} - \frac{\gamma S_0}{S}\right)^2 < 4\left(\frac{\gamma M + \mu}{S}\right)\left(\frac{\sigma}{M}\right), \tag{5.1a}$$

$$\frac{\beta}{(1+\alpha S)(1+\theta I)} < \frac{a_1 I}{S(1+b_1 I^2)}. \tag{5.1b}$$

Proof . *We definite the positive function*
$W_0(S,I,M) = (S - S_0 - S_0 \ln \frac{S}{S_0}) + I + (M - M_0 - M_0 \ln \frac{M}{M_0})$.

Clearly, $W_0 : R_+^3 \to R$ is a continuously differentiable function such that $W_0(S_0, 0, M_0) = 0$ and $W_0(S,I,M) > 0, \forall (S,I,M) \neq (S_0, 0, M_0)$. Further,

$$\frac{dw_0}{dt} = (\frac{S-S_0}{S})\left[\psi - \gamma SM - \frac{\beta SI}{(1+\alpha S)(1+\theta I)} + h(I) - \mu S\right] + \left[\frac{\beta SI}{(1+\alpha S)(1+\theta I)} - \frac{a_1 I^2}{1+b_1 I^2} - \mu I\right] + (\frac{M-M_0}{M})\left[\rho S - \sigma M\right]$$

Now, by doing some algebraic manipulation and using the conditions (5.1a) - (5.1b)*, we get*

$$\frac{dW_0}{dt} \leq -(\frac{\gamma M + \mu}{S})(S-S_0)^2 + (\frac{\rho}{M} - \frac{\gamma S_0}{S})(S-S_0)(M-M_0) - \frac{\sigma}{M}(M-M_0)^2 + \frac{\beta S_0 I}{(1+\alpha S)(1+\theta I)} - \frac{a_{1S_0} I^2}{S(1+b_1 I^2)} - \mu I.$$

Consequently, due to condition above $\frac{dW_0}{dt} < 0$. Thus E_0 is a globally asymptotically stable under the conditions (5.1a) *and* (5.1b). □

Theorem 5.2. *The E_1 is a globally asymptotically stable under $R_0 > 1$, and provided that below conditions*

$$\frac{\beta(1+\alpha S_1)S}{(1+\alpha S)(1+\theta I)(1+\alpha S_1)(1+\theta I_1)} < \mu + \frac{a_1(I+I_1)}{(1+b_1 I^2)(1+b_1 I_1^2)}, \tag{5.2}$$

$$q_{12}^2 < 2q_{11}q_{22}, \tag{5.3}$$

$$q_{13}^2 < 2q_{11}q_{33}. \tag{5.4}$$

Proof . *Consider the following positive definite function*

$$W_1(S,I,M) = \frac{(S-S_1)^2}{2} + \frac{(I-I_1)^2}{2} + \frac{(M-M_1)^2}{2}.$$

Clearly, $W_1 : R_+^3 \to R$ is a continuously differentiable function such that $W_1(S_1, I_1, M_1) = 0$ and $W_1(S,I,M) > 0, \forall (S,I,M)?R_+^3$ and $(S,I,M) \neq (S_1,, I_1, M_1)$.

Now, by the derivative with respect to system (3.2) *we get the resulting*

$$\frac{dW_1}{dt} = (S - S_1)\left[\psi - \gamma SM - \frac{\beta SI}{(1+\alpha S)(1+\theta I)} + \frac{a_1 I^2}{1+b_1 I^2} - \mu S\right] + (I - I_1).$$
$$\left[\frac{\beta SI}{(1+\alpha S)(1+\theta I)} - \frac{a_1 I^2}{1+b_1 I^2} - \mu I\right] + (M - M_1)\left[\rho S - \sigma M\right].$$

$$\begin{aligned}\frac{dW_1}{dt} &= -\left[\frac{q_{11}}{2}(S-S_1)^2 + q_{12}(S-S_1)(I-I_1) + q_{22}(I-I_1)^2\right] \\ &\quad - \left[\frac{q_{11}}{2}(S-S_1)^2 + q_{13}(S-S_1)(M-M_1) + q_{33}(M-M_1)^2\right]\end{aligned}$$

Clearly, by the conditions (5.2)- (5.4) *we get that*

$$\frac{dW_1}{dt} \leq -\left[\sqrt{\frac{q_{11}}{2}}(S-S_1) + \sqrt{q_{22}}(I-I_1)\right]^2 - \left[\sqrt{\frac{q_{11}}{2}}(S-S_1) + \sqrt{q_{33}}(M-M_1)\right]^2.$$

Such that

$$\begin{aligned}q_{11} &= \gamma M + \frac{\beta(1+\theta I)I_1}{(1+\alpha S)(1+\theta I)(1+\alpha S_1)(1+\theta I_1)} + \mu, \\ q_{12} &= \frac{\beta[(1+\alpha S_1)S - (1+\theta I)I_1]}{(1+\alpha S)(1+\theta I)(1+\alpha S_1)(1+\theta I_1)} - \frac{a_1(I+I_1)}{(1+b_1 I^2)(1+b_1 I_1^2)}, \\ q_{22} &= \frac{a_1(I+I_1)}{(1+b_1 I^2)(1+b_1 I_1^2)} + \mu - \frac{\beta(1+\alpha S_1)S}{(1+\alpha S)(1+\theta I)(1+\alpha S_1)(1+\theta I_1)}, \\ q_{13} &= \gamma S_1 - \rho \\ , q_{33} &= \sigma.\end{aligned}$$

It is easy see that, $\frac{dw_1}{dt} < 0$*. So,* E_1 *is globally asymptotically stable when the given conditions are satisfied.* $\square$

6. The Bifurcation Analysis

In the next theorem the conditions of bifurcation occur of system (2) is established. Mathematically bifurcation of the system means that if the change a parameter value in special cases then the solution of the system will change according to this parameter for all the time. Now, we can write system (3.2) in the formula: $\frac{dX}{dt} = F(X)$, here $X = (S, I, M)^T$ such that $F = (f_1, f_2, f_3)^T$ while $f_i; i = 1, 2, 3$ represent to the system (3.2). So according to the jacobian matrix of system (3.2), it is easy to verify that for any vector $V = (v_1, v_2, v_3)^T$, we have that second directional derivative

$$D^2F(S,C,M)(V,V) = \begin{bmatrix} 2\left\{\frac{\alpha\beta I v_1^2}{(1+\theta I)(1+\alpha S)^3} - \frac{\beta v_1 v_2}{(1+\theta I)^2(1+\alpha S)^2} - \gamma v_1 v_3 + \left[\frac{\theta\beta S}{(1+\alpha S)(1+\theta I)^3} + \frac{a_1 - 3a_1 b_1 I^2}{(1+b_1 I^2)^3}\right] v_2^2\right\} \\ 2\left\{\frac{-\alpha\beta I v_1^2}{(1+\theta I)(1+\alpha S)^3} + \frac{\beta v_1 v_2}{(1+\theta I)^2(1+\alpha S)^2} + \left[\frac{-\theta\beta S}{(1+\alpha S)(1+\theta I)^3} + \frac{3a_1 b_1 I^2 - a_1}{(1+b_1 I^2)^3}\right] v_2^2\right\} \\ 0 \end{bmatrix}. \tag{6.1}$$

6.1. The Bifurcation condition about E_0

Theorem 6.1. *If $R_0 = 1$, the transcritical bifurcation can occur at E_0 of system (3.2).*

Proof . *The $R_0 = 1$ when the parameter $\mu \equiv \mu^*, then Eq.(4.1) of system (3.2) at E_0$, has zero eigenvalue ($\lambda_0 = 0$).*

$$\mu = \mu^* = \frac{\beta S_0}{(1 + \alpha S_0)} \equiv R_0 = 1. \tag{6.2}$$

So, Eq.(4.1) of system (3.2) can be represent by

$$J_0 = J_0(\mu) = \begin{bmatrix} -\gamma M_0 - \mu & -\mu & -\gamma S_0 \\ 0 & 0 & 0 \\ \rho & 0 & -\sigma \end{bmatrix}.$$

Clearly, let $V^{[0]} = (v_1^{[0]}, v_2^{[0]}, v_3^{[0]})^T representstheeigenvectorsof$ J_0 *of* $\lambda_0 = 0$.
So

$$(J_0 - \lambda_0)V^{[0]} = 0, \quad \text{get that } V^{[0]} = (m_1 v_3^{[0]}, m_2 v_3^{[0]}, v_3^{[0]})^T,$$

suuch that, $m_1 = \frac{\sigma}{\rho}$, $m_2 = -\frac{(1+\alpha S_0)}{\beta S_0}\left[\gamma S_0 + \frac{\sigma(\gamma M_0+\mu)}{\rho}\right]$, *with* $v_3^{[0]} \neq 0$.

As well as, let $L^{[0]} = \left[l_1^{[0]}, l_2^{[0]}, l_3^{[0]}\right]^T$ *represents the eigenvectors of* $\lambda_0 = 0$ *of* J_0^T.

Obviously from

$$(J_0^T - \lambda_0)L^{[0]} = 0$$

. *We obtain* $L^{[0]} = \left[0, l_2^{[0]}, 0\right]^T$ *; such that* $l_2^{[0]} \neq 0$.

Now, consider

$$\frac{\partial F}{\partial \mu} = F_\mu(X, \beta) = [-S, -I, 0]^T$$

Thus, $F_\mu(E_0, \mu) = [-S_0, 0, 0]^T$ *which gives* $\left[L^{[0]}\right]^T F_\mu(E_0, \mu) = 0$.
So, by help of the Sotomyor's theorem bifurcation theory, the saddle nod bifurcation is not occur near E_0 of system (3.2). Furthermore, because we have

$$DF_\mu(X, \mu) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can see that

$$DF_\mu(E_0, \mu)V^{[0]} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sigma}{\rho} v_3^{[0]} \\ -\frac{(1+\alpha S_0)}{\beta S_0}\left[\gamma S_0 + \frac{\sigma(\gamma M_0+\mu)}{\rho}\right] v_3^{[0]} \\ 0 \end{bmatrix} \neq 0.$$

Moreover, by substituting E_0, μ and $V^{[0]}$ in (31) we get:

$$D^2F(E_0, \mu)(V^{[0]}, V^{[0]}) =$$
$$\begin{bmatrix} 2\left\{\left[\frac{\alpha\beta I m_1^2}{(1+\theta I)(1+\alpha S)^3} - \frac{\beta m_1 m_2}{(1+\theta I)^2(1+\alpha S)} - \gamma m_1\right](v_3^{[0]})^2 + \left[\frac{\theta\beta S}{(1+\alpha S)(1+\theta I)^3} + \frac{a_1 - 3a_1 b_1 I^2}{(1+b_1 I^2)^3}\right] m_2^2 (v_3^{[0]})^3\right\} \\ 2\left\{\left[\frac{\beta m_1 m_2}{(1+\theta I)^2(1+\alpha S)^2} - \frac{\alpha\beta I m_1^2}{(1+\theta I)(1+\alpha S)^3}\right] + \left[\frac{3a_1 b_1 I^2 - a_1}{(1+b_1 I^2)^3} - \frac{\theta\beta S}{(1+\alpha S)(1+\theta I)^3}\right] m_2^2\}(v_3^{[0]})^2\right\} \\ 0 \end{bmatrix}$$

Hence, it is obtaining

$$\left[L^{[0]}\right]^{T}\left[D^{2}F(E_{0},\mu)(V^{[0]},V^{[0]})\right]\neq 0.$$

Now, according to bifurcation theorem by Sotomayor's, system (3.2) *at* E_0 *with* $\mu \equiv \mu^*$ *provided that* $R_0 = 1$. $\square$

6.2. The Bifurcation condition about E_1

Theorem 6.2. *System* (3.2) *at* E_1 *has a saddle node bifurcation with the parameter value* $\gamma \equiv \tilde{\gamma}$ *such that*

$$\tilde{\gamma}=\frac{\sigma\left[b_{11}b_{22}-b_{12}b_{21}\right]}{\rho S_{1}b_{22}}. \tag{6.3}$$

Where

$$b_{11}=-\gamma M_{1}-\frac{\beta I_{1}}{(1+\theta I_{1})(1+\alpha s_{1})^{2}}-\mu; \qquad b_{12}=\frac{-\beta s_{1}}{(1+\alpha s_{1})(1+\theta I_{1})^{2}}+\frac{\{2a_{1}I_{1}\}}{(1+b_{1}I_{1}^{2})^{2}}$$
$$b_{21}=\frac{\beta I_{1}}{(1+\theta I_{1})(1+\alpha s_{1})^{2}}; \qquad b_{22}=\frac{-\beta s_{1}}{(1+\alpha s_{1})(1+\theta I_{1})^{2}}-\frac{\{2a_{1}I_{1}\}}{(1+b_{1}I_{1}^{2})^{2}}-\mu$$

Proof . *From Eq.* (4.6), *see that system* (3.2) *at* E_1 *has zero eigenvalue,* $\tilde{\lambda}_1 = 0$, *when* $\gamma \equiv \tilde{\gamma}$, *it is clearly that* $\tilde{\gamma} > 0$, *hence we can rewrite in below*

$$\tilde{J}_{1}=J_{1}(\tilde{\gamma})=\begin{bmatrix}-\tilde{\gamma}M_{1}-\frac{\beta I_{1}}{(1+\theta I_{1})(1+\alpha s_{1})^{2}}-\mu & \frac{-\beta s_{1}}{(1+\alpha s_{1})(1+\theta I_{1})^{2}}+\frac{\{2a_{1}I_{1}\}}{(1+b_{1}I_{1}^{2})^{2}} & -\tilde{\gamma}s_{1}\\ \frac{\beta I_{1}}{(1+\theta I_{1})(1+\alpha s_{1})^{2}} & \frac{-\beta s_{1}}{(1+\alpha s_{1})(1+\theta I_{1})^{2}}-\frac{\{2a_{1}I_{1}\}}{(1+b_{1}I_{1}^{2})^{2}}-\mu & 0\\ \rho & 0 & -\sigma\end{bmatrix} \tag{6.4}$$

Clearly, let $V^{[1]} = (v_1^{[1]}, v_2^{[1]}, v_3^{[1]})^T$ *represents the eigenvectors of*

$$\tilde{\lambda}_1 = 0$$

.

Thus $(\tilde{J}_1 - \tilde{\lambda}_1)V^{[1]} = 0$, *which gives:* $V^{[1]} = (v_1^{[1]}, r_1 v_1^{[1]}, r_2 v_1^{[1]})^T$, *where* $r_1 = \frac{-\beta I_1 \Omega_1 \Omega_3^2}{-\Omega_2(\beta S_1 \Omega_3^2 + 2a_1 I_1 \Omega_1 \Omega_2 + \mu \Omega_1^2 \Omega_2 \Omega_3^2)}$, $r_2 = \frac{\rho}{\sigma}$, $v_1^{[1]} \neq 0$.

Here

$$\Omega_1 = (1+\theta I_1),$$
$$\Omega_2 = (1+\alpha S_1),$$
$$\Omega_3 = (1+b_1 I_1^2).$$

Let $\Psi^{[1]} = \left[\psi_1^{[1]}, \psi_2^{[1]}, \psi_3^{[1]}\right]^T$ *represents the eigenvectors associated with* $\tilde{\lambda}_1 = 0$ *of* $\tilde{J}_1^T$. *So fromula* $(\tilde{J}_1^T - \tilde{\lambda}_1)\Psi^{[1]} = 0$. *We get*

$$\Psi^{[1]}=\left[\psi_{1}^{[1]},D_{1}\psi_{1}^{[1]},D_{2}\psi_{1}^{[1]}\right]^{T}$$

. *Such that* $\mathcal{D}_1 = \frac{\beta S_1 \Omega_3^2 - 2a_1 I_1 \Omega_1^2 \Omega_2}{-(\beta S_1 \Omega_3^2 + 2a_1 I_1 \Omega_1^2 \Omega_2 + \mu \Omega_1^2 \Omega_2 \Omega_3^2)}$, $\mathcal{D}_2 = -\frac{\gamma s_1}{\sigma}$, $\psi_1^{[1]}$ *is any nonzero real number.*

Now, consider

$$\frac{\partial F}{\partial\gamma}=F_{\gamma}(X,\gamma)=\left[-MS,0,0\right]^{T}.$$

So, $F_\gamma(E_1,\gamma) = [-M_1 S_1, 0, 0]^T$ *and then* $\left[\Psi^{[1]}\right]^T F_\gamma(E_1,\gamma) \neq 0$.
However, the system (3.2) *has no transcritical bifurcation. But the* $1st$ *condition of the bifurcation theorem about saddle-node type is satisfied. Now, since*

$$DF_\gamma(X,\gamma) = \begin{bmatrix} -M & 0 & -S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Where $DF_\gamma(X,\gamma)$ *represent the derivative of* $F_\gamma(X,\gamma)$ *with* $X = [S, I, M]^T$. *Moreover, it is clearly*

$$DF_\gamma(E_1,\tilde{\gamma})V^{[1]} = \begin{bmatrix} -M & 0 & -S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^{[1]} = \begin{bmatrix} -(M+r_2S)v_1^{[1]} \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\Psi^{[1]}\right]^T \left[DF_\gamma(E_1,\tilde{\gamma})V^{[1]}\right] = \left[\psi_1^{[1]}, D_1\psi_1^{[1]}, D_2\psi_1^{[1]}\right] \begin{bmatrix} -(M+r_2S)v_1^{[1]} \\ 0 \\ 0 \end{bmatrix} = -(M+r_2S)\psi_1^{[1]}v_1^{[1]} \neq 0.$$

Moreover, by substituting E_1, γ *and* $V^{[1]}$ *in* (6.1) *we get:*

$$D^2F(E_1,\gamma)(V^{[1]},V^{[1]}) = \begin{bmatrix} 2(v_1^{[1]})^2\{\frac{\alpha\beta I}{(1+\theta I)(1+\alpha S)^3} - \frac{\beta r_1}{(1+\theta I)^2(1+\alpha S)^2} - \gamma r_2 + \left[\frac{\theta\beta S}{(1+\alpha S)(1+\theta I)^3} + \frac{a_1-3a_1b_1I^2}{(1+b_1I^2)^3}\right] r_1^2\} \\ 2(v_1^{[1]})^2\{\frac{-\alpha\beta I}{(1+\theta I)(1+\alpha S)^3} + \frac{\beta r_1}{(1+\theta I)^2(1+\alpha S)^2} + \left[\frac{-\theta\beta S}{(1+\alpha S)(1+\theta I)^3} + \frac{3a_1b_1I^2-a_1}{(1+b_1I^2)^3}\right] r_1^2\} \\ 0 \end{bmatrix}.$$

Hence, it is obtain $\left[\Psi^{[1]}\right]^T \left[D^2F(E_1,\gamma)(V^{[1]},V^{[1]})\right] \neq 0$.
Hence, the system (3.2)*] has a saddle-node bifurcation at* E_1 *with parameter* γ. □

7. The numerical illustration of system (2.1)

The global dynamical behavior of system (2.1) will studied numerically for different sets of parameters and different sets of initial points in this section. The goal of such part are know the role of change the parameters values and confirm the analytical results. It is see that that, for the following biologically feasible set of parameters values:

$$\begin{aligned} &\psi = 50, \quad \gamma = 0.01, \quad \beta = 0.25, \quad \alpha = 0.05, \quad \theta = 0.1, \\ &a_1 = 0.3, \quad b_1 = 0.05, \quad \mu = 0.4, \quad \rho = 0.4, \quad \sigma = 0.3 \end{aligned} \tag{7.1}$$

The trajectory of system (2.1) convergent to E_1 see Fig.(1).

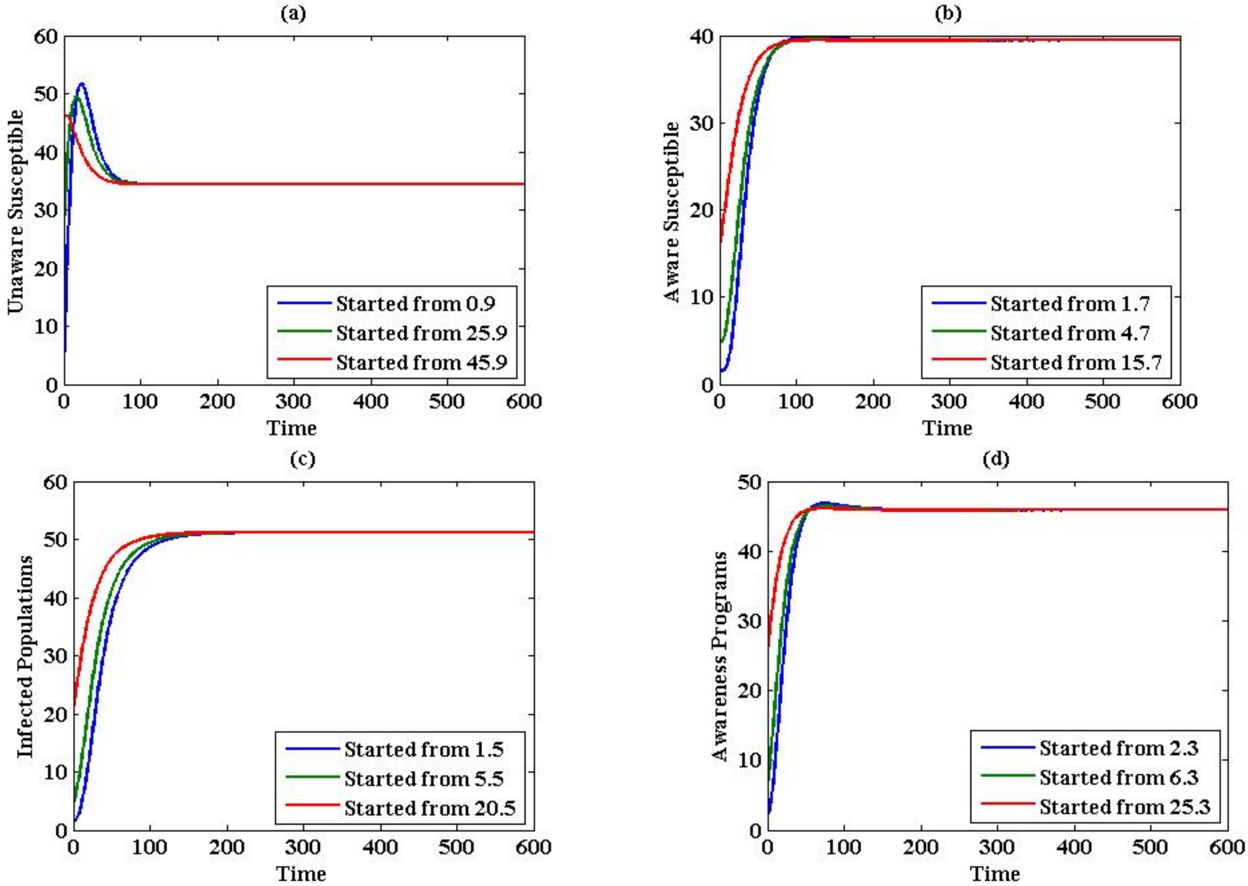
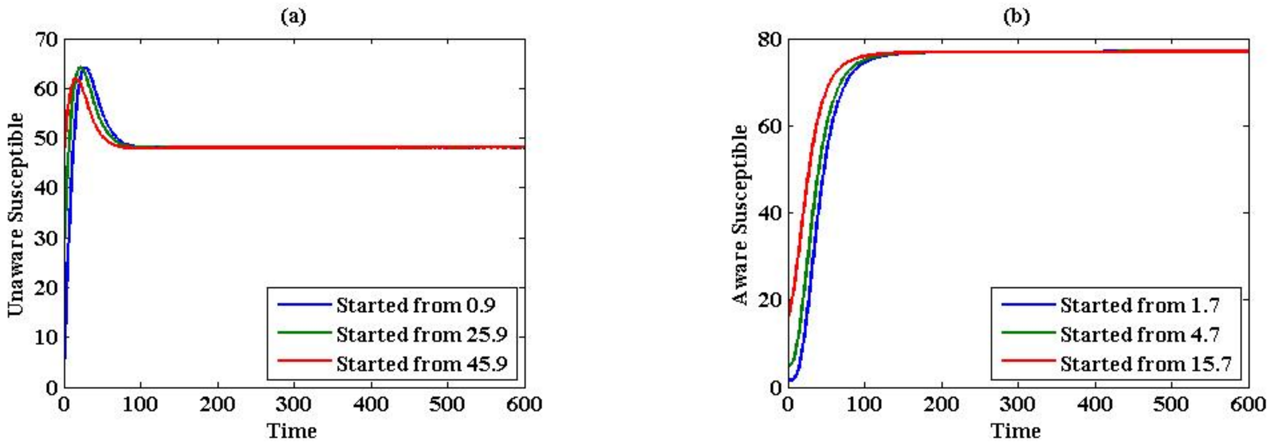


Figure 1: Time attractor of globally asymptotically stable to E_1 of system (2.1) and $R_0 = 4.61 > 1$.

However, for the data by equation (7.1) with $\beta = 0.025$ the trajectory of system (2.1) convergent to E_0 see Fig. (2).



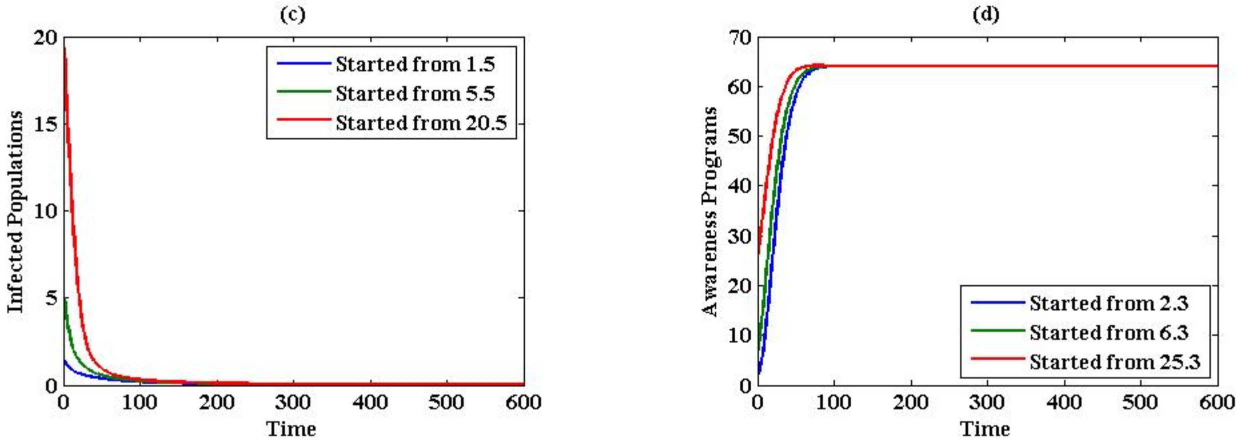


Figure 2: Time attractor of globally asymptotically stable to E_0 of system (2.1) and $R_0 = 0.46 < 1$.

Clearly, in order to discuss the impact of varying some parameter values on the dynamical behavior of system (2.1), the following results are observed. According to the Fig.3, it is clear that the trajectory of system (2.1) convergent to E_0 by applied the parameters values given in Eq. (7.1) with varying$\alpha > 0.5$ with $R_0 = 0.79 < 1$.

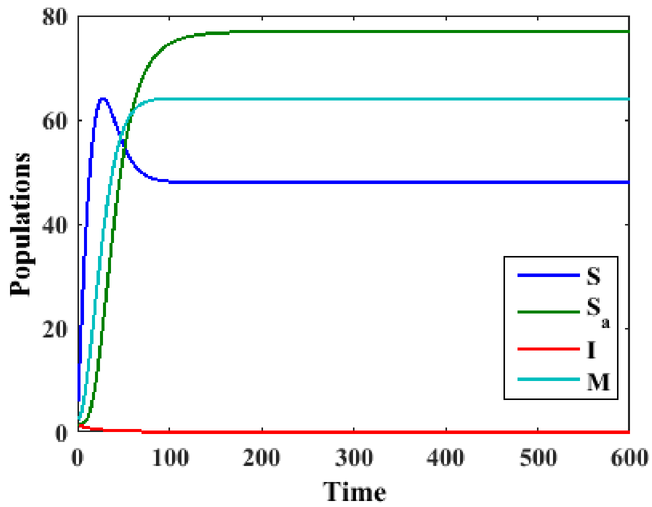


Figure 3: The measure of inhibition effect on the dynamical behavior of system (2.1).

Also we applied the parameters values given in Eq. (7.1) but putting $\mu \geq 2.6$ the trajctory of the system (2.1) convergent to $E_0 = (17.635, 1.595, 0, 23.514)$ and $R_0 = 0.65 < 1$, see Fig.(4) below.

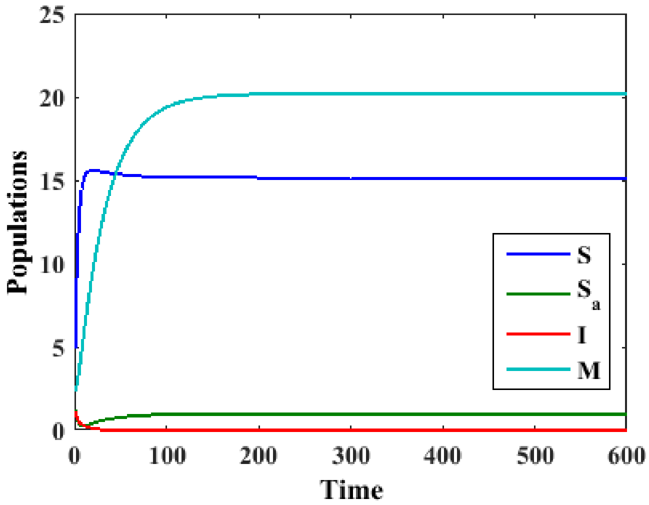


Figure 4: The death rate effect on the dynamical behavior of system (2.1).

Now, the effect of media coverage is discussion by Fig. 5, it easy see that when the level of awareness increasing due to the media coverage (say γ), we obtain the dynamical behavior of system (2.1) convergent to E_0 and that is mean the endemic equilibrium point becomes unstable when $\gamma \geq 0.6$. In addition we get similar result if the media rate increasing through $\rho \geq 20$.

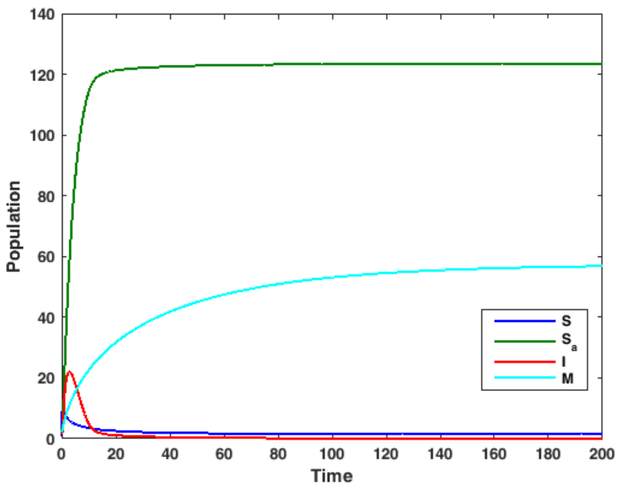


Figure 5: The response effect to media programs on the dynamical behavior of system (2.1).

8. Conclusion and Results

In this study, we have examined the epidemic model with SIS type under the effect of media programs, the Crowley-Martin formula to transmission of disease and holling type III treatment function. We have also provided the existence of all equilibrium points and calculated the basic reproduction number. The local stability results studied by applying the trace-determinant and Routh-Hurwitz Criterion. As well as, the global stability results studied by applying the technique and theorem of the Lyapunov function. In addition, we investigated the local bifurcation such as (transcritical, saddle node and pitchfork bifurcation) around all the equilibrium points of the proposed model according to Sotomayor's theorem. Through the numerical simulation, we confirm the analytical results that have been interpreted through graphical representation. The graphic representation provides rich details to the global health through which mathematical models can be an important factor in understanding the behavior of epidemic disease spread and control.

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